Bell Inequalities for Highly Entangled Four-Party Quantum States Beyond the Stabilizer Formalism

Absolutely maximally entangled (AME) states are multipartite pure quantum states that are maximally entangled across every balanced bipartition. They generalize Bell and GHZ states to larger numbers of parties and higher local dimensions. In the four-party case considered in this proposal, with local dimension $d \geqslant 3$, every two-party reduced state obtained by tracing out the remaining two subsystems is maximally mixed.

Many known examples of such states, denoted by AME$(4, d)$, arise from the stabilizer or graph-state formalism, where a state is defined as the unique joint +1 eigenstate of an appropriate subgroup of the generalized Pauli group. A particularly important exception is provided by the AME$(4, 6)$ class as none of its inequivalent representatives admits a stabilizer description and each exhibits a rich algebraic structure. Although the AME$(4, 4)$ and AME$(4, 5)$ classes also contain non-stabilizer examples, AME$(4, 6)$ is the first local dimension in this setting for which the entire class is non-stabilizer. Despite their complexity, these states are relevant for quantum error correction, multipartite teleportation, and fault-tolerant quantum computation. Recently, a concrete quantum-circuit implementation of a particular AME$(4, 6)$ state was proposed, being an important step toward their experimental realization.

The main objective of these preliminary studies is to pave the way toward deriving a four-party Bell inequality for which a specific AME$(4, 6)$ state achieves a provably nonclassical quantum value. Concretely, one selects local measurement settings and constructs a Bell functional, i.e., a linear combination of joint outcome correlators, such that its value on the AME$(4, 6)$ state violates the classical bound and is close to, or ideally saturates, the best available quantum upper bound. While analogous constructions for stabilizer states (like AME$(4, 3)$) are well established, the non-stabilizer formalism requires new methods and the relevant observables must be inferred from the internal algebraic and/or symmetry structure of the state.

Within the MINIATURA project, the main deliverable will be a tailored four-party Bell inequality, together with the tightest classical and quantum bounds attainable by the available methods. Full self-testing and robustness analyses will be deferred to future work. The research plan includes:

- Symmetry and structure analysis: Identify special symmetries or internal block structures of an AME$(4, 6)$ representative and encode them into the choice of observables and the Bell functional.
- Classical bound evaluation: Develop algebraic or combinatorial techniques to avoid brute-force and time-consuming routines, in order to compute the classical (local hidden-variable) bound efficiently.
- Quantum value computation: Use the Navascués-Pironio-Acín (NPA) hierarchy to obtain upper bounds on the quantum value of the chosen functional and verify if the AME$(4, 6)$ correlations saturate these bounds.

Deriving a Bell inequality tailored to a given pure entangled state is demanding. Since exact classical and quantum bounds are often out of reach, a minimal but crucial objective is to establish the tightest possible bounds. This will ensure the non-triviality of the inequality and verify that its quantum value exceeds its classical counterpart.

To gain additional knowledge, ideas and support, short-term visits to scientific institutions in Poland and abroad are planned. Discussions with experts on entanglement and nonlocality will help to explore new perspectives and acquire experience for developing as an independent researcher.

Achieving the predicted Bell violation would provide device-independent evidence of high-dimensional multipartite nonlocality arising from measurements on AME$(4, 6)$ states and would rule out any classical local strategy reproducing the observed correlations. Such a tailored four-party Bell test would be an important first step toward a full device-independent certification protocol for quantum processors operating beyond the qubit paradigm. Finally, it would provide a foundation for a broader research program on self-testing non-stabilizer AME states which would be a part of a more ambitious long-term project.