Objectivity in open quantum systems

A thesis submitted in partial fulfillment of the requirements for the doctoral degree in Physical Sciences by

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Dedicated to my father in heaven, my mother and my two sisters.

Abstract

Since the birth of quantum mechanics, the macroscopic world has not been consistently explained along with the microscopic world, despite our belief that microscopic objects are fundamental ingredients of the macroscopic world. In recent years, it has been one of the most popular views on a mechanism behind the quantum-to-classical transition that such apparent distinction between the macroscopic and microscopic worlds comes from a collective effect known as decoherence due to a large number of uncontrolled degrees of freedom.

This thesis examines an advanced decoherence mechanism in two simple but important theoretical models of open quantum systems: a harmonic oscillator interacting with either a collection of harmonic oscillators, called quantum Brownian motion (QBM) model, or spins, called a boson-spin model. Our interest is to find signatures of an emergent classicality through environmental interactions in the systems as proposed by the quantum Darwinism idea. More concretely, we look for "objectivity" emerging from quantum states, based on further development of quantum Darwinism, called spectrum broadcast structures (SBS). These are specific multipartite quantum state structures, encoding an operational notion of objectivity.

In quantum Darwinism, the process of classicalization through environmental interactions is parallel to the information transfer from a central system to the environment. We quantify this information transfer by the distinguishability for the quantum states of the environment. Due to such dualism, our focus in the dynamics are on the situation where the state of a central oscillator is close to a classical state and not much influenced by the environmental interactions, which is not the one usually adopted in open systems theory, e.g. in the master equation approaches such as the Born-Markov approximation. We choose realistic initial conditions of the environment such as the thermal state. In order to identify objectivity in the structure of a quantum state, we analyze the decoherence factor and the generalized overlap, which measure decoherence and the environmental state distinguishability, respectively.

We obtain two different length scales associated with decoherence and distinguishability for both systems. This may come as a surprise because the decoherence scale is usually treated as the classicalization scale. The consequences in both systems for objectivity are the following: i) the bigger the number of environmental subsystems are taken into account, the better objectivity occurs, ii) the decoherence factor and the generalized overlap form a complementary relation and iii) distinguishability is more difficult to obtain than decoherence. Especially, in the boson-spin model, we find that the initial momentum contribution of a central oscillator and a spin self-Hamiltonian play a crucial role in objectivity. We have also found an interesting application of the Floquet theory in this model.

Finally, we study objectivity from a quantum-information point of view. Working in the boson-spin model, we treat the information transfer to the environment as a quantum channel. We analyze the Holevo quantity, which describes the maximum information transfer into a spin environment. This provides an interesting example of application for the continuous variable version of the Holevo theorem to open quantum systems and the quantum-to-classical transition.

Streszczenie

Od narodzin mechaniki kwantowej, powiązanie świata makroskopowego z mikroskopowym wciąż nie zostało konsekwentnie wyjaśnione, pomimo naszego przekonania, że obiekty mikroskopowe są fundamentalnymi składnikami świata makroskopowego. W ostatnich latach popularny pogląd na mechanizm stojący za przejściem od kwantowo-klasycznym jest taki, że rozróżnienie między tymi światami wynika z efektu zbiorowego, znanego jako dekoherencja i spowodowanego oddziaływaniem z dużą liczbą niekontrolowanych stopni swobody.

Niniejsza praca doktorska bada zaawansowane mechanizmy dekoherencji w dwóch prostych, ale ważnych teoretycznych modelach otwartych układów kwantowych: oscylatora harmonicznego oddziałującego ze otoczeniem oscylatorowym, nazywanego modelem kwantowego ruchu Browna (QBM), albo z otoczeniem spinowym, nazywanego modelem bozon-spin. Celem pracy jest zbadanie śladów klasyczności, pojawiających się pod wpływem oddziaływania z otoczeniem, tak jak zaproponowano to w programie kwantowego darwinizmu. Bardziej konkretnie, szukamy „obiektywności" wyłaniającej się ze stanów kwantowych, opierając się na rozwinięciu idei kwantowego darwinizmu, zwanym strukturami rozgłoszeniowymi (SBS). Są to specyficzne wielocząstkowe struktury stanów kwantowych, które kodują operacyjne pojęcie obiektywności.

W kwantowym darwinizmie, proces uklasycznienia poprzez oddziaływanie z otoczeniem przebiega dzięki przekazywaniu informacji z układu do otoczenia. Z tego powodu przy badaniu dynamiki skupiamy się na sytuacji, w której stan centralnego oscylatora jest bliski stanowi klasycznemu i nie jest zbytnio zaburzany przez oddziaływanie z otoczeniem. Nie jest to typowym podejściem w teorii układów otwartych, np. tych opartych na równaniu master i przybliżeniu Born-Markov. Wybieramy realistyczne warunki początkowe otoczenia, takie jak stan termiczny. Do opisania przekazu informacji używamy rozróżnialności stanów kwantowych otoczenia. Nieco dokładniej mówiąc, aby zidentyfikować obiektywność w strukturze stanu kwantowego, wyprowadzamy i badamy współczynniki dekoherencji oraz uogólnione nakładanie się stanów (ang. generalized state overlap), które mierzą odpowiednio dekoherencję i stopień rozróżnialności stanów otoczenia.

Jednym z wyników pracy są uzyskane dwie różne skale długości związane z dekoherencją i rozróżnialnością w obu modelach. Może to wydać się zaskakujące, ponieważ zwyczajowo skala dekoherencji jest traktowana jako skala uklasycznienia. Konsekwencje dla obiektywności są następujące: i) im większa liczba podsystemów otoczenia jest branych pod uwagę, tym lepsza obiektywność, ii) czynniki dekoherencji i uogólnione nakładanie się spełniają relację typu komplementarności, iii) rozróżnialność jest trudniejsza do uzyskania niż dekoherencja. W szczególności w modelu bozon-spin początkowy pęd centralnego oscylatora oraz ewolucja swobodna spinów otoczenia odgrywają kluczową rolę w obiektywności. Przy okazji tego modelu, znaleźliśmy również interesujące zastosowanie teorii Floquet-a.

Na koniec badamy obiektywność z punktu widzenia procesów informacji kwantowej. Pracując w modelu bozon-spin, traktujemy przekaz informacji do otoczenia jako kanał kwantowy. Analizujemy wielkość Holevo, która opisuje maksymalny możliwy transfer informacji do środowiska spinowego, rozumiany jako górne ograniczenie na pojemność odpowiedniego kanału kwantowego. Stanowi to interesujący przykład zastosowania wersji twierdzenia Holevo dla zmiennych ciągłych do otwartych układów kwantowych i przejścia kwantowego-klasycznego.

Declaration

This dissertation is based the following list of articles, supplementing introductions, summaries and conclusions.

1. Tae-Hun Lee and Jarosław K. Korbicz, "Complementarity between decoherence and information retrieval from the environment", Phys. Rev. A **109**, 032221 (2024)

2. Tae-Hun Lee and Jarosław K. Korbicz, "Encoding position by spins: Objectivity in the boson-spin model", Phys. Rev. A **109**, 052204 (2024)

3. Tae-Hun Lee and Jarosław K. Korbicz, "Holevo bound and objectivity in the boson-spin model", [arXiv:2409.01186 [quant-ph]], doi:10.48550/arXiv.2409.01186, (currently under review in Physical Review A)

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Contents

Chapter 1

Introduction and preliminary notations

Quantum mechanics is a general framework to describe both microscopic and macroscopic worlds. Thus, the principles of quantum mechanics do not distinguish these two scales. However, they separate a measurement process, which is associated with a macroscopic device, out of the rest, without explaining any physical detail on it. This apparent inconsistency, i.e. the separation between the two scales in general, has still remained largely unresolved and is known as the quantum-to-classical transition problem. The reason that a full and satisfactory solution to the problem has been unknown for more than a century might be that in practice, solving such a problem could have been regarded as just for theoretical completeness without affecting the rest of physics. Nevertheless, this issue cannot stay completely isolated when it extends to explain the absence of macroscopic quantum mechanical effects such as the incompatibility and the superposition principle and so on.

This dissertation aims at answering how far the present quantum mechanics can fill the gap between the theory and our perceived nature by investigating simple but important models of open quantum systems, allowing us to ultimately find whether any missing parts in quantum mechanics could exist. "Quantum Darwinism" [1] is a mechanism within quantum mechanics, currently being one of the most popular mechanisms to explain the inconsistency gap between microscopic and macroscopic worlds and hence ultimately a measurement process using environmental effects. The dissertation is based on its further development, called "spectrum broadcast structures" (SBS) [2, 3, 4]. In this introduction we present relevant concepts to understand the subsequent main articles.

1.1 Classical world from quantum mechanics

In our daily views on the world, all we see outside seem to exist independent of our observation. To any extent, our observations should disturb the already existing reality, but our disturbance always seems to be well controlled and minimized as much as we wish, so as to extract precise information representing the truly existing reality. However, our observations on the microscopic world reveal that such a picture is completely different from the actual nature and should be replaced by a new law, "quantum mechanics", disallowing us to minimize our disturbance at will. If our daily view is believed to be composed of microscopic worlds, we should accept that the pictures of the daily perceived world, a so-called "classical world" are collective effects with the underlying law, "quantum mechanics". Despite this significant difference between two worlds, it has not been fully shown since the birth of quantum mechanics how our daily world can be derived within quantum mechanics. It sounds absurd that a new fundamental theory has not been able to find its own way to explain phenomena well explained by the old one. However, this uneasy circumstance does not necessarily means that quantum mechanics needs to be modified, unless either any discrepancy between predictions from the theory and experimental results are found or any logical contradiction is pinpointed out.

The difficulty of this problem may not come from the problem itself but rather from the fact that we have not been able to define the problem rigorously. The principles of quantum mechanics do not discriminate, e.g. between light and massive objects, so we expect to equally see microscopic phenomena for massive objects in our daily life, for instance a superposition state of macroscopic objects. However, such phenomena have not been observed yet in the macroscopic world.

Similarly, the principles of quantum mechanics do not include the measurement into dynamics, which is performed by interaction with a macroscopic object, i.e. a macroscopic device. This separation is inconsistent but it is not certain whether it is not possible that quantum mechanics is able to fill the gap between measurement and the rest or we have not yet found a way to construct suitable quantum mechanical models. This unexplained gap culminates and is specified in a so-called "measurement problem" stating that if all the quantum mechanical processes are explained by a quantum mechanical unitary evolution, measurement processes should be a unitary process, which, however, neither singles out one out of all possible outcomes in a quantum state nor describes the disappearance of quantum coherences.

At present it would be the first necessary task to investigate how far quantum mechanics can reduce the gap between a unitary evolution and a measurement.

1.2 Role of the environment in quantum mechanics

New nature in the microscopic scale described by quantum mechanics should be extended to the macroscopic scale and show that classical physics is an approximation of quantum physics. Therefore, we expect that all the quantum mechanical phenomena observed in the microscopic world should appear in the macroscopic world.

The laws of nature have developed based on our intuition formed by our daily experiences that the properties of physical systems are locally identified on fundamental physical objects in space and time. Such concept is further generalized to one that the fundamental laws of physics should exist independent of a size of a system and hence concentrated at a point in space and time. If such local fundamental laws and at the same time the microscopic world both represents our true nature, it is quite difficult to explain the disappearance of quantum properties in the macroscopic world. Rather, it is plausible to consider that our local image of the macroscopic world is emergent as collective phenomena. This idea considers environment not just a simple disturbance but a crucial ingredient to make local phenomena appear.

It turns out that the environment interacting with a quantum system plays a significant role removing quantum coherences towards a particular basis, so-called "pointer states" of a quantum system [5, 6]. This process is called "decoherence" [7, 8]. Its further developments called quantum Darwinism [1] and spectrum broadcast structures (SBS) [2, 3, 4] are more advanced mechanisms that aim to describe the emergent objectivity of the macroscopic world.

1.3 Decoherence

Decoherence [7, 8] is a process due to interactions between a central system and the environment, which alone damps quantum coherences in the reduced state ρ_S of the central system and selects a particular basis. It manifests vanishing of the "coherent" (superposition) parts, i.e. off-diagonal elements, so the most classical density matrix [9, 10] is in a diagonal form, i.e. a statistical mixture of orthonormal states $\{|i\rangle\langle i|\}$ with some probabilities p_i,

$$\rho_S = \sum_i p_i |i\rangle\langle i|. \tag{1.1}$$

Here $\{|i\rangle\}$ is called a pointer basis [5, 6], which, if exists, is uniquely determined by the Hamiltonian of the system. If the diagonalization (1.1) happens in the evolution, it is at least a partial explanation of classicalization within quantum mechanics. Decoherence cannot happen in a unitary evolution and needs a non-unitary process of averaging or ignoring some degrees of freedom, called a "partial trace". Also, in most cases, the diagonalization (1.1) happens asymptotically in time.

It is in general a difficult problem to derive conditions for decoherence and pointer states for a given general Hamiltonian. But it can be illustrated on an important example of the following Hamiltonian:

$$H = A \otimes \sum_k^N g_k B^{(k)}, \tag{1.2}$$

where g_k is a coupling constant, an operator A is an observable in the Hilbert space for a central system and an operator $B^{(k)}$ is an observable in the Hilbert space for the environment which we assume to be compounded, with a short notation $B^{(k)} = \mathbb{I}_1 \otimes \cdots \otimes B^{(k)} \otimes \cdots \otimes \mathbb{I}_N$. The Hamiltonian (1.2) is called a bi-linear interaction between an operator A for a central system S and $B^{(k)}$ for the kth environment E_k, where interactions among $B^{(k)}$ are not considered. Such interaction Hamiltonians are popular in open quantum models. We also use the conventional assumption that an initial density matrix is a product state:

$$\rho_{S:E}(0) = \rho_S(0) \otimes \bigotimes_k^N \rho_k(0). \tag{1.3}$$

We expand the initial states $\rho_S(0)$ and $\rho_k(0)$ with the eigenbasis of A and $B^{(k)}$,

$$\rho_S(0) = \sum_{a,a'} c_{a,a'} |a\rangle\langle a'|, \;\; \rho_k(0) = \sum_{b_k,b'_k} \gamma_{b_k,b'_k} |b_k\rangle\langle b'_k|, \tag{1.4}$$

where $A|a\rangle = a|a\rangle$ and $B^{(k)}|b_k\rangle = b_k|b_k\rangle$. The Hamiltonian (1.2) allows us to conveniently factor out the unitary evolution operator $U_{S:E}(t)$ into each systems as

$$U_{S:E}(t) = e^{-iHt} = \sum_a |a\rangle\langle a| \otimes \bigotimes_k^N U_k(a), \tag{1.5}$$

where the unitary evolution operator for the environment E_k $U_k(a)$

$$U_k(a) \equiv e^{-itag_k B^{(k)}}. \tag{1.6}$$

Here we note that for readers the unit $\hbar = 1$ is assumed in this thesis without mentioning. The final state $\rho_{S:E}(t)$ is obtained by a unitary evolution:

$$\begin{aligned}\rho_{S:E}(t) &= U_{S:E}(t)\rho_{S:E}(0)U^\dagger_{S:E}(t)\\ &= \sum_{a,a'} c_{a,a'}|a\rangle\langle a'| \otimes \bigotimes_k^N \sum_{b_k,b'_k} \gamma^{(k)}_{b_k,b_k} U_k(a)|b_k\rangle\langle b'_k|U^\dagger_k(a')\\ &= \sum_{a,a'} c_{a,a'}|a\rangle\langle a'| \otimes \bigotimes_k^N \sum_{b_k,b'_k} \gamma^{(k)}_{b_k,b_k} e^{-itg_k(ab_k - a'b'_k)}|b_k\rangle\langle b'_k|.\end{aligned} \tag{1.7}$$

A reduced state for a central system S, $\rho_S(t)$, is obtained by tracing out environmental degrees freedom of $\{E_k\}$:

$$\begin{aligned}\rho_S(t) &= \sum_{a,a'} c_{a,a'}|a\rangle\langle a'| \prod_k^N \sum_{b_k,b'_k} \gamma^{(k)}_{b_k,b_k} e^{-itg_k(ab_k - a'b'_k)} \mathrm{Tr}_B(|b_k\rangle\langle b'_k|)\\ &= \sum_{a,a'} \prod_k^N \sum_{b_k} c_{a,a'}\gamma^{(k)}_{b_k,b_k} e^{-itg_k(a-a')b_k}|a\rangle\langle a'|\\ &= \left(\prod_k^N \sum_{b_k} \gamma^{(k)}_{b_k,b_k}\right) \sum_a c_{a,a}|a\rangle\langle a| + \sum_{a\neq a'} \prod_k^N \sum_{b_k} c_{a,a'}\gamma^{(k)}_{b_k,b_k} e^{-itg_k(a-a')b_k}|a\rangle\langle a'|\\ &= \sum_a c_{a,a}|a\rangle\langle a| + \sum_{a\neq a'} c_{a,a'}\Gamma_{a\neq a'}(t)|a\rangle\langle a'|,\end{aligned} \tag{1.8}$$

where the normalization condition $\mathrm{Tr}\rho_k(0) = \sum_{b_k} \gamma^{(k)}_{b_k,b_k} = 1$ has been used. Here we have introduced a measure of decoherence, called the "decoherence factor", $\Gamma_{a\neq a'}$:

$$\Gamma_{a\neq a'}(t) \equiv \prod_k^N \sum_{b_k} \gamma^{(k)}_{b_k,b_k} e^{-itg_k(a-a')b_k}. \tag{1.9}$$

In general, the condition for vanishing $\Gamma_{a\neq a'}(t)$ depends on several factors, including the initial distribution $\gamma^{(k)}_{b_k,b_k}$ but apart from particular distributions of $\gamma^{(k)}_{b_k,b_k}$, if a large number of environmental degrees of freedom traced out, i.e. $N \to \infty$ and $t \to \infty$, then one usually expects:

$$|\Gamma_{a\neq a'}(t)| \to 0. \tag{1.10}$$

This shows that the basis $|a\rangle$, which is the eigenbasis of A, is selected to be a pointer basis by the interaction due to the decoherence. In general, due to the existence of a self-Hamiltonian, it is difficult to find a pointer basis [5, 6].

1.4 Pointer state

In the last section 1.3, we have just showed that it is the significant feature of decoherence that given the Hamiltonian for a joint system, it can determine a particular basis in which a reduced state of a central system is diagonalized. This basis state is called a "pointer basis" [5, 6]. The pointer basis is chosen by the dynamical stability condition [6]. Theses are the states least affected by the open evolution. For example, ideal pointers for $H = X \otimes A + \mathbb{I} \otimes H_E$ are obtained by expanding the operator X in its eigenbasis $\{|x\rangle\}$,

$$\begin{aligned} H &= X \otimes A + \mathbb{I} \otimes H_E \\ &= \sum_x |x\rangle\langle x| \otimes H_x, \end{aligned} \tag{1.11}$$

where the effective Hamiltonian for environment $H_x = xA + H_E$. Using (1.11) a unitary evolution operator $U_{S:E}(t)$ is expressed in the similar form as before in (1.5):

$$U_{S:E}(t) = e^{-iHt} = \sum_x |x\rangle\langle x| \otimes e^{-iH_x t}. \tag{1.12}$$

We can now easily see that the eigenbasis $|x\rangle$ is dynamically stable, i.e.

$$[|x\rangle\langle x|, H] = 0. \tag{1.13}$$

This is the physical condition that defines (ideal) pointer states [6]. The unitary operator $U_{S:E}(t)$ in (1.12) is the same form as (1.5) in the last section 1.3. For more general evolutions, when ideal pointer states are not available, there is a method called the predictability sieve [11] but we will not study it here.

Here we would just like to make a comment that for a general Hamiltonian as time goes on a pointer basis will get deviated from ideal pointers $\{|x\rangle\}$ as the eigenbasis of X in (1.11). We consider the unitary evolution only up to the first order in the interaction picture. We will show that the pointer states in the interaction picture are approximately given by the eigenbasis of $\int_0^t dt' X_I(t')$, where $X_I(t) \equiv e^{iH_S t} X e^{-iH_S t}$. It means that in the Schrödinger picture it is not the eigenbasis of X but for a short time it. For a more general Hamiltonian

$$H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + X \otimes A, \tag{1.14}$$

it is non-trivial to define a pointer basis from the given Hamiltonian due to the non-commutativity of operators in the Hamiltonian. But we can sketch how an approximate pointer state appears. As we have just shown in the ideal pointer basis, we write H in (1.14) in the interaction picture as $H_I(t) = e^{iH_S \otimes \mathbb{I}_E t} H e^{-iH_S \otimes \mathbb{I}_E t}$ given by

$$H_I(t) = X_I(t) \otimes A + \mathbb{I} \otimes H_E, \tag{1.15}$$

where

$$X_I(t) = e^{iH_S t} X e^{-iH_S t}. \tag{1.16}$$

Here, even if the eigenbasis for $X_I(t)$ for a fixed t is found, it is not a pointer basis since due to the non-commutativity $[H_I(t_1), H_I(t_2)] \neq 0$ for $t_1 \neq t_2$, in general, a unitary evolution $U_I(t)$ is given by the Dyson series:

$$U_I(t) = 1 - i \int_0^t dt_1 H_I(t_1) + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H_I(t_1) H_I(t_2) + \cdots . \tag{1.17}$$

Up to the first order, we approximately express $U_I(t)$ as $U_I(t) \approx e^{-i \int dt' H_I(t')}$. Using (1.15),

$$\begin{aligned} \int_0^t dt' H_I(t') &= \tilde{X}_I(t) \otimes A + \mathbb{I} \otimes H_E t \\ &= \sum_{\tilde{x}_I(t)} |\tilde{x}_I(t)\rangle\langle\tilde{x}_I(t)| \otimes H_{\tilde{x}_I(t)}, \end{aligned} \tag{1.18}$$

where

$$H_{\tilde{x}_I(t)} \equiv \tilde{x}_I(t) A + t H_E, \tag{1.19}$$

$\tilde{X}_I(t) \equiv \int_0^t dt' X_I(t')$ and $\tilde{X}_I(t)$ has been expanded in its eigenbasis $|\tilde{x}_I(t)\rangle$ satisfying

$$\tilde{X}_I(t)|\tilde{x}_I(t)\rangle = \tilde{x}_I(t)|\tilde{x}_I(t)\rangle. \tag{1.20}$$

The unitary evolution operator in the interaction picture $U_{I,S:E}(t)$ is approximately expressed in the same form as (1.5):

$$U_{I,S:E}(t) \approx e^{-i\int dt' H_I(t')} = \sum_{\tilde{x}_I(t)} |\tilde{x}_I(t)\rangle\langle\tilde{x}_I(t)| \otimes e^{-iH_{\tilde{x}_I(t)}}. \tag{1.21}$$

Following the same procedure to express the reduced state $\rho_S(t)$ in (1.8), we expect to have a similar form of the decoherence factor in (1.9). Considering the eigenbasis $\{|E_{\tilde{x}_I,t}\rangle\}$ of $H_{\tilde{x}_I(t)}$ in (1.19)

$$H_{\tilde{x}_I(t)}|E_{\tilde{x}_I,t}\rangle = E_{\tilde{x}_I,t}|E_{\tilde{x}_I,t}\rangle \tag{1.22}$$

and expanding an initial state of environment $\rho_E(0)$ in $\{|E_{\tilde{x}_I,t}\rangle\}$,

$$\rho_E(0) = \sum_{E_{\tilde{x}_I,t},E'_{\tilde{x}_I,t}} \gamma_{E_{\tilde{x}_I,t},E'_{\tilde{x}_I,t}} |E_{\tilde{x}_I,t}\rangle\langle E'_{\tilde{x}_I,t}|, \tag{1.23}$$

similarly to (1.9) we can write the decoherence factor $\Gamma_{\tilde{x}_I(t)\neq\tilde{x}'_I(t)}(t)$ at fixed t ,

$$\Gamma_{\tilde{x}_I(t)\neq\tilde{x}'_I(t)}(t) = \sum_{E_{\tilde{x}_I,t}} \gamma_{E_{\tilde{x}_I,t},E_{\tilde{x}_I,t}} e^{-i(\tilde{x}_I(t)-\tilde{x}'_I(t))E_{\tilde{x}_I,t}}. \tag{1.24}$$

From (1.19) and (1.22), $E_{\tilde{x}_I,t}$ is expected to get large as $t \to \infty$. Thus, in the same argument as in (1.9) and (1.10), $\Gamma_{\tilde{x}_I(t)\neq\tilde{x}'_I(t)}(t)$ can tend to vanish for large environments,

$$\lim_{t\to\infty} |\Gamma_{\tilde{x}_I(t)\neq\tilde{x}'_I(t)}(t)| \to 0. \tag{1.25}$$

Returning to the Schrödinger picture, the pointer basis $|x(t)\rangle_S$ would be approximately

$$|x(t)\rangle_S \approx e^{-iH_S t}|\tilde{x}_I(t)\rangle. \tag{1.26}$$

This shows that in this approximation, a pointer basis for a general Hamiltonian (1.14) is deviated from the eigenbasis for the operator X. We will further elaborate this argument with an example, the boson-spin model, where a central system is a harmonic oscillator in section 2.4.1 in the next chapter. For a harmonic oscillator, $X_I(t)$ is known and the basis $|\tilde{x}(t)\rangle_I$ is effectively just a position basis $|x\rangle$ for a short time.

In next section 1.5 we introduce the concept of "objectivity" before we are going to discuss a mechanism to make the classical world emerge.

1.5 Objectivity

It is our aim to derive our classical view emerging from quantum mechanics. In order to achieve the "classical world", it is necessary to satisfy one important concept, called "objectivity" [12]. Although a quantum state which is a statistical ensemble of orthogonal states in a density matrix description is regarded as the most classical, in addition, classical nature also requires the **preservation of physical information** under repetitive measurements, i.e. "**consensus information**". The objectivity concept can be understood analogously to a familiar fundamental concept in physics, the "invariance of physics" based on the philosophy that physics should be described by observer-independent quantities. The difference is that "objectivity" is the **preservation of information** under different and repetitive measurements while the "invariance of physics" means the existence of invariant quantities preserved under symmetry transformations over observers in different states.

The objectivity is defined below [12] as a major concept in quantum Darwinism.

"*...an objective property of the system of interest is (i) simultaneously accessible to many observers (ii) who are able to find out what it is without prior knowledge and (iii) who can arrive at a* ***consensus*** *about it without prior agreement.*"

More specifically, objectivity in a quantum state requires a statistical ensemble of orthogonal states and extra systems to have a large number of degrees of freedom interacting with a central system, called "environment". The former condition is based on the distinguishability in quantum mechanics that only orthogonal states can be perfectly distinguishable and stored into another system in a unitary evolution [13]. In quantum Darwinism, interactions for a system with environment select a particular basis for the reduced state of the remaining systems after tracing out the unobserved part of environment, throughout a decoherence process. Decoherence can lead to the orthogonal distribution not only on a central system but also on environmental side since information of a central system stored in environment needs to be distinguished due to measurement on environment. In addition, a large number of degrees of freedom in environment are required to store the same copies of the information about a central system and preserve the information in repetitive measurements.

1.6 Quantum Darwinism

Quantum Darwinism [1] is a mechanism that tries to explain how objectivity emerges and is maintained in quantum mechanics without modifying quantum mechanics. Using decoherence as a basic tool [14], it achieves two aspects required for classicality, a unitary evolution leads to **i) a pointer basis**: the natural selection of a particular state and **ii) information proliferation**: many copies of the same information about which pointer state the state of a central system is distributed in, to be embedded into environment. The proposed condition for quantum Darwinism [1] is formulated in terms of the quantum

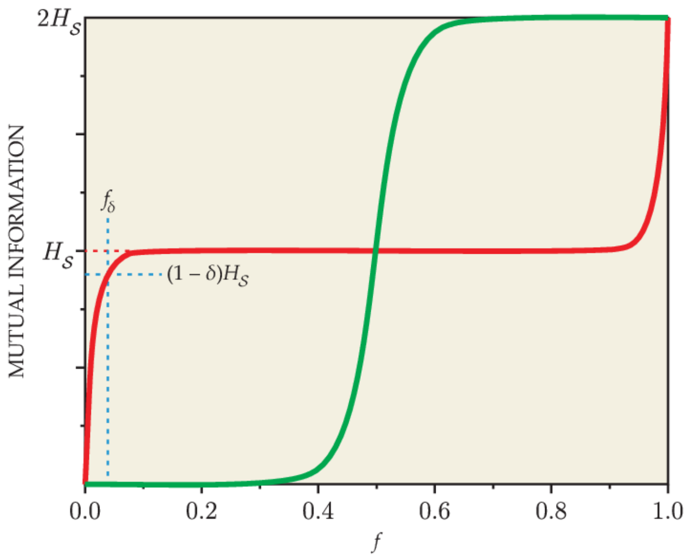

FIGURE 1.1: Mutual Information $I(S : oE)$ vs the fraction f for a remaining environment over the total number of environmental systems. $H_S = S(\rho_S)$ is the von Neumann entropy [10] for the central system. δ is called information deficit [15]. The red line represents a typical behaviour of the mutual information of a system-environment composite system through a decoherence process while the green line represents the quantum mutual information based on randomly chosen pure state of environment [15] (this plot is brought from [15]).

mutual information $I(S : oE)$ between a central system S and the observed environment oE, which measures the common information between S and oE after the unobserved environment uE traced out:

$$I(S : oE) = S(\rho_S) + S(\rho_{oE}) - S(\rho_{S:oE}), \tag{1.27}$$

where $S(\rho_S)$, $S(\rho_{oE})$ and $S(\rho_{S:oE})$ are the von Neumann entropy for the reduced states ρ_S, ρ_{oE} and the joint state $\rho_{S:oE}$, respectively. The condition reads:

$$I(S:oE) = S(\rho_S) \text{ for all } oE. \tag{1.28}$$

When this condition is satisfied, it leads to a characteristic behaviour of the so-called partial information plot. Figure 1.1 shows that as the observed environmental fraction f increases the mutual information is saturated to the entropy of a central system $S(\rho_S)$ and in principle could reach twice as much as $S(\rho_S)$ [15]. Note that a decoherence process requires

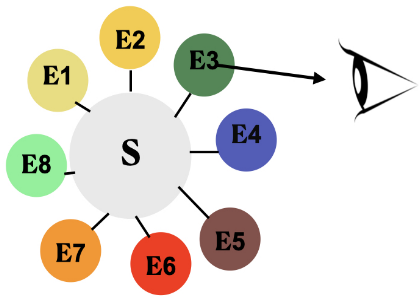

Figure 1.2: Extracting information about a system is done by measuring environmental fraction having the information transferred from a system.

a "partial tracing" which averages out the "unobserved" part of environment. As a central system and the environment interact, in general they get entangled with each other and after the unobserved environment traced out, a central system together with the rest environment, i.e. the "observed" environment, gets statistically mixed. The remaining system is a joint system of a central system and the observed environment. Quantum Darwinism interprets that the information of a central system is transferred into the environment. The same copies of information for a system are distributed to each of environmental systems. In this sense, in quantum Darwinism, the classicalization and information transfer simultaneously occur.

Our intuition often identifies a local physical state itself with information. This equivalence is unambiguous in classical physics. For instance, in the light postulate in special relativity, the "speed of light" should imply more precisely the speed of information carried by light. The statement assumes that physical reality can be identified with information. However, quantum Darwinism tells us that information is not perceived directly from a system of interest but rather relative statistical relations between environmental states which encodes the information of a system.

Classicality can be more specifically expressed by the objectivity concept [12]. Apparent physical reality perceived by observers should be universal and preserved over observers. In order to achieve objectivity in the quantum world, the environment requires a large number of degrees of freedom to destroy quantum coherences and stores a large number of the same copies of information. Each of our observations consumes the corresponding portion of environment (Figure 1.2 sketches the process of observation in quantum Darwinism). Thus, in quantum mechanics, information is not a quantum state itself but prevails throughout environment as a statistical mixture between quantum states. The environment in quantum Darwinism plays a crucial role forming redundant information of a system for the objective world. We introduce a more specific version of quantum Darwinism, the spectrum broadcast structure (SBS) [2, 3, 4] in section 1.7.

1.7 Spectrum broadcast structures(SBS)

It would be our ultimate goal to describe our classical view in quantum mechanical language. What is the best known quantum state based on quantum Darwinism corresponding to the classical world? What would be necessary conditions to specify objective quantum states in quantum Darwinism point of view?

We introduced the concept of objectivity in section 1.5. It is expressed into two familiar concepts for the notion of objectivity **i) non-disturbance** due to measurement and **ii) measurement agreement** over observers. The perfect non-disturbance of a given quantum state against measurement is obviously not possible due to the principle of the quantum measurement postulate unless the given state is the eigenstate of a measurement operator. Nevertheless, we can shift the non-disturbance concept for quantum states to information point of view, seeking for whether the same information of a state is accessible to many different observers.

This has lead to the formulation of spectrum broadcast structures (SBS) [2, 3, 4], which aim to explain quantum Darwinism at the level of quantum states. SBS is formed if after the unobserved environmental degrees of freedom traced out, the state of the central system S and observed environments labeled by k, $\rho_{S:oE}$, approaches a mixture:

$$\rho_{S:oE} = \sum_i p_i |i\rangle_S \langle i| \otimes \bigotimes_k \rho_{ki}, \tag{1.29}$$

where ρ_{ki} is the kth environmental state associated with a basis state $|i\rangle_S$ for a central system. We further specify the internal structure in $\rho_{S:oE}$ in (1.29). The basis $|i\rangle$ is the pointer basis. Taking measurement on any environmental fraction ρ_{ki} and estimating the probability p_i by repetitive measurements, we will make ρ_{ki} collapse. If we wish to preserve the same distinguishability as in a central state $\sum_i p_i|i\rangle\langle i|$, all ρ_{ki} must follow the same orthogonality condition:

$$\rho_{ki}\rho_{kj} = 0, \ \ (i \neq j). \tag{1.30}$$

In order to store the same information about a central system into the environment, it is not necessary how ρ_{ki} is internally structured as long as the orthogonality condition (1.30) is fulfilled. When (1.29) and (1.30) are satisfied, they define the SBS state [2, 3, 4]:

$$\rho_{SBS} = \sum_i p_i |i\rangle_S \langle i| \otimes \bigotimes_k \rho_{ki}, \tag{1.31}$$

where $\rho_{ki}\rho_{kj} = 0, \ (i \neq j)$. One can show that for the SBS state, the quantum Darwinism condition is satisfied:

$$I(\rho_{S:oE}) = S(\rho_S), \tag{1.32}$$

where $\rho_S = \sum_i p_i|i\rangle\langle i|$, $I(\rho_{S:oE})$ is the quantum mutual information for a joint system of a central system S and the observed environment oE and $S(\rho_S)$ is the von Neumann entropy of the reduced state ρ_S. However, the converse is not true in general [16]. In this sense, SBS is a stronger form of quantum Darwinism.

Now we show how the SBS state encodes objectivity. The SBS state has the following properties. Define the projectors Π_i^k:

$$\Pi_i^k \equiv \text{ orthogonal projection on supp } \rho_k, \tag{1.33}$$

where the environmental index k runs as $k = 1, \cdots N$ and the central system index i is associated with a basis state $|i\rangle$. In particular, we have

$$\Pi_j^k \rho_k \Pi_j^k = \delta_{ij} \rho_k \tag{1.34}$$

or equivalently:

$$\Pi_i^k \Pi_j^k = \delta_{ij} \Pi_i^k \tag{1.35}$$

by the orthogonality condition in (1.30). Thus, the sets $\{\Pi_i^k\}$ define the von Neumann measurements in each sub-environment E_k. The environmental observers will use those measurements to get the information about the state of the central system, that is index i (the observer measuring the central system directly will use $|i\rangle\langle i|$). Let any group of the observers measure independently their respective parts of the environment, say labeled by $k_1, k_2, \cdots$. Consider the probability of obtaining the results $i_1, i_2, \cdots$, $P(i_1, i_2, \cdots)$ on ρ_{SBS} in (1.31):

$$\begin{aligned} P(i_1, i_2, \cdots) &= \mathrm{Tr}[(\Pi_{i_1}^{k_1} \otimes \cdots) \rho_{SBS} (\Pi_{i_1}^{k_1} \otimes \cdots)] \\ &= \sum_i p_i \mathrm{Tr}(\Pi_{i_1}^{k_1} \otimes \cdots) |i\rangle\langle i| \otimes \bigotimes_k \rho_{ki} (\Pi_{i_1}^{k_1} \otimes \cdots) \\ &= \sum_i p_i \mathrm{Tr}(\Pi_{i_1}^{k_1} \rho_{k_1 i})(\Pi_{i_2}^{k_2} \rho_{k_2 i}) \cdots \mathrm{Tr}[|i\rangle\langle i| \otimes \bigotimes_{k \neq k_1, k_2, \cdots} \rho_{ki}] \\ &= \sum_i p_i \delta_{i i_1} \delta_{i i_2} \cdots = \begin{cases} p_i & \text{for } i_1 = i_2 = \cdots = i \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \tag{1.36}$$

where (1.34) and (1.35) have been used. Thus, this relation (1.36) implies the probability of independent observations vanishes unless it gives the same result. This is the agreement - all measure the same index i. As a remark, in the above derivation one can also include a 'superobserver' directly measuring the central system in the basis $|i\rangle$. The conclusion remains the same - the only possible event is when all the observers obtain the same outcome. Moreover, for each observer this index i is distributed with the same probability p_i. Indeed, the state seen by each observer reads:

$$\rho_{E_k} \equiv \mathrm{Tr}_{SE \setminus E_k}[\rho_{SBS}] = \sum_i p_i \rho_{ki}. \tag{1.37}$$

Let us look at the averaged post-measurement state (=unread measurement). From the first principles:

$$\begin{aligned}
\rho' &= \sum_{i_1,i_2,\cdots} P(i_1,i_2,\cdots)\frac{(\Pi_{i_1}^{k_1}\otimes\cdots)\rho_{SBS}(\Pi_{i_1}^{k_1}\otimes\cdots)}{\mathrm{Tr}[(\Pi_{i_1}^{k_1}\otimes\cdots)\rho_{SBS}(\Pi_{i_1}^{k_1}\otimes\cdots)]} \\
&= \sum_{i_1,i_2,\cdots} p_i\frac{(\Pi_{i_1}^{k_1}\otimes\cdots)\rho_{SBS}(\Pi_{i_1}^{k_1}\otimes\cdots)}{p_i} \\
&= \sum_{ij} p_j|j\rangle\langle j|\otimes\Pi_i^{k_1}\rho_{k_1 j}\Pi_i^{k_1}\otimes\Pi_i^{k_2}\rho_{k_2 j}\Pi_i^{k_2}\cdots\otimes\bigotimes_{k\neq k_1,k_2,\cdots}\rho_{kj} \\
&= \sum_{ij} p_j|j\rangle\langle j|\otimes\delta_{ij}\rho_{k_1 i}\otimes\delta_{ij}\rho_{k_2 j}\cdots\otimes\bigotimes_{k\neq k_1,k_2,\cdots}\rho_{kj} \\
&= \sum_{i} p_i|i\rangle\langle i|\otimes\bigotimes_{k}\rho_{ki},
\end{aligned} \tag{1.38}$$

where the definition ρ_{SBS} (1.31), the result $P(i_1,i_2,\cdots)$ (1.36), and the measurement property (1.34) have been used. Thus, the measurements defined by (1.33), and only those, do not disturb the state (1.31) on average. This shows an operational notion of agreement: only $i = i_1 = i_2\cdots$ cases contribute to the final state. The notion of non-disturbance is encoded in the statistical sense as the averaged state after measurement is unchanged. From (1.30) and (1.31), we see that SBS is characterized by two distinct conditions: decoherence needed to reach the form (1.31) and the perfect distinguishability which is equivalent to (1.30).

In the following section, two characterizing functions are introduced for decoherence and distinguishability, i.e. the "decoherence factor" and the "generalized overlap" [17], called the "objectivity markers" [18].

1.8 Objectivity measures

SBS requires decoherence and distinguishability in a quantum state, characterized by two quantities, the decoherence factor and the generalized overlap, which we call the "objectivity markers" [18]. It can be shown when they vanish, a quantum state is approaching SBS [19]. Vanishing objectivity markers depend on values of parameters in the given model, initial conditions of a central system and environment and the number of environments. Usually, the number of environments is the most essential condition. We will derive them below. This thesis is especially interested in the case that a central system is not much influenced by environment and hence the evolution is approximately separable:

$$U_{S:E}(t) \approx \sum_{X} U_S(t)|X\rangle\langle X| \otimes U_E(X,t). \tag{1.39}$$

For simplicity, we assume that an initial state of a central system and environment is a separable state as

$$\rho_{S:E}(0) = [\rho_S(0) \otimes \rho_{oE}(0)] \otimes \rho_{uE}(0). \tag{1.40}$$

Furthermore, assuming that interaction between environmental systems is ignored so that each of environmental systems evolves separately, the final state for a central system S in basis $|X\rangle$ and the observed environment in oE space can be written in a simple form:

$$\begin{aligned}\rho_{S:E}(t) = &\sum_{X,X'} U_S(t)|X\rangle\langle X|\rho_S(0)|X'\rangle\langle X'|U_S^{\dagger}(t)\\ &\otimes U_{oE}(X,t)\rho_{oE}(0)U_{oE}^{\dagger}(X',t) \otimes U_{uE}(X,t)\rho_{uE}(0)U_{uE}^{\dagger}(X',t).\end{aligned} \tag{1.41}$$

The reduced state for the joint system of a central system S and the observed environment oE is expressed as

$$\begin{aligned}\rho_{S:oE}(t) = &\sum_{X,X'} U_S(t)|X\rangle\langle X|\rho_S(0)|X'\rangle\langle X'|U_S^{\dagger}(t)\\ &\otimes U_{oE}(X,t)\rho_{oE}(0)U_{oE}^{\dagger}(X,t)\mathrm{Tr}[U_{uE}(X,t)\rho_{uE}(0)U_{uE}^{\dagger}(X,t)].\end{aligned} \tag{1.42}$$

The decoherence factor is defined by coefficients of off-diagonal elements in system space for the remaining state, for $X \neq X'$,

$$\begin{aligned}\Gamma_{X,X'} &\equiv \mathrm{Tr}[U_{uE}(X,t)\rho_{uE}(0)U_{uE}^{\dagger}(X',t)],\ (X \neq X')\\ &= \prod_{k\in uE} \mathrm{Tr}[U_{uE}^{(k)}(X,t)\rho_{uE}^{(k)}(0)U_{uE}^{(k)\dagger}(X',t)]\\ &= \prod_{k} \Gamma_{X,X'}^{(k)},\end{aligned} \tag{1.43}$$

where

$$\Gamma_{X,X'}^{(k)} \equiv \mathrm{Tr}[U_{uE}^{(k)}(X,t)\rho_{uE}^{(k)}(0)U_{uE}^{(k)\dagger}(X',t)],\ (X \neq X'). \tag{1.44}$$

It is worthwhile to notice that if the decoherence factor vanishes, the reduced density matrix after the unobserved environment uE in (1.42) traced out is diagonalized in the basis $U_S(t)|X\rangle$:

$$\rho_{S:oE}(t) = \sum_X U_S(t)|X\rangle\langle X|U_S^\dagger(t) \otimes U_{oE}(X,t)\rho_{oE}(0)U_{oE}^\dagger(X,t). \tag{1.45}$$

We define the generalized overlap or fidelity [17] for the environmental states:

$$B(\rho,\rho') \equiv \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right)^2, \tag{1.46}$$

where

$$\begin{aligned}\rho &= U_{oE}(X,t)\rho_{oE}(0)U_{oE}^\dagger(X,t),\\ \rho' &= U_{oE}(X',t)\rho_{oE}(0)U_{oE}^\dagger(X',t).\end{aligned} \tag{1.47}$$

1.8.1 Finite vs infinite systems

So far we implicitly treated systems having finite degrees of freedom. For finite systems, the decoherence factor is well defined. However, for systems having infinite degrees of freedom, there is a problem to define the decoherence factor due to undefined normalized quantum states for continuous spectrum observables. This is illustrated by a system with the Hamiltonian $H = X \otimes \sum_k B_k$, where X has an infinite continuous spectrum:

$$\begin{aligned}\rho_{S:oE}(t) &= \mathrm{Tr}_{uE}\rho_{S:E}(t)\\ &= \int dxdy\rho(x,y)\Gamma_t(y-x)|x\rangle\langle y| \otimes \bigotimes_{k\in oE} U_{k,x}\rho_k U_{k,y}^\dagger,\end{aligned} \tag{1.48}$$

where $U_{k,x} = e^{-itxB_k}$ and the decoherence factor $\Gamma_t(y-x)$:

$$\begin{aligned}\Gamma_t(y-x) &= \prod_{k\in uE} \mathrm{Tr}(U_{k,x}\rho_k U_{k,y}^\dagger)\\ &= \prod_{k\in uE} \mathrm{Tr}[\rho_k e^{it(y-x)B_k}].\end{aligned} \tag{1.49}$$

For finite systems it is known that the decoherence factor $\Gamma_{a,a'}$ is

$$0 \le \Gamma_{a,a'} \le 1. \tag{1.50}$$

However, in (1.48) for the perfect decoherence, $\Gamma_t(y-x)$ should be $\Gamma_t(y-x) = \delta(y-x)$, which needs to be infinite at $x = y$ but in (1.49) at $x = y$ $\Gamma_t(y-x) = 1$. This implies that for infinite dimensional systems we can expect only approximate decoherence and distinguishability, formulated through the decoherence and distinguishability length scales.

Chapter 2

Description of the articles

The purpose of this chapter is to provide preliminary information helpful to understand technical parts in the main articles attached in chapter 3. This chapter contains more specific technical descriptions from the main articles than the previous introductory chapter 1.

2.1 Quantum Brownian motion (QBM)

In the models studied in this thesis, a harmonic oscillator is chosen as the central system and either harmonic oscillators or spins are considered as the environment. The first model we studied is a quantum Brownian motion (QBM) model. QBM is a composite system of a central harmonic oscillator and a collection of harmonic oscillators as the environment. Linear QBM models with a bi-linear interaction in a position have been extensively studied [20, 21, 22, 23, 24]. The typical Hamiltonian for a linear QBM is given by

$$H = H = H_S + \sum_i H_E^{(i)} + \sum_i H_{\text{int}}^{(i)}, \tag{2.1}$$

where

$$\begin{aligned} H_S &= \frac{P^2}{2M} + \frac{1}{2} M \Omega^2 X^2, \\ H_E^{(i)} &= \frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2, \\ H_{\text{int}}^{(i)} &= C_i x_i X, \end{aligned} \tag{2.2}$$

M and Ω are a mass and an angular frequency of a central harmonic oscillator, m_i and ω_i are a mass and an angular frequency of the ith environmental harmonic oscillator, respectively and C_i is a bi-linear interaction coupling between a central harmonic oscillator and the ith environmental oscillator. The choice of $H_{int} = \sum_i C_i x_i X$ is physically motivated by the fact that our classical world appears to take a position as a pointer state. The goal of studying this popular model was to study the formulation of SBS and information transfer to the environment, generalizing earlier works [24, 25, 26].

2.2 Regimes of interest

Although the exact solution for the QBM dynamics can be obtained (it is a quadratic Hamiltonian), it is usually of little use. QBM is approached on appropriate approximation schemes for one's own purpose. We are mainly interested in finding the state of a central system being decohered and in parallel the state of the environment encoding information about the central system, based on the quantum Darwinism scenario. In this case, the Born-Markov approximation commonly used for solving the master equation for a state of a central system only [8], where the state of the environment remains approximately the same, is not appropriate to our interest. On the other hand, we look for the time regime where the state of a central system remains as it is after the quantum-classical transition.

2.2.1 Recoilless limit and classical trajectories

Our regimes of interest can be physically interpreted as follows. As we see in our daily life, macroscopic objects need to be decohered from their quantum superposition states and hence to be classical due to some underlying mechanism. Then it would be a right question of our interest to ask whether such classical states would be decohered further or return to superposition states. After such a quantum-classical transition, we assume that the state of a central system won't be significantly changed by interaction. For such a regime, the appropriate limit is the Born-Oppenheimer approximation [27], where a central system remains unchanged while environment is changed according to it. It is also called the "recoilless limit".

Unlike in the previous works [24, 25, 26] which relied on the Born-Oppenheimer type of an ansatz, here we approach the recoilless limit using the path intergrals. This is motivated by the situation that the central oscillator is expected to follow approximately a classical trajectory. The path integral kernel $K(X, X_0; x, x_0)$, a matrix element for a unitary operator, i.e. $\langle X, x|U_{S:E}(t)|X_0, x_0\rangle$, approximated by the dominance of classical paths, is factored out into the state of an intact central system from the environment and the dynamical state of the environment driven by classical paths,

$$
\begin{aligned}
K(X, X_0; x, x_0) &\equiv \langle X, x|U_{S:E}(t)|X_0, x_0\rangle \\
&= \int \mathcal{D}X \int \mathcal{D}x e^{iS_{sys}+iS_{env}+iS_{int}} \\
&\approx D_t(X_0, X) e^{iS_{sys}[X_{cl}(t)]} \int \mathcal{D}x e^{iS_{env}[x(t)]+iS_{int}[X_{cl}(t),x(t)]},
\end{aligned}
\tag{2.3}
$$

where S_{sys} and S_{env} are actions for a central system and the environment, respectively, S_{int} is the action for interactions and $X_{cl}(t)$ is an unperturbed classical trajectory of a harmonic oscillator. Here $D_t(X_0, X)$ is called the van Vleck determinant [28], a quantum contribution of a central system alone. The above was obtained using the no-recoil condition:

$$
\frac{C_k}{M\Omega^2} \ll 1. \tag{2.4}
$$

2.2.2 Effective dynamics for the environment

From the path integral representation (2.3) the effective evolution of the environment $U_{\text{eff}}(t)$ is driven by a time-dependent Hamiltonian:

$$
i\frac{\partial U_{\text{eff}}(t)}{\partial t} = (H_{env} + H_{int}[X_{cl}(t)])U_{\text{eff}}(t), \tag{2.5}
$$

where H_{env} and $H_{int}[X_{cl}]$ are given in (2.2). This equation can be solved giving in the interaction picture [29]:

$$
U_t^k = e^{i\zeta_k(t)} D\left(-i\frac{C_k}{\sqrt{2m_k\omega_k}} \int_0^t d\tau e^{i\omega_k\tau} X_{cl}(\tau)\right), \tag{2.6}
$$

where $\zeta_k(t)$ is some phase, which is irrelevant to the objective measures and the displacement operator $D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a}$. This leads to the approximate partially traced state [29]:

$$
\begin{aligned}
\langle X'|\rho_{S:oE}(t)|X\rangle \approx \int\int & dX_0 dX_0' \langle X_0'|\rho_S(0)|X_0\rangle K_t^{SC}(X, X_0) K_t^{SC}(X', X_0') \\
& \times F[X_{cl}(t), X_{cl}'(t)] U_t^k[X_{cl}] \rho_k(0) U_t^k[X_{cl}'],
\end{aligned}
\tag{2.7}
$$

where $K_t^{SC}(X', X_0')$ is the path integral kernel for an unperturbed harmonic oscillator and the influence functional $F[X_{cl}(t), X'_{cl}(t)]$ is given by

$$F[X_{cl}(t), X'_{cl}(t)] = \prod_{k \in uE} \mathrm{Tr}(U_t^k[X'_{cl}] U_t^k[X_{cl}] \rho_k(0)]). \tag{2.8}$$

2.3 Length scales

From the point of view of information transfer, the objectivization process can be conceptually divided into two parts. One is a selection process of a pointer basis via decoherence and the other is how environmental states become distinguishable. The former is measured by the decoherence factor (or the influence functional) and the latter by the generalized overlap. The decoherence factor measures how quantum coherences in the state of the central system are distributed, relative to some particular scale. Information about a central system is obtained via interactions with the environment after the environment absorbs the information from a central system during decoherence. Therefore, the decoherence scale is relevant to capability of measuring a central system from environment. The generalized overlap measures the distiguishability of environmental quantum states having information about a central system from a final measuring device. Therefore, it indicates a capability for observers to distinguish quantum states. As we find out, this distinguishability has its own scale.

For a position observable, the distinguishability due to decoherence is measured relatively to the "decoherence length", λ_{dec}. The decoherence length for QBM is found to be identical to the well-known thermal de Broglie wavelength λ_{dB}. It is a celebrated result [30]:

$$\lambda_{\mathrm{dec}} = \lambda_{\mathrm{dB}} = \frac{\hbar}{\sqrt{2Mk_BT}}. \tag{2.9}$$

On the other hand, the distinguishability in environment has another length scale, called the "distiguishability length" [29], λ_{dist}, which is the main result of the work:

$$\lambda_{\mathrm{dist}} = \sqrt{\frac{2k_BT}{M\Lambda^2}}, \tag{2.10}$$

where Λ is a cut-off frequency in the Lorentz-Drude spectral density [31]. It is important to notice the relation between λ_{dec} and λ_{dist}, called a "complementary relation", which was also discovered in [29]

$$\lambda_{\text{dec}}\lambda_{\text{dist}} = \frac{\hbar}{M\Lambda}. \tag{2.11}$$

This relation implies that in the studied regime of QBM, there exists some type of the relation between "information gain and disturbance". The former is described by distinguishability while the latter by decoherence.

2.4 Boson-spin model

Motivated by QBM, keeping a harmonic oscillator as a central system, here we consider especially environments having finite degrees of freedom, i.e. $\frac{1}{2}$-spins. This model is called the boson-spin model. Due to finite degrees of freedom in spins, it is expected that information about a harmonic oscillator, which has infinite degrees of freedom, can be only partially encoded into the spin environment. The Hamiltonian

$$H = H_S + \sum_i H_E^{(i)} + \sum_i H_{\text{int}}^{(i)}, \tag{2.12}$$

consists of

$$\begin{aligned} H_S &= \frac{P^2}{2M} + \frac{1}{2}M\Omega^2 X^2, \\ H_E^{(i)} &= -\frac{\Delta_i}{2}\sigma_x^{(i)}, \\ H_{\text{int}}^{(i)} &= g_i X \otimes \sigma_z^{(i)}, \end{aligned} \tag{2.13}$$

where g_i and Δ_i are the ith spin-environmental coupling and the ith spin self-energy (or often called the tunnelling matrix element). Assuming that a central harmonic oscillator already turned to be classical and here introducing the ansatz applied for QBM [24]:

$$U_{S:E}(t) = \int dX_0 e^{-iH_S t} |X_0\rangle\langle X_0| \otimes U_{\text{eff}}(X(t; X_0)), \tag{2.14}$$

we apply the same ansatz for the boson-spin model. Note that the use of the path integrals for spins is possible but could be problematic and will be studied elsewhere. Instead, we use the Born-Oppenheimer approximation [27]. The dynamics of the spin environment is described by:

$$i\frac{\partial U_{\text{eff}}(t)}{\partial t} = (H_{env} + H_{int}[X_{cl}(t)])U_{\text{eff}}(t)$$
$$= \sum_i \left[-\frac{\Delta_i}{2}\sigma_x^{(i)} + g_i X(t; X_0)\sigma_z^{(i)} \right] U_{\text{eff}}(t), \tag{2.15}$$

where $X_{cl}(t) = X(t; X_0)$ is chosen with an initial phase ϕ to be

$$X(t; X_0) = X_0 \cos(\Omega t + \phi). \tag{2.16}$$

2.4.1 Effective description: operator representation

In this section, for a heuristic purpose, using the operator representation we illustrate that our effective description motivated by the path integrals makes sense. We will show that for infinitesimal time a unitary evolution for a harmonic oscillator-environment joint system is in the same form as our ansatz (2.14).

We discussed the pointer state for a general Hamiltonian in the section 1.4, where we conveniently used the interaction picture. In the interaction picture, we can express a general Hamiltonian $H = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + X \otimes A$ in (1.14) as

$$H_I(t) = X_I(t) \otimes A_I(t), \tag{2.17}$$

where $X_I(t) = e^{iH_S t} X e^{-iH_S t}$ and $A_I(t) = e^{iH_E t} A e^{-iH_E t}$. Assuming that the initial state of a system is a product state of a harmonic oscillator and the environment, $|\Phi(0)\rangle = |\phi(0)\rangle \otimes |\psi(0)\rangle$, the state of the central system is expanded in coherent states [32], which is regarded as the most classical pure state, where an initial state of the central harmonic oscillator is $|\phi(0)\rangle = \sum_\alpha C_\alpha |\alpha\rangle$ and an initial state of the environment is $|\psi(0)\rangle$. Note that since initial states in both pictures are the same, we do not distinguish them. The coherent state is written as $|\alpha\rangle = |\langle x_\alpha\rangle, \langle p_\alpha\rangle\rangle$, where $\langle x_\alpha\rangle$ and $\langle p_\alpha\rangle$ are expectation values for X and P, respectively. Using H_I in (2.17), the relation $X_I(t) = e^{iH_S t} X e^{-iH_S t} = X \cos \Omega t + \frac{P}{M\Omega} \sin \Omega t$ and the property of a coherent state, $X|\langle x\rangle, \langle p\rangle\rangle \approx \langle x\rangle |\langle x\rangle, \langle p\rangle\rangle$ and

$P|\langle x\rangle,\langle p\rangle\rangle \approx \langle p\rangle|\langle x\rangle,\langle p\rangle\rangle$, the infinitesimal evolution is written by

$$\begin{aligned}
|\Phi(\delta t)\rangle_I &\approx |\Phi(0)\rangle - i\delta t\sum_\alpha C_\alpha\left[X\cos\Omega\delta t + \frac{P}{M\Omega}\sin\Omega\delta t\right]|\alpha\rangle\otimes A_I(\delta t)|\psi(0)\rangle \\
&\approx |\Phi(0)\rangle - i\delta t\sum_\alpha C_\alpha\left[\langle x_\alpha\rangle\cos\Omega\delta t + \frac{\langle p_\alpha\rangle}{M\Omega}\sin\Omega\delta t\right]|\alpha\rangle\otimes A_I(\delta t)|\psi(0)\rangle \\
&= \sum_\alpha C_\alpha|\alpha\rangle\otimes[|\psi(0)\rangle - i\delta t X(\alpha,\delta t)A_I(\delta t)|\psi(0)\rangle], \qquad (2.18)
\end{aligned}$$

where

$$\begin{aligned}
X(\alpha,\delta t) &\equiv \langle x_\alpha\rangle\cos\Omega\delta t + \frac{\langle p_\alpha\rangle}{M\Omega}\sin\Omega\delta t \\
&= \tilde{x}_\alpha\cos(\Omega\delta t + \phi_\alpha) \qquad (2.19)
\end{aligned}$$

with

$$\tilde{x}_\alpha \equiv \sqrt{\langle x_\alpha\rangle^2 + \frac{\langle p_\alpha\rangle^2}{M^2\Omega^2}},\ \tan\phi_\alpha \equiv \frac{\langle p_\alpha\rangle}{\langle x_\alpha\rangle M\Omega}. \qquad (2.20)$$

In (2.20), for a smaller amplitude, i.e. $\langle p_\alpha\rangle/M\Omega \sim \langle x_\alpha\rangle \ll 1$, $\tilde{x}_\alpha \approx \langle x_\alpha\rangle$. A coherent state $|\alpha\rangle$ can be expanded by $|X_0\rangle$ as $|\alpha\rangle = \int dX_0|X_0\rangle\langle X_0|\alpha\rangle$ and $\langle X_0|\alpha\rangle$ is the Gaussian, dominant around a peak $X_0 \approx \tilde{x}_\alpha$. (2.18) is expressed as

$$|\Phi(\delta t)\rangle_I \approx \int dX_0 C_{X_0}|X_0\rangle\otimes[1 - i\delta t X_0\cos(\Omega\delta t + \phi_{X_0})A_I(\delta t)]|\psi(0)\rangle, \qquad (2.21)$$

where $C_{X_0} \equiv \sum_\alpha C_\alpha\langle X_0|\alpha\rangle$ and ϕ_α has been replaced by the corresponding approximate ϕ_{X_0}. In general, ϕ_{X_0} differs for different X_0 but for simplicity we have fixed $\phi_{X_0} = \phi$. (2.21) is re-written with H_{eff}:

$$|\Phi(\delta t)\rangle_I \approx \int dX_0 C_{X_0}|X_0\rangle\otimes e^{-iH_{\text{eff}}\delta t}|\phi(0)\rangle_I, \qquad (2.22)$$

where

$$H_{\text{eff}}(\delta t) \equiv X_0\cos(\Omega\delta t + \phi)A_I(\delta t). \qquad (2.23)$$

Returning to the Schrödinger picture, the effective state at δt has been derived:

$$\begin{aligned}
|\Phi(\delta t)\rangle &\approx \int dX_0 C_{X_0}e^{-iH_S\delta t}|X_0\rangle\otimes e^{-iH_{\text{eff}}\delta t}|\phi(0)\rangle \\
&= U_{S:E}(\delta t)|\Psi(0)\rangle, \qquad (2.24)
\end{aligned}$$

where

$$U_{S:E}(\delta t) = e^{-iH_S\delta t} \otimes e^{-iH_{\text{eff}}\delta t} \tag{2.25}$$
$$= \int dX_0 e^{-iH_S\delta t}|X_0\rangle\langle X_0| \otimes e^{-iH_{\text{eff}}\delta t}. \tag{2.26}$$

This is the same form as the ansatz (2.14). If decoherence happens, comparing (2.25) with the approximate unitary evolution operator (1.21) in the interaction picture, the pointer state in the Schrödinger picture will be

$$|\text{pointer state}\rangle = e^{-iH_S\delta t}|X_0\rangle. \tag{2.27}$$

For the limit we used, $\langle p_\alpha\rangle/M\Omega \ll 1$ and $t = \delta t$ the basis $\{|X_0\rangle\}$ is the same as the basis $\{|\tilde{x}_I(t)\rangle\}$ in (1.20). Thus, the pointer basis (2.27) is the same as the approximate pointer state (1.26).

The overall unitary transformation $e^{-iH_S\delta t}$ in (2.27) indicates the pointer basis is time-dependent. We have just shown how the effective description can appears in a short time approximation.

We found that it would be quite non-trivial to precisely identify whether decoherence happens and hence in which basis objective quantum structures are formed. In the next section 2.5 we will introduce a systematic way to express approximate solutions of a unitary evolution operator and the objectivity markers.

2.5 Floquet theory

In this thesis the effective Hamiltonians for the environment resulting from the Born-Oppenheimer approximation [27] inherit the same periodicity from a central harmonic oscillator. In such a case, the Floquet theory [33, 34, 35] allows us to expand a unitary evolution operator in a more systematical manner along with the high frequency expansion [35]. Especially, it splits the evolution of the boson-spin model into two different characteristic frequencies.

The Floquet theorem states that if a time- dependent Hamiltonian $H(t)$ has a periodicity with a period $T = 2\pi/\Omega$, the corresponding unitary evolution operator can be written as a product of periodic unitary operators driven by a periodic time-dependent Hamiltonian

$K(t)$ with the same period $T = 2\pi/\Omega$ and a non-periodic unitary operator by a time-independent Hamiltonian H_F:

$$U(t, t_0) = e^{-iK(t)} e^{-i(t-t_0)H_F} e^{iK(t_0)}, \tag{2.28}$$

where t_0 is an initial time. Typically $K(t)$ is characterized by a micro-motion, which oscillates with high frequency Ω while H_F is responsible for a global behaviours with lower frequency which appears as a combination of parameters in $H(t)$.

Along with the Floquet theorem, one can expand those unitary operators assuming the frequency of a central system is a high frequency compared to other parameters, called the high-frequency expansion [35]. The high-frequency expansion is close to the Taylor expansion but not exactly. There is no known general convergence proof but the truncation of the series is justified for a sufficiently weak coupling. The advantage of the high-frequency expansion is that the weak coupling is systematically controlled. The periodic Hamiltonian $H(t)$ is written as

$$H(t) = H_0 + V(t) \tag{2.29}$$

The periodic time-dependent part $V(t)$ can be written with the Fourier modes as

$$V(t) = \sum_{j=1}^{\infty} (V^{(j)} e^{ij\Omega t} + V^{(-j)} e^{-ij\Omega t}). \tag{2.30}$$

The Hamiltonians H_F and $K(t)$ can be expanded by $1/\Omega$ [35],

$$\begin{aligned} H_F = H_0 &+ \frac{1}{\Omega} \sum_{j=1}^{\infty} \frac{1}{j} [V^{(j)}, V^{(-j)}] \\ &+ \frac{1}{2\Omega^2} \sum_{j=1}^{\infty} \frac{1}{j^2} ([[V^{(j)}, H_0], V^{(-j)}] + \text{H.c.}) + \cdots , \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} K(t) = &\frac{1}{i\Omega} \sum_{j}^{\infty} \frac{1}{j} (V^{(j)} e^{ij\Omega t} - V^{(-j)} e^{-ij\Omega t}) \\ &+ \frac{1}{i\Omega^2} \sum_{j}^{\infty} \frac{1}{j^2} ([V^{(j)}, H_0] e^{ij\Omega t} - \text{H.c.}) + \cdots . \end{aligned} \tag{2.32}$$

Note that in (2.32) the high frequency expansion looks similar to the Taylor expansion around $1/\Omega$ but different since the expansion coefficients are a function of Ω.

2.5.1 Objectivity measures and emergent scale

In the analysis of our objectivity measures, the decoherence factor and the the generalized overlap explicitly show two different scales due to two different frequencies with the help of the Floquet theory with the high frequency expansion. The decoherence factor $|\Gamma^1_{X_0,X_0'}|^2$ and the generalized overlap $B^1_{X_0,X_0'}$ for a single spin environment [18] are expanded with the weak coupling condition $\xi = gX_0/\Omega \ll 1$ with $\phi = 0$ in (2.16) as

$$|\Gamma^1_{X_0,X_0'}|^2 = \left[1 - \frac{\sin^2[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]}{\cosh^2(\beta\Delta/2)}\right]\cos^2[\delta\xi\sin\tau] \tag{2.33}$$

and

$$B^1_{X_0,X_0'} = 1 - E(\beta)^2\sin^2[\delta\xi\sin\tau], \tag{2.34}$$

where $\delta\xi = g(X_0 - X_0')/\Omega$, $\tau \equiv \Omega t$ and $\tilde{\Delta} \equiv \Delta/2\Omega$. As seen in the above expressions (2.33) and (2.34), $\sin\tau$ corresponds to a micro-motion while $\sin^2[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]$ in $|\Gamma^1_{X_0,X_0'}|^2$ to a large profile behaviour with $\tilde{\Delta} \ll 1$. In this approximate expansion the generalized overlap $B^1_{X_0,X_0'}$ does not show extra scale. The importance of the large scale behaviour for the objectivity in the central system becomes more clear when environment has a large number of degrees of freedom in. As new environmental systems are added, they can bring different frequencies and break the original periodicity. This causes destructive interference, leading to a decay as time goes large. The generalized overlap $B^1_{X_0,X_0'}$ having no separate frequency in (2.34) does not decay. However, we found that a phase in a classical trajectory of the central oscillator in (2.16) in the effective Hamiltonian for the environment is another source breaking the original periodicity [18]. We will further discuss objectivity conditions in the next section 2.5.2.

2.5.2 Objectivity conditions

There are many factors affecting the objectivity in the boson-spin model. The initial state of the environment, which we choose as the thermal state, can be controlled by temperature. On the other hand, the strengths of couplings are another. Especially, it is important to emphasize the existence of factors breaking a periodicity in the objectivity. If a periodicity appearing in the objectivity measures is the only periodicity, objectivity cannot be achieved. There are two possible ways to break a periodicity. One is that different environments have different frequencies. The other is that from the initial state of a central harmonic oscillator, initial phases will change phases in dynamics of environment. Finally, the sizes of the observed and unobserved environments play a crucial role. In [18] we performed averaging over the functions and presented a detailed analysis of when SBS can and cannot be approached.

2.6 Holevo quantity and objectivity

The concepts of information are quite different according to which view of the world we take. In an ontological view of the world, information is identified with a physical reality but in a statistical point of view, information is identified by how many possible states are available and how evenly they are distributed. Mostly information is referred to the latter view due to the present philosophy based on quantum mechanics. For the latter case, when a physical state is completely determined, it is said that there is no information in the physical state. However, when possible statistical distinct states are evenly distributed, the information is maximal. This measure of information is known as entropy.

It is important to quantify how much information in a statistical point of view can be transferred from one system to another. In this sense the von Neumann entropy is commonly used to count the information in quantum systems. In practice, it is particularly interesting to see how much information between two systems is in common. The quantum mutual information $I(X : Y)$ between two systems X and Y, is defined as difference between the information of the joint system, $S(\rho_{X:Y})$ and the sum of the information for X and Y, $S(\rho_X) + S(\rho_Y)$,

$$I(X : Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{X:Y}), \tag{2.35}$$

where $S(\rho_X)$ and $S(\rho_Y)$ are the von Neumann entropies for reduced density matrices, ρ_X and ρ_Y for X and Y, respectively and $S(\rho_{X:Y})$ is the von Neumann entropy for the (X, Y) joint system. In reality, quite frequently classical information X is prepared into a quantum state ρ_X with a probability p_X from a classical system X to Y. In this quantum communication setting, the bound for the mutual information $\chi(\rho)$ is called the Holevo quantity [36, 37],

$$\begin{aligned} I(X : Y) &\leq \chi(\rho) \\ &\equiv S\left(\sum_X p_X \rho_X\right) - \sum_X p_X S(\rho_X). \end{aligned} \tag{2.36}$$

In [38] we apply this channel view of the objectivization process to the results of [18]. This view was initiated in the works [39, 40, 41, 42, 43, 44, 45]. We apply it to a statistical state as seen by a single spin in the environment:

$$\begin{aligned} \rho^{(1)}(t) &= \mathrm{Tr}_{SE\setminus E_1}\rho_{S:E}(t) \\ &= \int dX_0 p(X_0)\rho^{(1)}(X_0), \end{aligned} \tag{2.37}$$

where $p(X_0) \equiv \langle X_0|\rho_S(0)|X_0\rangle$ is the probability distribution of the initial position and

$$\rho^{(1)}(X_0) \equiv U_1(X_0,t)\rho_E^{(1)}(0)U_1^{\dagger}(X_0,t). \tag{2.38}$$

We find that the Holevo bound can be surprisingly larger even for a single spin. The analysis for many spins would be also interesting but it is a technical challenge and we leave it for the future.

As seen in previous sections, the objectivization process is classified into two processes decoherence and distinguishability through interactions. The classicalization process is in parallel with information transfer in quantum Darwinism. Here since decoherence is assumed, only distinguishability is relevant to information transfer.

Chapter 3

Collections of articles

This chapter consists of three sections based on three articles. Each section is composed of three subsections of summary, work contribution and the attached publication.

3.1 Complementarity between decoherence and information retrieval from the environment

3.1.1 Summary

In history of physics, a measurement process has been regarded as the abstract concept or an isolated point-like event. Although due to its extremely short time scale its detail seems to be completely hidden, fortunately or not, its internal mechanism seems to be completely separated from the rest of the world within current available scale. "To obtain information" from a quantum system through a measurement device or our senses, means to us that its quantum state at the measurement stage needs be in a particular form to be best mapped into our daily picture, which allows us to depict an image about the world. Therefore, to reduce a quantum state to such a state, say a classical state, is equivalent to obtain "information", i.e., the reduction (or evolution) from a quantum state to a classical state is physically parallel to information transfer to a measurement device or any other systems, called environment. When a quantum state evolves to an ideal state to give a "perfect" classical picture, here we will call such a quantum state an "objective state"[1].

The main purpose of this article is to point out two distinct notions in accessing the information about a system of interest. After a system is first decohered by interaction with environment, in an observer's point of view, the available information remains at best the ensemble of quantum states in statistical distribution. Then, when we take a measurement on the environment, not a system directly, it is required to distinguish between possible different quantum states in the distribution. The first process is called decoherence and the second, distinguishability. These two notions tell us how much "objectivity" [1] a final system has. Here we show that limitations in accessible information are expressed by two distinct associated length scales. The model of our interest is a coupled system of a central harmonic oscillator and a collection of harmonic oscillators with a bi-linear interaction, called quantum Brownian motion (QBM) [20].

In this work we compute and analyze two objectivity measures for QBM, the influence functional [20] (or the decoherence factor in general) and the generalized overlap for decoherence and distinguishability, respectively. Since we are interested in the recoilless limit and the systems have continuous degrees of freedom, this approximation may be more suitably represented in the path integral method.

Since the analytical solutions for the system are way to complicated to express in terms of our common basis, we adopt several approximation schemes relevant to our interest. First, we are interested in the moment that a central oscillator gets stabilized to be a classical state and is governed approximately only by its own dynamics without being mechanically influenced by the environment, by the so-called recoilless approximation. Then, it is a suitable question whether a system in such an approximate classical state would evolve further to a more classical state. Technically, this postulate allows us to factor out the unitary evolution for the environmental oscillators from the total unitary evolution. Second, the environmental oscillators also weakly interact relative to energy of a central harmonic oscillator so that the higher order interactions are ignored in order to avoid the technical complications. Third, the initial state for the environmental harmonic oscillators are considered the thermal state. Fourth, we take a certain spectral density assigning how the interaction couplings between a central harmonic oscillator and environmental harmonic oscillators are distributed with respect to environmental frequencies. We choose the "Lorentz-Drude form" [31]. This distribution is motivated by the physical argument that for a large frequency it decays to zero and needs a cut-off frequency. Fifth, in order to obtain typical length scales for decoherence and distinguishability, we choose the energy hierarchy among temperature T, the cut-off energy Λ and an angular frequency of a central harmonic oscillator Ω, $k_B T \gg \hbar\Lambda \gg \hbar\Omega$, known as the Caldeira-Leggett limit [46].

The decoherence length is given by the "thermal de Broglie wavelength", $\lambda_{\mathrm{dB}}^2 = \hbar^2/2Mk_BT$

and obtained from an exponential decay width in the influential functional of a path difference in a central oscillator. It turns out that the generalized overlap also appears in an exponential decay form of the path difference with the width, $\lambda_{\text{dist}}^2 = 2k_B T/M\Lambda^2$. It can be noticeable that $\lambda_{\text{dist}} \gg \lambda_{\text{dB}}$ and they form the complementarity relation $\lambda_{\text{dist}}\lambda_{\text{dB}} = \hbar/M\Lambda$. This consequence is physically interpreted as although a quantum state of a system appears to be decohered enough to be classicalized above a certain distance, the information in the environment for the a system always become less clear encoded in the environment.

3.1.2 Work contribution

Upon starting my PhD, I joined the project initially set by Prof. Jarosław Korbicz some years ago in order to compute the objectivity markers in the path integral formulation. My contribution to this article is following:

1. Computing the action for a coupled harmonic oscillators to derive the influence functional and the generalized overlap.
2. Working on a caustic problem in the path integral and the reduced density matrix for the environment.
3. Computing the density matrix expressions, fermionic Matsubara representations in Appendix A, B and C.
4. Expressing the higher order corrections for the environmental reduced density matrix to (B23) in terms of known statistical variables.
5. Helping in deriving and interpreting the distinguishability length, which is the main result.
6. Helping in preparing the manuscript.

Complementarity between decoherence and information retrieval from the environment

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We address the problem of the fundamental limitations of information extraction from the environment in open quantum systems. We derive a model-independent, hybrid quantum-classical solution of open dynamics in the recoilless limit, which includes environmental degrees of freedom. Specifying to the celebrated Caldeira-Leggett model of hot thermal environments, ubiquitous in everyday situations, we reveal the existence of a new lengthscale, called the distinguishability length, different from the well-known thermal de Broglie wavelength that governs the decoherence. Interestingly, a new integral kernel, called quantum Fisher information kernel, appears in the analysis. It complements the well-known dissipation and noise kernels and satisfies disturbance-information gain type of relations, similar to the famous fluctuation-dissipation relation. Our results complement the existing treatments of the Caldeira-Legget model from a non-standard and highly nontrivial perspective of information dynamics in the environment. This leads to a full picture of how the open evolution looks like from both the system and the environment points of view, as well as sets limits on the precision of indirect observations.

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I. INTRODUCTION

One of the perpetual questions is if what we perceive is really "out there?" While the ontology of quantum mechanics is still a matter of a debate (see, e.g., [1–3]), it is nowadays commonly accepted following the seminal works of Zeh [4] and Zurek [5,6] that interactions with the environment and the resulting decoherence processes lead to an effective emergence of classical properties, like position [7–9]. It is then usually argued, using idealized pure-state environments, that the decoherence efficiency corresponds directly to the amount of information recorded by the environment (see, e.g., [9]). The more the environment learns about the system, the stronger it decoheres it. On the other hand, we perceive the outer world by observing the environment and the information content of the latter determines what we see.

Here we show that there is a gap between the two: What the environment learns about the system, as determined by the decohering power, and what can be extracted from it via measurements. Some part of information stays bounded. We show it for physically most relevant thermal environments in the Caldeira-Leggett regime [10], which is the universal choice for high-temperature environments, ubiquitous in real-life situations. One of the most emblematic results of the entire decoherence theory states that spatial coherences decay on the lengthscale given by the thermal de Broglie wavelength λ_{dB} and on the timescale $t_{dec} \sim \gamma^{-1}(\lambda_{dB}/d)^2$, where d is a separation and γ^{-1} is related to the relaxation time [10–12]. We complement this celebrated result by analyzing information extractable from the environment as quantified by the state distinguishability [13]. We show that it is governed by a new lengthscale, which we call the distinguishability length, larger than the decoherence length. Thus the resolution with which the system's position can be readoff from the environment is worse than the decoherence resolution; see Fig. 1. A part of the information gained by the environment during the

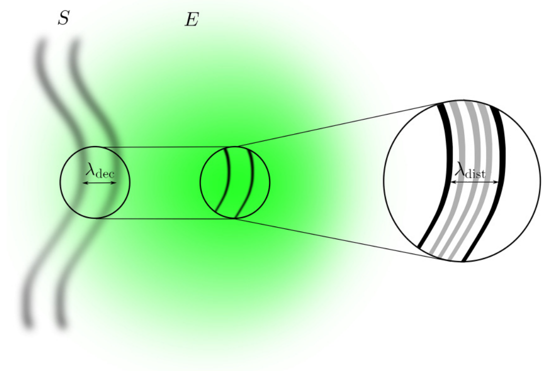

FIG. 1. The environment decoheres the central system at a lengthscale λ_{dec} (equal to the thermal de Broglie wavelength λ_{dB} in the studied example) as a result of dynamical buildup of correlations and information leakage into the environment. However, not all of that information is accessible, the retrieval is limited by its own resolution, the distinguishability length λ_{dist}. Since $\lambda_{dist} \gg \lambda_{dec}$, part of the information stays bounded in the environment. Decoherence and distinguishability are complementary to each other as reflected by information gain versus disturbance type of a relation: $\lambda_{dist}\lambda_{dec} =$ const. The accompanying timescales satisfy $t_{dist}/t_{dec} \sim (\lambda_{dist}/\lambda_{dec})^2$, so that reaching a given information retrieval resolution takes a much longer time than reaching the same decoherence resolution.

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decoherence is bounded in it in the thermal noise, similarly to, e.g., the bounding a part of thermodynamic energy as thermal energy, unavailable for work, or to bounding quantum entanglement [14]. We obtain the corresponding distinguishability timescale and introduce a new integral kernel, quantum Fisher information (QFI) kernel, similar to the well-known noise and dissipation kernels [7–9] and governing the distinguishability process. The discovery of this phenomenon was possible due to the paradigm change in studies of open quantum systems initiated by quantum Darwinism [15–20] and spectrum broadcast structures (SBS) [21–25] programs. They recognize the environment as a carrier of useful information about the system, rather than just the source of noise and dissipation, and study its information content in the context of the quantum-to-classical transition. The existence of the gap can be, in principle, deduced from the existing literature on quantum Darwinism, e.g., [20], and the corresponding timescale separation was shown in, e.g., [26]. However, those studies were performed in finite-dimensional settings. Here we study a continous variable system, which called for new methods. Some hints on the effect were also obtained in earlier studies of SBS, especially in the quantum Brownian motion (QBM) model [23], where state distinguishability and its temperature dependence was analyzed, but the limited, numerical character of the studies did not reveal the existence of the distinguishability length and the gap to the decoherence lengthscale. Our results, apart from showing intrinsic limitations of indirect observations, also characterize decoherence, which has become one of the key paradigms of modern quantum science [27–33], from a little studied perspective of the "receiver's end." Last but not least, we uncover an interesting new feature of the venerable Caldeira-Leggett model

Although there exist powerful methods of analysis of quantum open systems, such as the Bloch-Redfield [34,35] or Davies-Gorini-Kossakowski-Lindblad-Sudarshan [36–38] equations, they describe the evolution of the central system alone, neglecting the environment completely. This will not tell us anything about the information acquired by the environment and we instead derive an approximate solution method focusing on the evolution of the environment. To this end, we divide the environment into two parts, one denoted $E_{\rm uno}$ is assumed to be unobserved and hence traced over, while the remaining part, denoted $E_{\rm obs}$ is assumed to be under observation by an external observer. Our main object of study will be the so-called partially traced state, obtained by tracing out only the unobserved part of the environment [21–23,25]

$$\varrho_{S:E_{\rm obs}} = {\rm Tr}_{E_{\rm uno}} \varrho_{S:E}, \tag{1}$$

Here $S : E_{\rm obs}$ denotes that the resulting state is a joint state of the system S and the observed part of the environment $E_{\rm obs}$. As the first approximation, it is enough to consider the recoiless limit, where the central system S influences the environment E but is massive enough not to feel the recoil. It is a version of the Born-Oppenheimer approximation and the opposite, and less studied, limit to the commonly used Born-Markov approximation [7–9], where the influence $S \to E$ is completely cut out and is thus useless for our purposes.

II. HYBRID QUANTUM-CLASSICAL DYNAMICS IN THE RECOILLESS LIMIT

We follow the treatment of Feynman and Vernon [39] using path integrals. The full system-environment propagator reads

$$K_t(X, X_0; x, x_0) = \int_{\substack{x(0)=x_0\\ x(t)=x}} \mathscr{D}x(t) \int_{\substack{X(0)=X_0\\ X(t)=X}} \mathscr{D}X(t) \times \exp\frac{i}{\hbar}\{S_{\rm sys}[X(t)] + S_{\rm env}[x(t)] + S_{\rm int}[X(t), x(t)]\}, \tag{2}$$

where $S_{\rm sys}, S_{\rm env}, S_{\rm int}$ are the actions of the system, environment, and interaction, respectively; $X(t)$ is the system trajectory with the initial condition $X(0) = X_0$, similarly $x(t)$ is the environment trajectory with $x(0) = x_0$. For a massive enough central system we may neglect the recoil of the environment

$$\left|\frac{\delta S_{\rm sys}}{\delta X(t)}[X(t)]\right| \gg \left|\frac{\delta S_{\rm int}}{\delta X(t)}[X(t), x(t)]\right|. \tag{3}$$

and expand the parts containing $X(t)$ around a classical trajectory $X_{\rm cl}(t; X_0)$ (in what follows we drop the dependence on the initial position X_0, it will be self-understood), which satisfies the unperturbed equation $\delta S_{\rm sys}/\delta X(t)[X(t)] \approx 0$. The standard Gaussian integration around $X_{cl}(t)$ gives [40]

$$K_t \approx e^{\frac{i}{\hbar}S_{\rm sys}[X_{cl}(t)]} \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S_{\rm env}[x(t)]} e^{\frac{i}{\hbar}S_{\rm int}[X_{\rm cl}(t), x(t)]} \times D_t[X_0, X; x(t)], \tag{4}$$

where $D_t[X_0, X; x(t)]$ is the van Vleck determinant [41] for $S_{\rm sys} + S_{\rm int}$. It depends on $x(t)$ through $\delta^2 S_{\rm int}/\delta X(t)\delta X(t')$. This is a quantum leftover of the $E \to S$ back-reaction, which we also neglect, assuming

$$\left|\frac{\delta^2 S_{\rm sys}}{\delta X(t)\delta X(t')}[X_{\rm cl}(t)]\right| \gg \left|\frac{\delta^2 S_{\rm int}}{\delta X(t)\delta X(t')}[X_{cl}(t), x(t)]\right|, \tag{5}$$

which, e.g., holds trivially for linearly coupled systems, when $\delta^2 S_{\rm int}/\delta X(t)\delta X(t') = 0$. Then $D_t[X_0, X; x(t)]$ reduces to the van Vleck propagator for S alone, $D_t(X_0, X)$ and can be pulled out of the integral in (4)

$$K_t \approx e^{\frac{i}{\hbar}S_{\rm sys}[X_{\rm cl}(t)]} D_t(X_0, X) \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S_{\rm env}[x(t)]} e^{\frac{i}{\hbar}S_{\rm int}[X_{\rm cl}(t), x(t)]}, \tag{6}$$

where the first two terms define the semi-classical propagator for the central system $K_t^{\rm sc}(X, X_0) \equiv e^{\frac{i}{\hbar}S_{\rm sys}[X_{\rm cl}(t)]} D_t(X_0, X)$. The remaining path integral can be represented using the standard operator formalism

$$\int \mathscr{D}x(t) e^{\frac{i}{\hbar}S_{\rm env}[x(t)]} e^{\frac{i}{\hbar}S_{\rm int}[X_{\rm cl}(t), x(t)]} \equiv \langle x|\hat{U}_t[X_{\rm cl}]|x_0\rangle, \tag{7}$$

where $\hat{U}_t[X_{\rm cl}]$ is the effective unitary evolution of the environment with $X_{\rm cl}(t)$ acting as a classical force

$$i\hbar\frac{d\hat{U}_t}{dt} = (\hat{H}_{\rm env} + \hat{H}_{\rm int}[X_{\rm cl}(t)])\hat{U}_t, \tag{8}$$

where $\hat{H}_{\rm env}$, $\hat{H}_{\rm int}$ are the Hamiltonians corresponding to the actions $S_{\rm env}, S_{\rm int}$, respectively. In what follows we use hats to denote operators. Thus from (4), (5), (7), and (8) we obtain the propagator in the recoilless limit

$$K_t(X, X_0; x, x_0) \approx K_t^{\rm sc}(X, X_0)\langle x|\hat{U}_t[X_{\rm cl}]|x_0\rangle. \tag{9}$$

Assuming a product initial state $\varrho(0) = \varrho_{0S} \otimes \varrho_{0E}$, we can use (9) to construct the approximate solution for the full system-environment state $\varrho_{S:E}$. We obtain it in the form of partial matrix elements between the position states of the central system

$$\langle X'|\varrho_{S:E}(t)|X\rangle \approx \iint dX_0 dX_0' \langle X_0'|\varrho_{0S}|X_0\rangle K_t^{\rm sc}(X, X_0)^* K_t^{\rm sc}(X', X_0')\hat{U}_t[X_{\rm cl}]\varrho_{0E}\hat{U}_t[X_{\rm cl}']^\dagger. \tag{10}$$

We next specify to the most common situation when the environment is composed out of a number of subenvironments or modes, denoted E_k, e.g., a collection of harmonic oscillators. Furthermore, we assume there are not direct interactions between the parts of the environment, only the central interactions so that $\hat{H}_{\rm env} = \sum_k \hat{H}_k$ and $\hat{H}_{\rm int} = \sum_k \hat{H}_{\rm int}^k$, where k labels the subenvironments, $\hat{H}_k$, $\hat{H}_{\rm int}^k$ are the self-energy and interaction Hamiltonians of the kth subenvironment, respectively. It is immediate to see that, due to that, the effective evolution of the environment has a product structure $\hat{U}_t[X_{\rm cl}] = \bigotimes_k \hat{U}_t^k[X_{\rm cl}]$, where each $\hat{U}_t^k[X_{\rm cl}]$ satisfies the corresponding equation (8). Substituting it into (10) and tracing out subenvironments assumed to be unobserved, $E_{\rm uno}$, we obtain the desired solution for the partially traced state (1)

$$\langle X'|\varrho_{S:E_{\rm obs}}(t)|X\rangle \approx \iint dX_0 dX_0' \langle X_0'|\varrho_{0S}|X_0\rangle K_t^{\rm sc}(X, X_0)^* K_t^{\rm sc}(X', X_0') F[X_{\rm cl}(t), X_{\rm cl}'(t)] \bigotimes_{k\in E_{\rm obs}} \hat{U}_t^k[X_{\rm cl}]\varrho_{0k}\hat{U}_t^k[X_{\rm cl}']^\dagger. \tag{11}$$

Here we assume the usual product initial state $\varrho_{0E} = \bigotimes_k \varrho_{0k}$, where ϱ_{0k} is the initial state of the kth subenvironment. Moreover, $X_{\rm cl}(0) = X_0$, $X_{\rm cl}(t) = X$, $X_{\rm cl}'(0) = X_0'$, $X_{\rm cl}'(t) = X'$ and

$$F[X_{\rm cl}(t), X_{\rm cl}'(t)] \equiv \prod_{k\in E_{\rm uno}} {\rm Tr}\big(\hat{U}_t^k[X_{\rm cl}']^\dagger \hat{U}_t^k[X_{\rm cl}]\varrho_{0k}\big) \tag{12}$$

is the influence functional [39,40]. This is a general, hybrid solution with effectively classical central system, driving quantum environment. The, admittedly coarse, approximation (11) is enough for our purposes.

In what follows we will specify to one of the paradigmatic models of open quantum systems, linear quantum Brownian motion model (see, e.g., [42–44]), as an example. It is described by Lagrangeans: $L_{\rm sys} = 1/2(M\dot{X}^2 - M\Omega^2X^2)$, $L_{\rm env} = \sum_k 1/2(m_k\dot{x}_k^2 - m_k\omega_k^2x_k^2)$, $L_{\rm int} = -X\sum_k C_kx_k$, and the corresponding actions. It is easy to see that the no-recoil condition (3) will be satisfied when

$$\frac{C_k}{M\Omega^2} \ll 1, \tag{13}$$

so that $X_{\rm cl}(\tau)$ are the ordinary oscillator trajectories and the condition (5) is trivial due to the linearity in X of the interaction term.

The influence functional for QBM was first calculated in [39] for the physically most relevant situation of the thermal environments and has the well-known form [7,8,10,39]

$$F[X_{\rm cl}(t), X_{\rm cl}'(t)] = \exp\left\{-\frac{1}{\hbar}\int_0^t d\tau \int_0^\tau d\tau' \Delta(\tau)\nu(\tau-\tau')\Delta(\tau')\right\} \tag{14}$$

$$\times \exp\left\{-\frac{i}{\hbar}\int_0^t d\tau \int_0^\tau d\tau' \Delta(\tau)\eta(\tau-\tau')\bar{X}_{\rm cl}(\tau')\right\}, \tag{15}$$

where $\Delta(\tau) \equiv X_{\rm cl}(\tau) - X_{\rm cl}'(\tau)$ is the trajectory difference, $\bar{X}_{\rm cl}(\tau) \equiv (1/2)(X_{\rm cl}(\tau) + X_{\rm cl}'(\tau))$ is the trajectory average, and $\nu(\tau)$, $\eta(\tau)$ are the noise and dissipation kernels, respectively [7–10,39],

$$\nu(\tau) \equiv \int d\omega J_{\rm uno}(\omega){\rm cth}\left(\frac{\hbar\omega\beta}{2}\right)\cos\omega\tau, \tag{16}$$

$$\eta(\tau) \equiv \int d\omega J_{\rm uno}(\omega)\sin\omega\tau, \tag{17}$$

with $\beta = 1/(k_BT)$ denoting the inverse temperature and $J_{\rm uno}(\omega) \equiv \sum_{k\in E_{\rm uno}} C_k^2/(2m_k\omega_k)\delta(\omega-\omega_k)$ is the spectral density of the unobserved part of the environment. The modulus of $F[X_{\rm cl}(t), X_{\rm cl}'(t)]$ controls the decoherence process.

III. DISTINGUISHABILITY OF LOCAL STATES AND QUANTUM FISHER INFORMATION KERNEL

To understand what information about S is available locally in the the environment, we need the local states of each subenvironment E_k, as these are the states that fully determine the results of local measurements for each of the observers. $\varrho_k(t) = {\rm Tr}_{E_1\ldots\cancel{E_k}\ldots}{\rm Tr}_S\varrho_{S:E_{\rm obs}}$. The detailed calculation, relying on (13), is presented in Appendix B. The result is $\varrho_k(t) \approx \int dX_0 p(X_0)\varrho_t^k[X_{\rm cl}^0]$, and similarly for the entire observed fraction of the environment $E_{\rm obs}$:

$$\varrho_{E_{\rm obs}}(t) \approx \int dX_0 p(X_0) \bigotimes_{k\in E_{\rm obs}} \varrho_t^k[X_{\rm cl}^0], \tag{18}$$

where

$$\varrho_t^k[X_{\rm cl}] \equiv \hat{U}_t^k[X_{\rm cl}]\varrho_{0k}\hat{U}_t^k[X_{\rm cl}]^\dagger, \tag{19}$$

are conditional states of E_k, $p(X_0) \equiv \langle X_0|\varrho_{0S}|X_0\rangle$, and $X_{\rm cl}^0$ is the classical trajectory with the endpoint 0: $X_{\rm cl}^0(0) = X_0$, $X_{\rm cl}^0(t) = 0$. In the case of linear QBM, the evolution law (8), satisfied by $\hat{U}_t^k[X_{\rm cl}]$, describes a harmonic oscillator forced along the classical trajectory $X_{\rm cl}$. It has a well-known solution, which in the interaction picture reads (we present it in Appendix C for completeness)

$$\hat{U}_t^k[X_{\rm cl}] = e^{i\zeta_k(t)}\hat{D}\left(-\frac{iC_k}{\sqrt{2\hbar m_k\omega_k}}\int_0^t d\tau e^{i\omega_k\tau}X_{\rm cl}(\tau)\right), \tag{20}$$

where $\zeta_k(t)$ is an irrelevant phase factor and $\hat{D}(\alpha) \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the standard optical displacement operator. The local states of E_k are mixtures of oscillator states (19), forced along $X_{\rm cl}^0$. They are parametrized by the central system's initial position X_0 (cf. [23]), spread with the probability $p(X_0)$.

The information content of the fragment E_k is determined by the distinguishability of the local states $\varrho_t^k[X_{\rm cl}]$ for different trajectories. We will consider a general $X_{\rm cl}(\tau)$, which can later be specified to $X^0_{\rm cl}(\tau)$. Among the available distinguishability measures [13], a particularly convenient one is the generalized overlap $B(\varrho,\sigma)\equiv {\rm Tr}\sqrt{\sqrt{\varrho}\sigma\sqrt{\varrho}}$. It provides both lower and upper bounds for such operational quantities as the probability of error and to the quantum Chernoff information [45] and is a good compromise between computability and a clear operational meaning. We define the generalized overlap for the conditional states of the kth environment

$$B_k[\Delta,t]\equiv B\big(\varrho_t^k[X_{\rm cl}],\varrho_t^k[X'_{\rm cl}]\big) \tag{21}$$

[from the definition and (19) and (20), B depends only on the difference of trajectories Δ]. For thermal ϱ_{0k}, the overlap (21) was found in [23] (see Appendix C)

$$B_k[\Delta,t]=\exp\left\{-\frac{C_k^2}{4\hbar m_k\omega_k}{\rm th}\left(\frac{\hbar\omega_k\beta}{2}\right)\left|\int_0^t d\tau e^{i\omega_k\tau}\Delta(\tau)\right|^2\right\}. \tag{22}$$

A single environmental mode typically carries a vanishingly small amount of useful information. To decrease the discrimination error, it is beneficial to combine the modes into groups, called macrofractions [21], scaling with the total number of observed modes. In our case, we consider the entire observed environment $E_{\rm obs}$. Since there are no direct interactions in the environment, the conditional states of the observed fraction are products, cf. (18): $\varrho_t^{\rm obs}[X_{\rm cl}]\equiv\bigotimes_{k\in E_{\rm obs}}\varrho_t^k[X_{\rm cl}]$. The generalized overlap factorizes w.r.t. the tensor product, so there is no quantum metrological advantage here [46], but still there is a classical one [21]

$$\begin{aligned}B_{\rm obs}[\Delta,t]&\equiv B\big(\varrho_t^{\rm obs}[X_{cl}],\varrho_t^{\rm obs}[X'_{\rm cl}]\big)=\prod_{k\in E_{\rm obs}}B_k[\Delta,t]\\&=\exp\left\{-\sum_{k\in E_{\rm obs}}\frac{C_k^2}{4\hbar m_k\omega_k}{\rm th}\left(\frac{\hbar\omega_k\beta}{2}\right)\left|\int_0^t d\tau e^{i\omega_k\tau}\Delta(\tau)\right|^2\right\}\\&\equiv\exp\left\{-\frac{1}{\hbar}\int_0^t d\tau\int_0^\tau d\tau'\Delta(\tau)\phi(\tau-\tau')\Delta(\tau')\right\},\end{aligned} \tag{23}$$

where, in the last step, we passed to the continuum limit and introduced a new kernel

$$\phi(\tau)\equiv\int_0^\infty d\omega J_{\rm obs}(\omega){\rm th}\left(\frac{\hbar\omega\beta}{2}\right)\cos\omega\tau, \tag{24}$$

called the quantum Fisher information kernel. Here $J_{\rm obs}(\omega)\equiv\sum_{k\in E_{\rm obs}}C_k^2/(2m_k\omega_k)\delta(\omega-\omega_k)$ is the spectral density corresponding to the observed environment. Note that, quite surprisingly, the QFI kernel and the overlap (23) have an almost identical structure to the noise kernel (16) and the real part of the influence functional [7,8,10,39], the only difference being in the reversed temperature dependence [23,47]. It can be intuitively understood by recalling that here the higher the temperature the more efficient the decoherence but also the more noisy the environment. The name QFI kernel comes from the observation that the QFI of the X_0 phase imprinting: $\varrho_k(X)\equiv e^{-i/\hbar X_0C_k\hat{x}_k}\varrho_{0k}e^{i/\hbar X_0C_k\hat{x}_k}$ is proportional to the integrand of (24) (see, e.g., [48]).

We want a fair comparison of the decohering power and the information content of the observed environment, so we assume equal spectral densities for the unobserved and the observed fractions $J_{\rm obs}=J_{\rm uno}\equiv J(\omega)$ and choose it to be in the Lorenz-Drude form [7–9]

$$J(\omega)=\frac{2M\gamma}{\pi}\omega\frac{\Lambda^2}{\Lambda^2+\omega^2}, \tag{25}$$

where Λ is the cutoff frequency and γ is the effective coupling strength.

IV. INFORMATION GAP

For our demonstration it is enough to use the highly popular Caldeira-Leggett limit [9,10], $k_BT/\hbar\gg\Lambda\gg\Omega$, which is the high-temperature, hight-cutoff limit. The behavior of the influence functional in this limit is emblematic to the entire decoherence theory and can be obtained, e.g.. by approximating ${\rm cth}x\approx x^{-1}$ in the noise kernel (16) and then using $\Lambda e^{-\Lambda\tau}\approx\delta(\tau)$ valid for $\tau\gg\Lambda^{-1}$ (or using the Matsubara representation [49]). This leads to the celebrated result that decoherence becomes efficient at lengths above the thermal de Broglie wavelength $\lambda_{\rm dB}^2=\hbar^2/2Mk_BT$ [8,10–12]

$$|F[X_{\rm cl}(t),X'_{\rm cl}(t)]|\approx\exp\left[-\frac{\gamma}{\lambda_{\rm dB}^2}\int_0^t d\tau\Delta(\tau)^2\right], \tag{26}$$

and for times larger than the decoherence time [12] $t_{\rm dec}=1/\gamma(\lambda_{\rm dB}/d)^2$, where d is a given separation and γ^{-1} is related to the relaxation time.

The QFI kernel can be studied in the similar way, approximating ${\rm th}x\approx x$ in (24) and passing to a large $\Lambda\tau$,

$$\phi(\tau)\underset{\hbar\Lambda\beta\ll1}{\approx}\frac{\gamma M\hbar\beta\Lambda^2}{\pi}\int_0^\infty d\omega\frac{\omega^2}{\omega^2+\Lambda^2}\cos(\omega\tau) \tag{27}$$

$$=\frac{\gamma M\hbar\beta\Lambda^2}{\pi}\left(\int_0^\infty d\omega\cos(\omega\tau)-\Lambda^2\int_0^\infty d\omega\frac{\cos(\omega\tau)}{\omega^2+\Lambda^2}\right) \tag{28}$$

$$=\gamma M\hbar\beta\left(\Lambda^2\delta(\tau)-\frac{1}{2}\Lambda^3e^{-\Lambda|\tau|}\right). \tag{29}$$

We define $f(\tau) \equiv \Lambda e^{-\Lambda\tau}$ for $\tau > 0$ so that $\Lambda^3 e^{-\Lambda(\tau-\tau')} = d^2/d\tau'^2 f(\tau - \tau')$. We can then calculate the integral with the second term of (29) integrating by parts two times

$$\int_0^\tau d\tau' \Lambda^3 e^{-\Lambda(\tau-\tau')} \Delta(\tau') = \int_0^\tau d\tau' \ddot{f}(\tau - \tau')\Delta(\tau') \quad (30)$$

$$= \dot{f}(0_+)\Delta(\tau) - \dot{f}(\tau)\Delta(0) - f(0)\dot{\Delta}(\tau) + f(\tau)\dot{\Delta}(0) \quad (31)$$

$$+ \int_0^\tau d\tau' f(\tau - \tau')\ddot{\Delta}(\tau'), \quad (32)$$

here the dot means $d/d\tau'$, so that $\dot{f}(0_+) = +\Lambda^2$. In the large cutoff limit it is justified to assume $\tau \gg \Lambda^{-1}$, i.e., we consider a timescale much larger than the one set by the cutoff, similarly as it is done analyzing the influence functional [10]. Then the boundary terms containing $f(\tau)$ and $\dot{f}(\tau)$ can be neglected and moreover we can substitute $f(\tau) \approx \delta(\tau)$ in the last integral to obtain

$$\int_0^\tau d\tau' \Lambda^3 e^{-\Lambda|\tau-\tau'|} \Delta(\tau') \underset{\Lambda\tau\gg 1}{\approx} \Lambda^2\Delta(\tau) - \Lambda\dot{\Delta}(\tau) + \ddot{\Delta}(\tau) \quad (33)$$

$$= \Lambda^2\Delta(\tau) - \Lambda\Omega\Delta\left(\tau + \frac{\pi}{2\Omega}\right) - \Omega^2\Delta(\tau) \quad (34)$$

$$\underset{\Lambda\gg\Omega}{\approx} \Lambda^2\Delta(\tau), \quad (35)$$

where we use the fact that $\Delta(\tau)$ is a difference of two oscillator trajectories, so that it satisfies $\dot{\Delta}(\tau) = \Omega\Delta(\tau + \pi/2\Omega)$ and $\ddot{\Delta}(\tau) = -\Omega^2\Delta(\tau)$. Finally, we neglect the Ω terms because $\Lambda \gg \Omega$. More generally, $\dot{\Delta}(\tau)$, $\ddot{\Delta}(\tau)$ are inverse proportional to the system's evolution timescale, which is the slowest timescale, and hence those terms can be omitted compared to the Λ^2 term. Using (35) and (29) we obtain our main result: the generalized overlap between the environmental macrostates. It is both remarkably simple and similar to (26)

$$B[\Delta, t] \approx \exp\left[-\frac{\gamma}{\lambda_{\rm dist}^2}\int_0^t d\tau\, \Delta(\tau)^2\right]. \quad (36)$$

The expression (36) immediately implies that the distinguishability process is described by its own lengthscale, which we call the distinguishability length

$$\lambda_{\rm dist}^2 \equiv \frac{2k_BT}{M\Lambda^2} = \frac{\hbar}{M\Lambda}\left(\frac{2k_BT}{\hbar\Lambda}\right) \quad (37)$$

and happens on the associated distinguishability timescale

$$t_{\rm dist} = \frac{1}{\gamma}\left(\frac{\lambda_{\rm dist}}{d}\right)^2. \quad (38)$$

Surprisingly, (37) does not depend on $\hbar$ in the leading order. It is the lengthscale at which the energy of the "cutoff oscillator" of mass M and frequency Λ equals the thermal energy: $M\Lambda^2\lambda_{\rm dist}^2/2 = k_BT$. The cutoff dependence of (37) can be understood recalling that the cutoff defines the shortest lengthscale in the environment. Indeed, (37) can be expressed as the characteristic length of the cutoff oscillator, $\sqrt{\hbar/M\Lambda}$, rescaled by the ratio of the thermal energy to the cutoff energy $\sqrt{2k_BT/\hbar\Lambda}$. Of course the higher-order terms in the th$(\hbar\beta\omega/2)$ expansion in (23) and (24) will contribute $O(\hbar^2)$ terms to (37). There is clearly a competition in (37) between the temperature T, which degrades the discriminating ability of the environment and the cutoff frequency Λ which increases it. The relative difference between the two lengthscales

$$\frac{\lambda_{\rm dist} - \lambda_{\rm dB}}{\lambda_{\rm dB}} \approx 2\frac{k_BT}{\hbar\Lambda} \gg 1, \quad (39)$$

shows that there is a "resolution gap" between the decoherence and the distinguishability accuracy [50]. The environment decoheres the system at shorter lengthscales than those at which information can be extracted from it, i.e., a part of the information stays bounded in the environment. The timescales are separated even more strongly

$$\frac{t_{\rm dist}}{t_{\rm dec}} = 4\left(\frac{k_BT}{\hbar\Lambda}\right)^2, \quad (40)$$

meaning the distinguishability process takes much longer time than the decoherence for the same separation. This is in accord with the earlier results for generic finite-dimensional systems; see, e.g., [26]. For molecular environments $\Lambda \sim 10^{13}$ Hz and at $T \sim 300$ K, $\lambda_{\rm dist}/\lambda_{\rm dec} \sim 10$, $t_{\rm dist}/t_{\rm dec} \sim 100$, which is still orders of magnitude shorter for macroscopic bodies than typical relaxation times [12].

As a by-product we obtain a type of information gain versus disturbance relation (see, e.g., [51]), where the disturbance is represented by the decoherence efficiency

$$\lambda_{\rm dB}\lambda_{\rm dist} = \frac{\hbar}{M\Lambda}. \quad (41)$$

The right-hand side does not depend on the state of the environment (encoded in the temperature) and is the square of the characteristic length of the cutoff oscillator. More generally, passing to the Fourier transforms of the noise, dissipation, and QFI kernels, denoted by the tilde, we obtain the following relations, true for thermal environments:

$$\tilde{\phi}(\omega) = \tilde{\nu}(\omega)\text{th}^2\left(\frac{\hbar\omega\beta}{2}\right), \quad (42)$$

$$\tilde{\phi}(\omega) = i\tilde{\eta}(\omega)\text{th}\left(\frac{\hbar\omega\beta}{2}\right). \quad (43)$$

They resemble the celebrated fluctuation-dissipation relation [43,52,53]

$$\tilde{\nu}(\omega) = i\tilde{\eta}(\omega)\text{cth}\left(\frac{\hbar\omega\beta}{2}\right), \quad (44)$$

but connect dissipation and noise to information accumulation in the environment. These interesting relations will be investigated further elsewhere.

V. CONCLUSION

We showed here, using the celebrated model of Caldeira and Leggett as an example, that there is an information gap between what environment learns, decohering the system, and what can be extracted from it via measurements, i.e., some information stays bounded in the environment. For that, we developed a series of rather nontrivial and nonstandard tools, including a path-integral recoiless limit, a hybrid quantum-classical extended state solution (11), and quantum Fisher information kernel. Our results uncover the existence of a new lengthscale, determining the information content of the environment and complementary to the celebrated thermal de Broglie decoherence lengthscale.

The unorthodox point of view taken here, i.e., that of the environment instead of the the central system, was motivated by the modern developments of the decoherence theory [15,17,21], explaining the apparent objectivity of the macroscopic world through redundantly stored information in the environment. From this perspective, the solution (11) can approximate an SBS state, storing an objective position of the central system, in the semi-classical approximation. We hypothesize that the approach to objectivity is possible only in such a limit, when the central system is macroscopic enough, making objectivity a macroscopic phenomenon.

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APPENDIX A: ALTERNATIVE DERIVATION OF THE HYBRID SOLUTION

In the particular case of a linear QBM model, the hybrid $SE_{\rm obs}$ solution from the main text

$$\langle X'|\varrho_{S:E_{\rm obs}}(t)|X\rangle \approx \iint dX_0 dX_0' \langle X_0'|\varrho_{0S}|X_0\rangle K_t^{\rm sc}(X,X_0)^* K_t^{\rm sc}(X',X_0') F[X_{\rm cl}(t),X_{\rm cl}'(t)] \bigotimes_{k\in E_{\rm obs}} \hat{U}_t^k[X_{\rm cl}]\varrho_{0k}\hat{U}_t^k[X_{\rm cl}']^\dagger \tag{A1}$$

can be also obtained in the following way: Forgetting for a moment the evolution of the environment, the central system is a forced harmonic oscillator with a well-known solution for the propagator [39]. It is determined by the action

$$S[x(t)] = \frac{M\Omega}{2\sin\Omega t}\left[\left(X^2+X_0^2\right)\cos\Omega t - 2XX_0\right] + \frac{1}{\sin\Omega t}\sum_k C_k \int_0^t d\tau [X\sin\Omega\tau + X_0 \sin\Omega(t-\tau)]x_k(\tau)$$
$$- \frac{\Omega}{\sin\Omega t}\sum_{k,l}\frac{C_k C_l}{M\Omega^2}\int_0^t d\tau \int_0^\tau d\tau' \sin\Omega\tau' \sin\Omega(t-\tau)x_k(\tau)x_l(\tau'). \tag{A2}$$

Neglecting the last term, using the resulting action to construct the propagator for the global state, and changing to the operator picture for the environmental degrees of freedom, we obtain the solution (11).

APPENDIX B: TRACING OVER THE CENTRAL SYSTEM

Here we calculate the trace over the central system S of the hybrid solution (11). We first assume, for simplicity, only one observed environment and one unobserved. Generalization to multiple environments in both groups will be obvious and we present it at the end. The main idea is to rewrite the trace using the no-recoil condition

$$\frac{C_k}{M\Omega^2} \ll 1. \tag{B1}$$

First, we take the matrix elements w.r.t. the environment and comeback from the operator form of the environment part of (11) to the path integral one using

$$\int \mathscr{D}x(t) e^{\frac{i}{\hbar}S_{\rm env}[x(t)]} e^{\frac{i}{\hbar}S_{\rm int}[X_{\rm cl}(t),x(t)]} \equiv \langle x|\hat{U}_t[X_{\rm cl}]|x_0\rangle. \tag{B2}$$

This gives

$$\int dX \langle X;x'|\varrho_{S:E_{\rm obs}}|X;x\rangle$$
$$= \int dX_0 dX_0' dx_0 dx_0' \langle X_0'|\varrho_{0S}|X_0\rangle \langle x_0'|\varrho_{0E}|x_0\rangle \tag{B3}$$
$$\times \int dX K_t^{\rm sc}(X,X_0)^* K_t^{\rm sc}(X,X_0') F[X_{\rm cl},X_{\rm cl}'] \tag{B4}$$
$$\times \int \mathscr{D}x\mathscr{D}x' \exp\frac{i}{\hbar}(S_{\rm env}[x]-S_{\rm env}[x']$$
$$+ S_{\rm int}[X_{\rm cl},x] - S_{\rm int}[X_{\rm cl}',x']). \tag{B5}$$

Because of the tracing, the classical trajectories have the same endpoints $X_{\rm cl}(t) = X_{\rm cl}'(t) = X$, and $x(0)=x_0,\ x(t)=x$ and similarly for $x'(\tau)$. Let us analyze the above expression term by term. It is well known that the semiclassical propagator $K_t^{sc}(X,X_0)$ for the harmonic oscillator is equal to the full quantum one. We thus have

$$K_t^{\rm sc}(X,X_0)^* K_t^{\rm sc}(X,X_0') \tag{B6}$$
$$= \frac{M\Omega}{2\pi\hbar|\sin\Omega t|} e^{\frac{iM\Omega}{2\hbar\sin\Omega t}[(X_0'^2-X_0^2)\cos\Omega t + 2X\Delta X_0]}. \tag{B7}$$

There are X-dependent and X-independent parts.

The influence functional may be written using path integrals as [40]

$$F[X_{\rm cl},X_{\rm cl}'] = \int d\tilde{y}dy_0 dy_0' \int \mathscr{D}y\mathscr{D}y' \langle y_0'|\tilde{\varrho}_{0E}|y_0\rangle$$
$$\times e^{\frac{i}{\hbar}(S_{\rm env}[y]-S_{\rm env}[y']+S_{\rm int}[X_{\rm cl},y]-S_{\rm int}[X_{\rm cl}',y'])}, \tag{B8}$$

where $\tilde{\varrho}_{0E}$ is the initial state of the unobserved part of the environment $E_{\rm uno}$, which can be different from the initial state of the observed part $E_{\rm obs}$, ϱ_{0E} in (B4). The boundary conditions are $y(0)=y_0,\ y'(0)=y_0',\ y(t)=y'(t)=\tilde{y}$. The generalization to multiple unobserved environments is straightforward: the combined influence functional will be a product over $j\in E_{\rm uno}$ of the terms (B8) for each mode j with ϱ_{0j} initial state, $F=\prod_{j\in E_{\rm uno}}F_j$

The terms of the form $S_{\rm env}[x]-S_{\rm env}[x']$, appearing both in (B5) and (B8), do not depend on the integration variable X and thus can pulled in from of the integral over X. The interaction terms $S_{\rm int}[X_{\rm cl}, y] - S_{\rm int}[X'_{\rm cl}, y']$ from (B5) and (B8) will have both X-dependent and X-independent parts. To separate them, let us parametrize the classical trajectory $X_{\rm cl}$ satisfying the apropriate boundary conditions $X_{\rm cl}(0) = X_0$, $X_{\rm cl}(t) = X$, as below

$$X_{\rm cl}(\tau) = X_0 \cos\Omega\tau \left[1 - \frac{\sin\Omega\tau}{\sin\Omega t}\right] + X\frac{\sin\Omega\tau}{\sin\Omega t} \tag{B9}$$

$$\equiv X_0 a(\tau) + X\frac{\sin\Omega\tau}{\sin\Omega t}. \tag{B10}$$

Then it is easy to see that

$$S_{\rm int}[X_{\rm cl}, x] - S_{\rm int}[X'_{\rm cl}, x']$$

$$= -C\int_0^t d\tau a(\tau)[X_0 x(\tau) - X'_0 x'(\tau)] \tag{B11}$$

$$-\frac{CX}{\sin\Omega t}\int_0^t d\tau \sin\Omega\tau[x(\tau) - x'(\tau)]. \tag{B12}$$

We now combine the X-dependent factors from (B5), (B7), and (B8) and integrate them over X:

$$\frac{M\Omega}{2\pi\hbar|\sin\Omega t|}\int dX \exp\left\{\frac{iX}{\hbar\sin\Omega t}\left[M\Omega\Delta X_0 - C\int_0^t d\tau \sin\Omega\tau\big[x(\tau) - x'(\tau) + y(\tau) - y'(\tau)\big]\right]\right\} \tag{B13}$$

$$= \frac{M\Omega}{2\pi\hbar|\sin\Omega t|}\int dX \exp\left\{\frac{iXM\Omega^2}{\hbar\sin\Omega t}\left[\frac{\Delta X_0}{\Omega} - \frac{C}{M\Omega^2}\int_0^t d\tau \sin\Omega\tau\big[x(\tau) - x'(\tau) + y(\tau) - y'(\tau)\big]\right]\right\} \tag{B14}$$

$$\approx \frac{M\Omega}{2\pi\hbar|\sin\Omega t|}\int dX \exp\left(\frac{iXM\Omega\Delta X_0}{\hbar\sin\Omega t}\right) = \delta(\Delta X_0), \tag{B15}$$

where, in the crucial step, we used the recoilless condition (13) and neglected the action integral. We can now comeback to the main integral (B4) and (B5). The delta function (B15) forces the trajectories $X_{\rm cl}(\tau)$ and $X'_{\rm cl}(\tau)$ to be equal as it forces $X_0 = X'_0$ (the endpoints are the same in this calculation as we are calculating the trace over X). This immediately forces the influence functional (B8) to be equal to 1 since

$$F[X_{\rm cl}, X_{\rm cl}] = {\rm Tr}(\hat{U}_t[X_{\rm cl}]\tilde{\varrho}_{0E}\hat{U}_t[X_{\rm cl}]^\dagger) = 1. \tag{B16}$$

The X-independent part of (B7) will be equal to 1 as well since $X_0'^2 - X_0^2 = 0$. We are thus left with the following integral:

$$\int dX\langle X; x'|\varrho_{S:E_{\rm obs}}|X; x\rangle = \int dX_0 p(X_0)\int dx_0 dx'_0\langle x'_0|\varrho_{0E}|x_0\rangle \tag{B17}$$

$$\times\int \mathscr{D}x\mathscr{D}x' \exp\frac{i}{\hbar}\left(S_{\rm env}[x] - S_{\rm env}[x'] - C\int_0^t d\tau a(\tau)[X_0 x(\tau) - X'_0 x'(\tau)]\right) \tag{B18}$$

$$= \int dX_0 p(X_0)\int dx_0 dx'_0\langle x'_0|\varrho_{0E}|x_0\rangle\int \mathscr{D}x\mathscr{D}x' e^{\frac{i}{\hbar}(S_{\rm env}[x] + S_{\rm int}[X^0_{\rm cl}, x] - S_{\rm env}[x'] - S_{\rm int}[X^0_{\rm cl}, x'])} \tag{B19}$$

$$= \int dX_0 p(X_0)\langle x'|U_t\big[X^0_{\rm cl}\big]\varrho_{0E}U_t\big[X^0_{\rm cl}\big]^\dagger|x\rangle, \tag{B20}$$

where in (B19) we came back to the operator picture using (B2) and introduce

$$p(X_0) \equiv \langle X_0|\varrho_{0S}|X_0\rangle, \tag{B21}$$

which is the initial distribution of the central system's position. Above, $X^0_{\rm cl}$ is the classical trajectory with the endpoint 0:

$$X^0_{\rm cl}(0) = X_0,\ X^0_{\rm cl}(t) = 0. \tag{B22}$$

It appears by comparing the action integral in the exponent of (B18) to (B10) with $X = 0$. Having the result (B20) for a single degree of freedom of the observed environment, we can now apply it to the initial task with multiple environments. Performing the above calculations for each degree of freedom j we finally obtain

$$\varrho_k(t) = {\rm Tr}_{E_1\dots\not{E_k}\dots}\int dX\langle X|\varrho_{S:E_{\rm obs}}|X\rangle \tag{B23}$$

$$\approx \int dX_0 p(X_0){\rm Tr}_{E_1\dots\not{E_k}\dots}\bigotimes_{j\in E_{obs}}\hat{U}^j_t\big[X^0_{\rm cl}\big]\varrho_{0j}\hat{U}^j_t\big[X^0_{\rm cl}\big]^\dagger$$

$$= \int dX_0 p(X_0)\hat{U}^k_t\big[X^0_{\rm cl}\big]\varrho_{0k}\hat{U}^k_t\big[X^0_{\rm cl}\big]^\dagger \tag{B24}$$

$$\equiv \int dX_0 p(X_0)\varrho^k_t\big[X^0_{\rm cl}\big], \tag{B25}$$

where the approximation signalizes that we have used the no-recoil condition (13) and we define

$$\varrho^k_t[X_{\rm cl}] \equiv \hat{U}^k_t[X_{\rm cl}]\varrho_{0k}\hat{U}^k_t[X_{\rm cl}]^\dagger. \tag{B26}$$

APPENDIX C: GENERALIZED OVERLAP FOR THERMAL QBM

For completeness' sake, we present here the derivation of the generalize overlap (21) from [23]. We first solve the effective dynamics for the environmental modes, resulting from (8). In the case of the linear QBM considered here, the effective Hamiltonian $\hat{H}_{\rm eff} \equiv \hat{H}_{\rm env} + \hat{H}_{\rm int}[X_{\rm cl}]$ decomposes of course w.r.t. the subenvironments and for the kth subeviron-ment has a simple form

$$\hat{H}^k_{\rm eff} = \frac{\hat{p}_k^2}{2m_k} + \frac{m_k\omega_k^2\hat{x}_k^2}{2} - C_k X_{\rm cl}(t)\hat{x}_k, \tag{C1}$$

where $\hat{x}_k$, $\hat{p}_k$ are the canonical observables. This is a standard forced harmonic oscillator. It can be solved in many ways, the fastest being by passing to the interaction picture

$$\hat{H}^k_{\rm eff}(t) = -C_k X_{\rm cl}(t)\hat{x}_k(t), \tag{C2}$$

where $\hat{x}_k(t) = \sqrt{\hbar/2m_k\omega_k}(e^{-i\omega_k t}\hat{a} + e^{i\omega_k t}\hat{a}_k^\dagger)$ with $\hat{a}_k, \hat{a}_k^\dagger$ being the corresponding annihilation and creation operators. Using the fact that (C2) commute for different times to a number $[\hat{H}^k_{\rm eff}(t), \hat{H}^k_{\rm eff}(t')] = ic(t,t')$, one can use the Baker-Campbell-Hausdorff formula formula to calculate the evolution via $\lim_{n\to\infty}(\prod_{r=1}^n \exp[-i/\hbar\hat{H}^k_{\rm eff}(t_r)\Delta t])$, $\Delta t \equiv t/n$, $t_r \equiv r\Delta t$:

$$\lim_{n\to\infty}\left(\prod_{r=1}^n \exp[-\frac{i}{\hbar}\hat{H}^k_{\rm eff}(t_r)\Delta t]\right)$$

$$= e^{i\zeta(t)}\exp\left(-\frac{i}{\hbar}\int_0^t d\tau \hat{H}^k_{\rm eff}(\tau)\right) \tag{C3}$$

$$= e^{i\zeta(t)}\exp\left[-i\frac{C_k}{\sqrt{2\hbar m_k\omega_k}}\left(\int_0^t d\tau X_{\rm cl}(\tau)e^{i\omega_k\tau} + \text{c.c.}\right)\right], \tag{C4}$$

where $\zeta(t)$ is some phase factor, that, as we will see below, will be unimportant for our calculations. Defining

$$\alpha(t) \equiv -\frac{iC_k}{\sqrt{2\hbar m_k\omega_k}}\int_0^t d\tau X_{\rm cl}(\tau)e^{i\omega_k\tau}, \tag{C5}$$

the exponent in (C4) becomes the standard displacement operator $\hat{D}[\alpha(t)]$, so that in the interactrion picture we obtain

$$\hat{U}_t^k[X_{cl}] = e^{i\zeta(t)}\hat{D}[\alpha(t)], \tag{C6}$$

which is the expression (8).

We now calculate the single system generalized overlap (21). To simplyfy the notation, we will drop the index k in all the objects that define (21) since the calculation is the same for every mode. Using the definition of the generalized overlap together with the operator identity, following from the spectral theorem $\sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$, we obtain

$$B[\Delta, t]$$

$$= \text{Tr}\sqrt{\sqrt{\varrho_0}\hat{U}_t[X'_{\rm cl}]^\dagger\hat{U}_t[X_{\rm cl}]\varrho_0\hat{U}_t[X_{\rm cl}]^\dagger\hat{U}_t[X'_{\rm cl}]\sqrt{\varrho_0}}, \tag{C7}$$

where we pulled the extreme left and right unitaries out of the both square roots and used the cyclic property of the trace to cancel them out. From (C6) we obtain that modulo phase factors, which cancel in (C7)

$$\hat{U}_t[X'_{\rm cl}]^\dagger\hat{U}_t[X_{\rm cl}] \simeq \hat{D}[\alpha(t) - \alpha'(t)] \equiv \hat{D}(\eta_t), \tag{C8}$$

where we could use the interaction picture expression (C6) since the free evolutions cancel and we introduced $\eta_t \equiv \alpha(t) - \alpha'(t)$ for a later convenience. Next, assuming that ϱ_0 is a thermal state, we use the well-known coherent state representation for the middle ϱ_0 under the square root in (C7)

$$\varrho_0 = \int \frac{d^2\gamma}{\pi\bar{n}}e^{-|\gamma|^2/\bar{n}}|\gamma\rangle\langle\gamma|, \tag{C9}$$

where $\bar{n} = 1/(e^{\hbar\beta\omega} - 1)$ is the mean excitation number at the inverse temperature β. Denoting the Hermitian operator under the square root in (C7) by $\hat{A}_t$, we obtain

$$\hat{A}_t = \int \frac{d^2\gamma}{\pi\bar{n}}e^{-|\gamma|^2/\bar{n}}\sqrt{\varrho_0}\hat{D}(\eta_t)|\gamma\rangle\langle\gamma|\hat{D}(\eta_t)^\dagger\sqrt{\varrho_0} \tag{C10}$$

$$= \int \frac{d^2\gamma}{\pi\bar{n}}e^{-|\gamma|^2/\bar{n}}\sqrt{\varrho_0}|\gamma + \eta_t\rangle\langle\gamma + \eta_t|\sqrt{\varrho_0}. \tag{C11}$$

To perform the square roots above, we now use the Fock representation of the thermal state

$$\varrho_0 = \sum_n \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|, \tag{C12}$$

so that

$$\hat{A}_t = \int \frac{d^2\gamma}{\pi\bar{n}}e^{-\frac{|\gamma|^2}{\bar{n}}}\sum_{m,n}\sqrt{\frac{\bar{n}^{m+n}}{(\bar{n}+1)^{m+n+2}}}$$

$$\times \langle n|\gamma + \eta_t\rangle\langle\gamma + \eta_t|m\rangle|n\rangle\langle m| \tag{C13}$$

and the scalar products above are given by the well-known expressions of the coherent states in the Fock basis

$$\langle n|\gamma + \eta_t\rangle = e^{-|\gamma+\eta_t|^2/2}\frac{(\gamma + \eta_t)^n}{\sqrt{n!}}. \tag{C14}$$

The strategy is now to use this relation and rewrite each sum in (C13) as a coherent state but with a rescaled argument, and then try to rewrite (C13) as a single thermal state (with a different mean excitation number than ϱ_0). To this end we note that

$$e^{-\frac{1}{2}|\gamma+\eta_t|^2}\sum_n\left(\frac{\bar{n}}{\bar{n}+1}\right)^{\frac{n}{2}}\frac{(\gamma+\eta_t)^n}{\sqrt{n!}}|n\rangle \tag{C15}$$

$$= e^{-\frac{1}{2}\frac{|\gamma+\eta_t|^2}{\bar{n}+1}}\left|\sqrt{\frac{\bar{n}}{\bar{n}+1}}(\gamma + \eta_t)\right\rangle. \tag{C16}$$

Substituting this into (C13) and reordering gives

$$\hat{A}_t = \frac{1}{\bar{n}+1}e^{-\frac{|\eta_t|^2}{1+2\bar{n}}}\int\frac{d^2\gamma}{\pi\bar{n}}e^{-\frac{1+2\bar{n}}{\bar{n}(\bar{n}+1)}|\gamma + \frac{\bar{n}}{1+2\bar{n}}\eta_t|^2}$$

$$\times\left|\sqrt{\frac{\bar{n}}{\bar{n}+1}}(\gamma + \eta_t)\right\rangle\left\langle\sqrt{\frac{\bar{n}}{\bar{n}+1}}(\gamma + \eta_t)\right|. \tag{C17}$$

Note that since we are interested in $\text{Tr}\sqrt{\hat{A}_t}$ rather than $\hat{A}_t$ itself, there is a freedom of rotating $\hat{A}_t$ by a unitary operator, in particular, by a displacement. We now find such a displacement as to turn (C17) into the thermal form. Comparing the exponential under the integral in (C17) with the thermal form (C9), we see that the argument of the subsequent coherent states should be proportional to $\gamma + (\bar{n}\eta_t)/(1+2\bar{n})$. Simple algebra gives

$$\left|\sqrt{\frac{\bar{n}}{\bar{n}+1}}(\gamma+\eta_t)\right\rangle \simeq \hat{D}\left(\sqrt{\frac{\bar{n}}{\bar{n}+1}}\frac{\bar{n}+1}{1+2\bar{n}}\eta_t\right)\left|\sqrt{\frac{\bar{n}}{\bar{n}+1}}\left(\gamma+\frac{\bar{n}}{1+2\bar{n}}\eta_t\right)\right\rangle, \tag{C18}$$

where we omitted the irrelevant phase factor as we are interested in the projector on the above state. Inserting the above relation into (C17), dropping the displacements, and changing the integration variable $\gamma \to \sqrt{\bar{n}/(\bar{n}+1)}[\gamma+(1+2\bar{n})\eta_t]$ gives

$$B[\Delta,t] = e^{-\frac{1}{2}\frac{|\eta_t|^2}{1+2\bar{n}}}\frac{1}{\sqrt{1+2\bar{n}}}\mathrm{Tr}\sqrt{\varrho_{\rm th}[\bar{n}^2/(1+2\bar{n})]}, \tag{C19}$$

where $\varrho_{\rm th}(\bar{n})$ is the standard thermal state with the mean excitation number $\bar{n}$. We use the Fock expansion (C12) for $\varrho_{\rm th}[\bar{n}^2/(1+2\bar{n})]$ and obtain

$$B[\Delta,t] = e^{-\frac{|\eta_t|^2}{2+4\bar{n}}}\frac{1}{\sqrt{1+2\bar{n}}} \tag{C20}$$

$$\times\left(1+\frac{\bar{n}^2}{1+2\bar{n}}\right)^{-\frac{1}{2}}\sum_n\left(\frac{\bar{n}^2/(1+2\bar{n})}{1+\bar{n}^2/(1+2\bar{n})}\right)^{\frac{n}{2}} \tag{C21}$$

$$= \exp\left[-\frac{1}{2}|\alpha(t)-\alpha'(t)|^2 \mathrm{th}\left(\frac{\hbar\beta\omega}{2}\right)\right], \tag{C22}$$

where in the last step, we used the definition of η_t from (C8) and $\bar{n}=1/(e^{\hbar\beta\omega}-1)$. Finally, using (C5), we obtain the desired result (22). It is interesting that the overlap factor looks very similar to the real part of the influence functional but with the inverse temperature dependence.

APPENDIX D: MATSUBARA REPRESENTATION OF THE QFI KERNEL

Just like in the case of the noise kernel [7], one can derive a formal analytic expression for the QFI kernel using the fermionic Matsubara representation [49]

$$\mathrm{th}\left(\frac{\beta\hbar\omega}{2}\right) = \frac{4}{\beta\hbar\omega}\sum_{n=0}^{\infty}\frac{1}{1+(\nu_n/\omega)^2}, \tag{D1}$$

with fermionic frequencies

$$\nu_n = \frac{(2n+1)\pi}{\hbar\beta}. \tag{D2}$$

Substituting this into the QFI definition (24) and integrating term by term, we find

$$\phi(\tau) = \frac{4M\gamma\Lambda}{\hbar\beta}\sum_{n=0}^{\infty}\frac{e^{-\Lambda|\tau|}-(\nu_n/\Lambda)e^{-\nu_n|\tau|}}{1-(\nu_n/\Lambda)^2}, \tag{D3}$$

which looks identical to the expansion of the noise kernel $\nu(\tau)$ [7], except that, instead of the bosonic frequencies $\nu_n = 2n\pi/(\hbar\beta)$, we now have the fermionic (D2). In particular, now $\nu_0 = \pi/(\hbar\beta) \neq 0$ so that $\nu_n/\Lambda \gg 1$ for any n, including $n=0$. This complicates the analysis compared to the bosonic case, describing the influence functional, as there is now an interplay between Λ, ν_0, and τ. The double integral in (23), which we denote by $\Phi[\Delta,t]$:

$$\Phi[\Delta,t] \equiv \int_0^t d\tau\int_0^\tau d\tau'\Delta(\tau)\phi(\tau-\tau')\Delta(\tau') \tag{D4}$$

can be formally calculated term by term, using (D3) and an explicit expression for the trajectories difference

$$\Delta(\tau) \equiv X_{\rm cl}(\tau) - X'_{cl}(\tau) = \Delta X_0\frac{\sin[\Omega(t-\tau)]}{\sin\Omega t} + \Delta X\frac{\sin\Omega\tau}{\sin\Omega t}. \tag{D5}$$

We first slightly rearrange the expression (D3)

$$\phi(\tau) = \frac{4M\gamma}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\frac{\Lambda^2}{1-(\Lambda/\nu_n)^2} \times\left(e^{-\nu_n|\tau|}-\frac{\Lambda}{\nu_n}e^{-\Lambda|\tau|}\right). \tag{D6}$$

We now calculate term by term the double integral

$$\Phi[\Delta,t] \equiv \int_0^t d\tau\int_0^\tau d\tau'\Delta(\tau)\phi(\tau-\tau')\Delta(\tau'), \tag{D7}$$

using elementary integrals and obtain

$$\Phi[\Delta,t] = \frac{4M\gamma}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\frac{1}{1-(\Lambda/\nu_n)^2}\frac{1}{\sin^2\Omega t} \times\big[\Lambda^2[c_t(\nu_n)-(\Lambda/\nu_n)c_t(\Lambda)]\big(\Delta X_0^2+\Delta X^2\big) + \Lambda^2[d_t(\nu_n)-(\Lambda/\nu_n)d_t(\Lambda)]\Delta X_0\Delta X\big], \tag{D8}$$

with the coefficient defined as

$$c_t(\nu_n) = \frac{1}{1+(\Omega/\nu_n)^2}\left[\frac{t}{2\nu_n}-\frac{1}{4\nu_n\Omega}\sin(2\Omega t)-\frac{\sin^2\Omega t}{2\nu_n^2}\right]+\frac{1}{[1+(\Omega/\nu_n)^2]^2}\left[\frac{\Omega^2}{\nu_n^4}-e^{-\nu_n t}\left(\frac{\Omega}{\nu_n^3}\sin\Omega t+\frac{\Omega^2}{\nu_n^4}\cos\Omega t\right)\right], \tag{D9}$$

$$d_t(\nu_n) = \frac{1}{1+(\Omega/\nu_n)^2}\left[-\frac{t}{\nu_n}\cos\Omega t+\frac{1}{\Omega\nu_n}\sin\Omega t\right]-\frac{1}{[1+(\Omega/\nu_n)^2]^2}\left\{2\frac{\Omega^2}{\nu_n^4}\cos\Omega t+e^{-\nu_n t}\left[\frac{\Omega^2}{\nu_n^4}+\left(\frac{\Omega}{\nu_n^2}\cos\Omega t+\frac{1}{\nu_n}\sin\Omega t\right)^2\right]\right\}, \tag{D10}$$

and analogously for $c_t(\Lambda)$ and $d_t(\Lambda)$. The quantities that are small in the Caldeira-Leggett model are

$$\Lambda/\nu_n \ll 1, \; \Omega/\nu_n \ll 1, \; \Omega/\Lambda \ll 1, \tag{D11}$$

which can be used to simplify the above expressions.

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3.2 Encoding position by spins: Objectivity in the boson-spin model

3.2.1 Summary

Intuitively, information in classical physics can be defined as an identification with the state of a system itself. It is localized at each spacetime point as physical reality and exists independently of observation as physical reality itself. On the other hand, in quantum mechanics, the information of a system needs to be realized throughout interaction and understood as it is transferred through physical processes. In this way, it may not be appropriate to identify information with a quantum state itself, more specifically, due to the fact that a unitary evolution cannot copy a general state, by the well-known theorem, called the no-cloning theorem [13, 47, 48] and a quantum measurement fundamentally disturbs an original quantum state. In addition, it is known that only orthogonal states can be recorded to another state in a unitary evolution. Due to such distinct quantum properties, it is a challenge to derive a mechanism that leads to classical objectivity with its unlimited cloning of information. One such mechanism in [1], called quantum Darwinism and its stronger version, SBS in [2, 3, 4] were proposed.

In this article we apply the SBS analysis to the "boson-spin model", a joint system of a harmonic oscillator and the bi-linearly interacting spin-$\frac{1}{2}$ environment in the thermal state. This model is related to so-called Dicke models [49, 50]. Here we investigate how the objective structure of the system due to the interaction with the environment appears.

The objective structure in a quantum state is measured by the objectivity measures, called the objectivity markers, the decoherence factor and the generalized overlap, in parallel with confirming how well information of a central system is transferred into environment. The boson-spin model is particular in that continuous information of a harmonic oscillator is encoded in a finite dimensional system, the spin environment, while in the quantum Brownian motion (QBM) model both systems have continuous degrees of freedom. We use the same approximation used in early studies of the QBM model [24], called the Born-Oppenheimer approximation, i.e. that the state of a harmonic oscillator is hardly mechanically influenced by interaction with spins. This is also called the recoilless limit of the model.

The Born-Oppenheimer approximation simplifies the problem so that the spin environment is under a periodic effective Hamiltonian where a position operator for a harmonic oscillator is replaced by its classical trajectory. This periodicity allows us to use the Floquet theory [33] and the high-frequency expansion [34, 35]. The advantage of these techniques is to make approximations analytic and controllable in a series expansion. Similarly as in QBM, using a weak coupling limit defined by $gX_0/\Omega \ll 1$, where g is an interaction coupling and a position amplitude X_0 for a harmonic oscillator, the decoherence factor and the generalized overlap are expressed approximately in an exponential decay form in position space, which allows us to define two length scales, the decoherence length and the distiguishability length as obtained in QBM. It also confirms that the distiguishability length is larger than the decoherence length. It implies that to distinguish information in different positions requires a longer distance than the distance where the decoherence occurs.

The Floquet theory clearly shows the distinct roles of two frequencies from the harmonic oscillator and the spin self-Hamiltonian. The periodicity in the decoherence factor and the generalized overlap prevents them from decaying as time goes large. The period from the self-Hamiltonian is much larger than one from a harmonic oscillator. In order to achieve a large scale decay with the help of collective degrees of freedom, it is necessary to break the periodicity. The existence of the spin self-Hamiltonian plays an important role in this process.

The self-Hamiltonian dependence of the objectivity makers is related to the initial trajectory of the harmonic oscillator, especially visible in distinguishability. When the amplitude of a harmonic oscillator is initially the maximum in the effective Hamiltonian for spins, the generalized overlap does not depend on a spin self-coupling in the lowest order, which disables a generalized overlap to decay in large scale even with multi-spin degrees of freedom. However, with the initial non-maximal amplitude, the large scale frequency due to a spin self-coupling appears in the generalized overlap. Even a small amount of contribution of the large scale frequency breaks a periodicity in the generalized overlap and finally leads to its decay in long time scale with a large number of spins. The decay is maximized when the initial amplitude of a harmonic oscillator is zero.

In conclusion, in the boson-spin model, distinguishability is harder to achieve than decoherence due to its temperature dependence. In general, a large number of degrees of freedom in the environment are necessary for objectivity but a spin self-Hamiltonian plays a crucial role in its appearance in the objectivity markers, especially in the generalized overlap. The initial trajectory of a harmonic oscillator affects its appearance in a generalized overlap.

3.2.2 Work contribution

My contribution to this article is following:

1. Giving the idea and implementing the Floquet theory and the high frequency expansion.
2. Performing almost all of the analytical calculations, in particular calculating all the objectivity markers and their approximations.
3. Performing all the numerical calculations.
4. Creating large parts of the manuscript.

Encoding position by spins: Objectivity in the boson-spin model

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We investigate quantum objectivity in the boson-spin model, where a central harmonic oscillator interacts with a thermal bath of spin-$\frac{1}{2}$ systems. We analyze how information about a continuous position variable can be encoded into discrete, finite-dimensional environments. More precisely, we study conditions under which the so-called spectrum broadcast structures can be formed in the model. These are multipartite quantum-state structures, representing a mode-refined form of decoherence. Working in the recoil-less limit, we use the Floquet theory to show that despite its apparent simplicity, the model has a rich structure with different regimes, depending on the motion of the central system. In one of them, the faithful encoding of the position and hence objectivity are impossible irrespectively of the resources used. In another, large enough collections of spins will faithfully encode the position information. We derive the characteristic length scales, corresponding to decoherence and precision of the encoding.

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I. INTRODUCTION

Although quantum mechanics is believed to be the most fundamental theory of Nature, it is mysterious and puzzling that we still have not fully succeeded in explaining our daily observed world by quantum mechanics. How is it possible for all counterintuitive quantum natures like superposition, interference, disturbance, nonlocality, etc., clearly disappear in our macroscopic world? Although many alternative approaches compete for explaining the quantum-to-classical transition, none of them has been agreed on so far. In this situation it is important to restrict ourselves to the question how far the classical-quantum discrepancy can be explained within the current-state quantum mechanics. One of the aspects is the problem is the objective character of the macroscopic world as first noted by Zurek [1,2]. Objectivity may be viewed as an observer independence bears some resemblance to the relativity theory. But due to the inevitable disturbances introduced by observations in quantum mechanics, it is not *a priori* clear how to achieve the observer independence, at least at the basic level of measurement results.

In the history of physics, there has been an orthodox view that the world existing outside of the system of interest plays a role of a source of noise which can be, at least in principle, perturbatively controlled, so when it is minimized, the "true nature" will get more and more approachable. But quantum mechanics changed that view: our macroscopic reality can be considered a consequence of interaction between a system and the rest of the world, as emphasized by the decoherence theory [3,4]. Within this view there has been an idea developed, called quantum Darwinism, aimed at explaining the apparent observer independence in the macroscopic world [1,2,5,6]. It postulates that interactions between a system and an environment redundantly transfer the information of a system to the environment during decoherence. The idea has opened a new field of objectivity studies (see, e.g., [7–11] for some of the most recent developments). Although quantum Darwinism does not completely explain the nonunitary collapse process, the famous measurement problem, it is still remarkable that within quantum mechanics some form of objective classicality can be derived.

A further development of the quantum Darwinism idea is represented by spectrum broadcast structures (SBS) [6,12,13], which are specific quantum-state structures, encoding an operational form of objectivity. SBS are a stronger form of quantum Darwinism in a sense that SBS formation implies the original quantum Darwinism conditions but not vice versa [7]. Under appropriate conditions, SBS have been shown to form in almost all the canonical decoherence models [4], i.e., a collisional decoherence [12], quantum Brownian motion (QBM) in the recoil-less limit [14,15], a spin-spin model [16,17], and a spin-boson model [14,18]. The only one left is a boson-spin model, which we analyze in this work. The central system is a massive oscillator interacting with a thermal bath of spin-$\frac{1}{2}$ systems. We use here a recoil-less approximation, similarly as in the QBM studies [14,15], where the harmonic oscillator influences the spin environment, but the recoil is suppressed. This approximation leads to the same form of Hamiltonian as for, e.g., a two-level atom interacting with linearly polarized light [19]. As the effective Hamiltonian for the spin environment is time periodic, this allows us to use the Floquet theory and the high-frequency expansion. The most interesting question arising in this model, and absent in previous studies, is how finite-dimensional environments encode a continuous variable (the central oscillator's amplitude). We show that depending on the state of motion of a central oscillator there can be either only a momentary formation of SBS states or a permanent or asymptotic one. Interestingly, this

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behavior is opposite to the QBM model, showing once again the stark difference between spin and oscillator environments. We derive the length scales corresponding to decoherence and faithful information encoding in the environment, which scale $1/\sqrt{N}$, where N is the environment fraction size that is used to store the information or decohere the system.

We will be interested in a so-called partially reduced state $\rho_{S:oE}$, where a fraction of the environment, assumed unobserved and denoted by uE, was traced out as an unavoidable loss, but the remaining fraction, denoted by oE, is kept for observation. The spectrum broadcast structure (SBS) or an objective quantum state is then defined as follows [6,12]:

$$\rho_{S:fE} = \sum_i p_i |i\rangle_S\langle i| \otimes \rho_i^{E_1} \otimes \cdots \otimes \rho_i^{E_{fN}}, \tag{1}$$

where

$$\rho_i^{E_k}\rho_j^{E_k} = 0 \tag{2}$$

for $i \neq j$, which is equivalent to the states $\rho_i^{E_k}$ having orthogonal supports and thus being perfectly distinguishable. After unobserved degrees of freedom traced out, the SBS structure of a total density matrix is in an orthogonal convex combination form. The approach to the SBS structure is marked by vanishing quantum coherence (off-diagonal elements) and a perfect distinguishability (diagonal elements), corresponding to a vanishing decoherence factor and a vanishing generalized overlap (state fidelity), respectively [16]. These are the objectivity markers that we will analyze in various regimes.

II. DYNAMICS OF SYSTEM

The total Hamiltonian H for a simple harmonic oscillator, bilinearly interacting with spin-$\frac{1}{2}$ environment [4]

$$H = H_S + \sum_i H_E^{(i)} + \sum_i H_{\rm int}^{(i)}, \tag{3}$$

where

$$\begin{aligned} H_S &= \frac{\hat{P}^2}{2M} + \frac{1}{2}M\Omega^2\hat{X}^2,\\ H_E^{(i)} &= -\frac{\Delta_i}{2}\sigma_x^{(i)},\\ H_{\rm int}^{(i)} &= g_i\hat{X}\otimes\sigma_z^{(i)}, \end{aligned} \tag{4}$$

where M and Ω are a mass and an angular frequency of an oscillator, respectively, and g_i and Δ_i are a spin-environmental coupling constant and a self-energy (also called the tunneling matrix element) for the ith spin system, respectively. Here only bipartite interactions $H_{\rm int}^{(i)}$ between the ith spin and the harmonic oscillator are considered, without mutual interactions among the spins. Despite its simple form, the total Hamiltonian (3) is difficult to solve directly. For the purpose of our analysis it is, however, enough to use the so-called recoil-less limit, at least as a first approximation. In this limit, the central oscillator is assumed to be massive enough not to feel the recoil of the environment, while each of the environmental spins is affected by the motion of the central oscillator, which acts as a classical force. This is an opposite limit to the much more popular Born-Markov limit, where it is an environment that is assumed not to be affected by the system. The justification for such a choice comes from the fact that we are primarily interested in an information leakage from a system to an environment as cutting the influence of the system on the environment would also cut the information leakage. Hence, it leads us to a study of the opposite, recoil-less limit. It can be viewed as a version of the Born-Oppenheimer approximation [20] and it was already used in the objectivity studies in [14,21,22]. In the recoil-less limit, the system S evolves unperturbed, according to its own dynamics H_S. It influences the environment via the interaction Hamiltonian where the system's position operator $\hat{X}$ can be approximated by the classical trajectory $X(t;X_0)$, starting at some initial position $X(0) = X_0$. The resulting approximate solution is given by the following ansatz [14]:

$$|\Psi_{S:E}\rangle = \int dX_0\phi_0(X_0)e^{-i\hat{H}_S t}|X_0\rangle\hat{U}_{\rm eff}(X(t;X_0))|\psi_0\rangle. \tag{5}$$

Here the $\hbar = 1$ convention has been used and will be applied to the entire presentation. $\hat{U}_{\rm eff}(X(t;X_0))$ is the evolution generated by

$$H_{\rm eff} = \sum_i\left(-\frac{\Delta_i}{2}\sigma_x^{(i)} + g_iX(t;X_0)\sigma_z^{(i)}\right), \tag{6}$$

and $|\phi_0\rangle$ and $|\psi_0\rangle$ are initial states of S and E, respectively. Formally, (5) is generated by a controlled-unitary evolution

$$\hat{U}_{S:E}(t) = \int dX_0e^{-i\hat{H}_S t}|X_0\rangle\langle X_0| \otimes U_{\rm eff}(X(t;X_0)), \tag{7}$$

acting on the initial state $|\phi_0\rangle|\psi_0\rangle$. For simplicity, we will limit ourselves to trajectories obtained when the system is initially in the displaced squeezed vacuum state (for a general solution of a boson-boson model see [15]): $|\phi_0\rangle = \hat{D}(\alpha)\hat{S}(r)|0\rangle$, where $\hat{D}(\alpha)$ is the displacement operator and $\hat{S}(r) \equiv e^{r(\hat{a}^2-\hat{a}^{\dagger 2})/2}$. Especially interesting is a highly momentum squeezed state due to its large coherences in the position. Due to a position coupling to a central oscillator, we expect a strong decoherence in the position basis. We may then assume that the initial velocity of each trajectory is zero and initial positions are distributed according to the corresponding squeezed vacuum wave function, so that

$$X(t;X_0) = X_0\cos(\Omega t). \tag{8}$$

The analysis of the high initial position squeezing, for which we may assume $X(t=0) = 0$ and take $X(t;X_0) = X_0\sin(\Omega t)$, is analogous.

Assuming a fully product initial state

$$\rho_{S:E}(0) = \rho_S(0) \otimes \bigotimes_i \rho_E^{(i)}(0), \tag{9}$$

which is motivated here by the fact that we wish to study buildup of the system-environment correlations, the full solution is easily obtained from (7):

$$\begin{aligned}\rho_{S:E}(t) = &\int dX_0dX_0'\rho(X_0,X_0')e^{-iH_S t}|X_0\rangle\langle X_0'|e^{iH_S t}\\ &\bigotimes_{i=1}^N U_i(X_0,t)\rho_E^{(i)}(0)U_i^\dagger(X_0',t),\end{aligned} \tag{10}$$

where

$$\rho(X_0, X_0') \equiv \langle X_0|\rho_S(0)|X_0'\rangle \tag{11}$$

are the initial coherences and the conditional evolutions of the ith spin, $U_i(X_0, t)$, are generated by

$$H_i = -\frac{\Delta_i}{2}\sigma_x^{(i)} + g_i X_0 \cos(\Omega t)\sigma_z^{(i)}. \tag{12}$$

This allows us to find the effective evolution of spin states. The above Hamiltonian is well known and describes, e.g., an interaction of a linearly polarized light with a two-level atom [19]. The periodicity in time allows us to use the standard methods of the Floquet theory (see [23] for the historical work and, e.g., [24,25] for modern expositions) to find approximate solutions.

The Floquet theorem states that a unitary evolution for a periodic Hamiltonian can be written as a product of a unitary evolution driven by a periodic time-dependent Hamiltonian $K(t)$ with the same period of the Hamiltonian and a unitary evolution by a time-independent Hamiltonian H_F:

$$U(t, t_0) = e^{-iK(t)}e^{-i(t-t_0)H_F}e^{iK(t_0)}. \tag{13}$$

H_F is responsible for a slow dynamics forming an overall profile while $K(t)$ for a fast dynamics forming internal profile with the same oscillator periodicity $K(t) = K(t+T)$ as the given periodic Hamiltonian (14). The operators $K(t)$ and H_F can be perturbatively identified by using the high-frequency expansion [24,25]. One Taylor expands H_F in $1/\Omega \ll 1$, while $K(t)$ is the remaining part after H_F has been taken out. In general, convergence of the expansion is not always guaranteed [24,25]. By imposing conditions on the rest of parameters with Ω, the convergence of the series can be controlled, as we remark below. Fourier expanding the Hamiltonian (14),

$$H_i = H_0 + \sum_{j=1}^{\infty}(V^{(j)}e^{ij\Omega t} + V^{(-j)}e^{-ij\Omega t}), \tag{14}$$

where $H_0 = -(\Delta/2)\sigma_x$ and $V^{(1)} = V^{(-1)} = (gX_0/2)\sigma_z$, the rest being zero, one finds the high-frequency expansion of H_F and $K(t)$ (we omit the environment index i for simplicity):

$$H_F = H_0 + \frac{1}{\Omega}\sum_{j=1}^{\infty}\frac{1}{j}[V^{(j)}, V^{(-j)}] + \frac{1}{2\Omega^2}\sum_{j=1}^{\infty}\frac{1}{j^2}([[V^{(j)}, H_0], V^{(-j)}] + \text{H.c.}) + \cdots \tag{15}$$

and

$$K(t) = \frac{1}{i\Omega}\sum_{j}^{\infty}\frac{1}{j}(V^{(j)}e^{ij\Omega t} - V^{(-j)}e^{-ij\Omega t}) + \frac{1}{i\Omega^2}\sum_{j}^{\infty}\frac{1}{j^2}([V^{(j)}, H_0]e^{ij\Omega t} - \text{H.c.}) + \cdots . \tag{16}$$

For the purpose of this work, we take the lowest-order terms only and $t_0 = 0$. This gives

$$tH_F = -\tilde{\Delta}(1-\xi^2)\tau\sigma_x, \tag{17}$$

$$K(t) = \xi\sigma_z \sin\tau, \tag{18}$$

where we introduced dimensionless position ξ, tunneling energy $\tilde{\Delta}$, and time τ:

$$\xi \equiv \frac{gX_0}{\Omega},\ \tilde{\Delta} \equiv \frac{\Delta}{2\Omega},\ \tau \equiv \Omega t. \tag{19}$$

Picking the initial time $t_0 = 0$, there is no initial kick $K(0) = 0$. The convergence of the expansion is guaranteed for $\tilde{\Delta}, \xi \ll 1$. Using (17) and (18), the unitary evolutions defined in (13) are easily found:

$$e^{-i(t-t_0)H_F} = \cos[\tilde{\Delta}(1-\xi^2)\tau] + i\sigma_x \sin[\tilde{\Delta}(1-\xi^2)\tau],$$
$$e^{-iK(t)} = \cos[\xi\ \sin\tau] - i\sigma_z \sin[\xi\ \sin\tau]. \tag{20}$$

The first is the slow motion part of the dynamics, while the second is the fast motion part (the micromotion), with the time-periodic frequency proportional to $\sin\tau$. They lead to the following effective evolution, modulo $O(\Omega^{-2})$ terms:

$$U_k(X_0, t) = \left[\cos(\xi_k \sin\tau) - i\sigma_z^{(k)}\sin(\xi_k \sin\tau)\right] \times \left[\cos\left[\tilde{\Delta}_k(1-\xi_k^2)\tau\right] + i\sigma_x^{(k)}\sin\left[\tilde{\Delta}_k(1-\xi_k^2)\tau\right]\right]. \tag{21}$$

III. OBJECTIVE QUANTUM STATES

We now investigate possibilities of a formation of the SBS-like state (1). The form of the evolution (7) dictates that the corresponding pointer state eigenvalues are the initial oscillator position X_0, equivalently its amplitude, that controls the evolution of the environment and hence leaks into it. Following the general quantum Darwinism setup, we assume that part of the environment, called oE, is under observation while the rest, called uE, is unobserved. We are thus interested in the partial trace of (10) over uE:

$$\rho_{S:oE}(t) = \text{Tr}_{uE}\,\rho_{S:E}(t) = \int dX_0 dX_0' \rho(X_0, X_0')\Gamma_{X_0,X_0'}e^{-iH_S t}|X_0\rangle\langle X_0'|e^{iH_S t} \bigotimes_{i\in oE} U_i(X_0, t)\rho_E^{(i)}(0)U_i^{\dagger}(X_0', t), \tag{22}$$

where

$$\Gamma_{X_0,X_0'} \equiv \prod_{k\in uE}\text{Tr}\left[U_k(X_0, t)\rho_E^{(k)}(0)U_k^{\dagger}(X_0', t)\right] = \prod_k \Gamma^{(k)}_{X_0,X_0'} \tag{23}$$

is the decoherence factor, associated with the unobserved part of the environment uE. We note that since the decoherence factor is a function, which magnitude is always less than one, it can never be $\sim\delta(X_0 - X_0')$, and hence a full decoherence and a strict $S : E$ disentanglement cannot happen, though they take place for discrete variables, but they happen rather in an existence of some decoherence length, below which coherences are preserved [15,26].

To identify under such conditions the candidates for the information-encoding states $\rho_i^{E_k}$ from (1), we recall that in the Darwinism setup, the observers monitor the system only indirectly, via portions of the environment. Since in realistic conditions, a single environment will in general carry a vanishingly small information about the system, we assume that each observer has an access to a collection of environments, called macrofraction [12], scaling with the total number of environments N. The state of a macrofraction is obtained from (22) by tracing out everything except for the given macfrofraction:

$$\rho_{\text{mac}}(t) = \text{Tr}_{SoE\setminus\text{mac}}\rho_{S:oE}(t) = \int dX_0 p(X_0)\rho_{\text{mac}}(X_0), \quad (24)$$

where $p(X_0) \equiv \langle X_0|\rho_S(0)|X_0\rangle$ is the probability distribution of the initial position and

$$\rho_{\text{mac}}(X_0) \equiv \bigotimes_{k\in\text{mac}} U_k(X_0,t)\rho_E^{(k)}(0)U_k^\dagger(X_0,t) \quad (25)$$

is the conditional state corresponding to X_0. Thus, to know X_0, each observer must be able to distinguish the states (25) for different X_0. There are different scenarios to study state distinguishability [27], e.g., the quantum Chernoff bound used already in some studies of quantum Darwinism [28,29] or, more appropriate here due to the continuous parameter, quantum metrology [30]. Here, like in the the previous SBS studies, we will follow a simple form of the latter and will study the generalized overlap (state fidelity), which is the integral of the quantum Fisher information [31]:

$$B(\rho,\rho') \equiv [\text{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}]^2. \quad (26)$$

Note $B(\rho,\rho')$ here is defined by squaring, which is different from the usual definition. We note that $B(\rho,\rho')$ vanishes if and ony if the states have orthogonal supports, $\rho\rho'=0$, providing a measure of distinguishability. Although it produces less tight bounds on the probability of discrimination error than the quantum Chernoff bound [28,32], it has the property of factorizing with respect to the tensor product, making its calculation easier and similar to the decoherence factor:

$$B^{\text{mac}}_{X_0,X_0'} \equiv B[\rho_{\text{mac}}(X_0),\rho_{\text{mac}}(X_0')] = \prod_{k\in\text{mac}} B^{(k)}_{X_0,X_0'}. \quad (27)$$

From the metrological point of view, encoding X_0 into a fully product state (25) leads to a rather uninteresting classical scenario [30], but encoding efficiency is not our goal here and we postpone a study of more advanced scenarios with entangled macrofraction states to a future research. Just like with the decoherence, we expect that distinguishability will be achieved only at some characteristic length scale.

Summarizing, the approach to the SBS structure will be characterized by two quantities [16], the decoherence factor (23) and the generalized overlap (27). We will call them “objectivity markers.” We note that their factorized character, i.e., a total measure is a product of measures for individual environmental systems, which is due to the uncorrelated initial state. As one expects, a single factor corresponding to a single environmental spin will be oscillatory and of course will not lead anywhere close to the SBS structure. However, due to the factorized character, we expect that for a sufficiently large group of spins a considerable dephasing will take place at some length scales, leading to an approximate SBS structure.

IV. CALCULATION OF OBJECTIVITY MARKERS

We first focus on a single term in each expression (23) and (27), dropping the environment indices for brevity. State fidelity has a particularly simple form for spin-$\frac{1}{2}$ states. Let $M \equiv \sqrt{\rho}\rho'\sqrt{\rho}$, then

$$B(\rho,\rho') = \text{Tr}M + 2\sqrt{\det M}. \quad (28)$$

Since in our case ρ and ρ' have the same initial state ρ_0 but different evolutions U and U', we obtain

$$M = U\sqrt{\rho_0}U^\dagger U'\rho_0 U'^\dagger U\sqrt{\rho_0}U^\dagger$$

and, finally,

$$B_{X_0,X_0'} = \text{Tr}\big[U^\dagger_{X_0,X_0'}\rho_0 U_{X_0,X_0'}\rho_0\big] + 2\det\rho_0,$$

where we defined a relative evolution operator:

$$U_{X_0,X_0'} \equiv U^\dagger(X_0',t)U(X_0,t). \quad (29)$$

We note that the decoherence factor also depends on it.

Before calculating the markers for the evolution (21), we first derive general expressions. We will find it convenient to use the Bloch representation, decomposing any operator into identity and Pauli matrices. This representation will give a nice geometrical interpretation for the decoherence factor and the generalized overlap and their complementarity relation. Let us decompose the initial state and the relative evolution in the Pauli basis:

$$\rho_0 = \tfrac{1}{2}(1+\vec{a}\cdot\vec{\sigma}), \quad (30)$$

where $|\vec{a}| \leqslant 1$ and

$$U_{X_0,X_0'} = u_0 + i\vec{u}\cdot\vec{\sigma}, \quad (31)$$

where

$$u_0^2 + |\vec{u}|^2 = 1. \quad (32)$$

Then it is easy to obtain the decoherence factor

$$\Gamma_{X_0,X_0'} = u_0 + i\vec{a}\cdot\vec{u} \quad (33)$$

and its modulus

$$|\Gamma_{X_0,X_0'}|^2 = u_0^2 + (\vec{a}\cdot\vec{u})^2, \quad (34)$$

which controls the decoherence process in the position basis. Calculation of the generalized overlap, in turn, leads to (see Appendix A for the details)

$$B_{X_0,X_0'} = 1 - |\vec{a}\times\vec{u}|^2. \quad (35)$$

Combining Eqs. (34) and (35) the relation between $|\Gamma_{X_0,X_0'}|^2$ and $B_{X_0,X_0'}$ is expressed by

$$B_{X_0,X_0'} - |\Gamma_{X_0,X_0'}|^2 = (1-|\vec{a}|^2)(1-u_0^2). \quad (36)$$

This relation can be interpreted as a sort of complementarity between decoherence factor and distinguishability.

Using (34) and (36), we can express the decoherence factor and the generalized overlap in the high-frequency expansion of Sec. II. As seen, the decoherence factor and the generalized overlap are a function of only a vector $\vec{a}$ representing an initial density matrix for a spin ρ_0 and a relative unitary operator

$U_{X_0,X_0'}$ defined in Eq. (29). Using (21), we obtain in the first order of the high-frequency expansion that

$$U_{X_0,X_0'} = U_F^\dagger(\tau;\xi')U_K(\tau;\xi-\xi')U_F(\tau;\xi) = u_0 + i\vec{u}\cdot\vec{\sigma}, \tag{37}$$

where

$$\begin{aligned} u_0 &= \cos[\tilde{\Delta}(\xi^2-\xi'^2)\tau]\cos[\delta\xi\ \sin\tau],\\ u_1 &= -\sin[\tilde{\Delta}(\xi^2-\xi'^2)\tau]\cos[\delta\xi\ \sin\tau],\\ u_2 &= \sin[\tilde{\Delta}(2-\xi^2-\xi'^2)\tau]\sin[\delta\xi\ \sin\tau],\\ u_3 &= -\cos[\tilde{\Delta}(2-\xi^2-\xi'^2)\tau]\sin[\delta\xi\ \sin\tau], \end{aligned} \tag{38}$$

with the notation $\delta\xi \equiv \xi - \xi'$ and the dimensionless parameters defined in (19). We recall that for the trajectories (8) there is no initial kick at $t_0 = 0$ [cf. (18)]. Each u_i in (38) is a product of a fast and a slow moving part, as one would expect from the Floquet theory. The frequency of the slow motion is proportional to the tunneling energy $\tilde{\Delta}$, while the fast-moving terms are independent of it and have the time-dependent frequency $\delta\xi\ \sin\tau$. There is also a distinction between (u_0, u_1) and (u_2, u_3). The (u_0, u_1) pair has large overall sinusoidal patterns with small internal vibrations while in (u_2, u_3) the overall profiles are comparable to internal vibrations. As we will see, the fast-moving parts are unwanted for objectivity.

To proceed further, we will assume that the environment is initially in the thermal state, i.e.,

$$\rho_E^{(k)}(0) = \frac{e^{-\beta H_E^{(k)}}}{\mathrm{Tr}[e^{-\beta H_E^{(k)}}]} = \frac{1}{2}\left[1 + \sigma_x \tanh\left(\frac{\beta\Delta_k}{2}\right)\right], \tag{39}$$

where $\beta = 1/k_B T$, so that the parameters of the initial state are given by $\vec{a} = (E(\beta), 0, 0)$, where we introduced

$$E(\beta) \equiv \tanh\left(\frac{\beta\Delta}{2}\right) = \frac{\langle E\rangle}{\Delta/2}, \tag{40}$$

denoting the average thermal energy, rescaled by the tunneling energy. We obtain the following single-factor expressions for the decoherence and generalized overlap:

$$\left|\Gamma^1_{X_0,X_0'}\right|^2 = u_0^2 + E(\beta)^2 u_1^2, \tag{41}$$

$$B^1_{X_0,X_0'} = 1 - E(\beta)^2 + E(\beta)^2\left(u_0^2 + u_1^2\right), \tag{42}$$

which leads to

$$\left|\Gamma^1_{X_0,X_0'}\right|^2 = \left[1 - \frac{\sin^2[\tilde{\Delta}(\xi^2-\xi'^2)\tau]}{\cosh^2(\beta\Delta/2)}\right]\cos^2[\delta\xi\ \sin\tau], \tag{43}$$

$$B^1_{X_0,X_0'} = 1 - E(\beta)^2\sin^2[\delta\xi\ \sin\tau]. \tag{44}$$

The time dependence of the decoherence factor is given by the slow motion and the fast motion modulating each other. In contrast, the generalized overlap depends only on the fast oscillating part. We note that the decoherence factor depends here on the temperature, contrasting the result in a boson-spin system mapped from that of the quantum Brownian motion in the Born-Markov approximation [4]. In particular, at zero temperature the slow-motion part vanishes. Sample plots of the markers as the functions of the rescaled time τ are presented in Figs. 1 and 2.

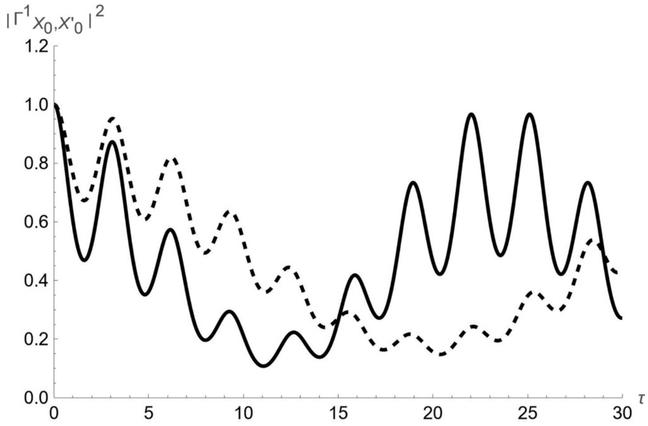

FIG. 1. Time dependence of decoherence factors for a single-spin environment (43). The solid and dashed lines stand for $\xi = 0.9$ and 0.6, respectively, while $\xi' = 0.1$ is fixed. The rest of the parameters are $\tilde{\Delta} = \frac{1}{6}$, $\beta\Delta = 1$. Both slow and fast oscillations are clearly visible.

The full decoherence and overlap functions are products of the above factors [cf. (23) and (27)]. We begin their analysis by first assuming small dimensionless amplitudes of the central oscillator $\xi, \xi' \ll 1$, which allows to expand the trigonometric functions. In particular, $\sin^2[\tilde{\Delta}(\xi^2-\xi'^2)\tau] \approx (\xi^2-\xi'^2)^2 t^2\Delta^2/4 + O(\xi^8)$, which is valid for times

$$t \ll \frac{2\Omega^2}{g^2\Delta\left|X_0^2 - X_0'^2\right|}. \tag{45}$$

Similarly, we expand the fast-motion factors containing $[(\xi - \xi')\sin\tau]$. Keeping the terms at most quadratic in ξ, we obtain

$$\left|\Gamma^1_{X_0,X_0'}\right|^2 = 1 - \delta\xi^2\sin^2\tau + O(\xi^4) \approx \exp\left[-\frac{g^2\delta X_0^2}{\Omega^2}\sin^2\Omega t\right], \tag{46}$$

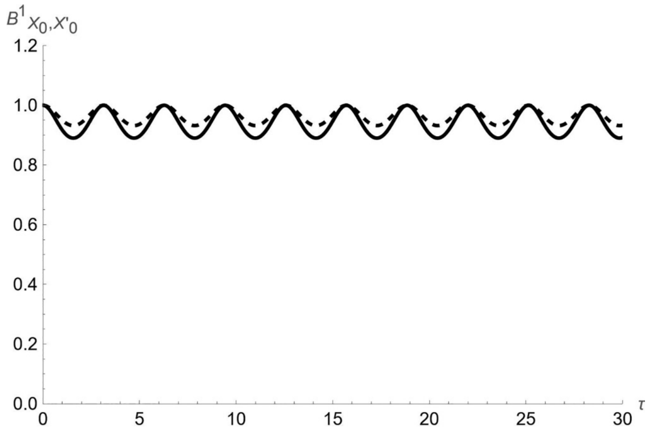

FIG. 2. Time dependence of generalized overlaps for a single-spin environment (44). The solid and dashed lines stand for $\xi = 0.9$ and 0.6, respectively, while $\xi' = 0.1$ is fixed. The rest of the parameters are the same as in Fig. 1 for a better comparison: $\tilde{\Delta} = \frac{1}{6}$, $\beta\Delta = 1$.

where we came back to the original variable using (19) and defined $\delta X_0 \equiv X_0 - X_0'$. The temperature dependence, the whole slow oscillating part, disappears in the lowest order and appears only in the ξ^4 terms and higher. The behavior of the overlap is different in this respect and depends strongly on the temperature even in the lowest order in ξ:

$$B^1_{X_0,X_0'} = 1 - \delta\xi^2 E(\beta)^2 \sin^2\tau + O(\xi^4)$$
$$\approx \exp\left[-\frac{g^2\delta X_0^2}{\Omega^2}E(\beta)^2\sin^2\Omega t\right]. \quad (47)$$

To calculate the products (23) and (27), we will assume that the constants g_k and Δ_k are identically and independently distributed according to some probability distributions and there are sufficiently many terms in the resulting sums in the exponentials to apply the law of large numbers:

$$\left|\Gamma_{X_0,X_0'}\right|^2 = \prod_{k\in uE}\left|\Gamma^{(k)}_{X_0,X_0'}\right|^2 = \exp\left[-\frac{\delta X_0^2}{\Omega^2}\sum_{k\in uE} g_k^2\sin^2\Omega t\right] \quad (48)$$

$$\approx \exp\left[-N_u\frac{\langle g^2\rangle\delta X_0^2}{\Omega^2}\sin^2\Omega t\right], \quad (49)$$

where N_u, is the size of the unobserved fraction uE of the environment and $\langle g^2\rangle$ is the average of g_k over the uE. This procedure [16] can be viewed as a form of an introduction of spectral density, but we keep control of the number of spins in the fractions. Similarly for the generalized overlap,

$$B_{X_0,X_0'} = \prod_{k\in\mathrm{mac}} B^{(k)}_{X_0,X_0'}$$
$$\approx \exp\left[-N_{\mathrm{mac}}\frac{\langle g^2\rangle\delta X_0^2}{\Omega^2}\langle E(\beta)^2\rangle\sin^2\Omega t\right], \quad (50)$$

where N_{mac} is the observed macrofraction size and we assumed that the distributions of g's and Δ's are independent; $\langle E(\beta)^2\rangle$ is understood as the average over the Δ [cf. (40)]. Due to the periodicity of the markers (49) and (50), it is immediately obvious that in the small displacement limit, there are complete recoherences at the turning points $t_n = n\pi/\Omega$ and there is no asymptotic behavior as $t\to\infty$. We can thus speak of the approach to the objective state only in the time intervals between the turning points. As anticipated, this approach is governed by two length scales, controlling the decoherence and the distinguishability processes:

$$\lambda_{\mathrm{dec}} \equiv \frac{\Omega}{\sqrt{N_u\langle g^2\rangle}}, \quad (51)$$

$$\lambda_{\mathrm{dist}} \equiv \frac{\Omega}{\sqrt{N_{\mathrm{mac}}\langle g^2\rangle\langle E(\beta)^2\rangle}}. \quad (52)$$

Their dependencies on the fraction sizes, or equivalently on the Hilbert-space dimensionalities of the fractions, mean that shorter distances are resolved as more spins are taken into account. We call the above length scales the decoherence and the distinguishability length scales. The distinguishability length scale is temperature dependent and grows with the temperature approximately linearly for high temperatures. This is intuitively clear as hotter environment is closer to the totally mixed state and thus its information-carrying capabilities are worse. Moreover, for nonzero temperatures, $\lambda_{\mathrm{dist}} > \lambda_{\mathrm{dec}}$ for the same fraction sizes, meaning that one can extract the position X_0 from the environment with a worse resolution than one at which the environment decoherences the central system. This phenomenon of "bound information" in the environment was observed in the QBM model in [15].

Let us now analyze the objectivity markers beyond the small-amplitude approximation. We first look at the generalized overlap:

$$B_{X_0,X_0'} = \prod_{k\in\mathrm{mac}}[1 - E_k(\beta)^2\sin^2(\delta\xi_k\sin\tau)]. \quad (53)$$

It is immediately clear that the time-periodic frequency $\sin\tau$ dictates the periodic character of $B_{X_0,X_0'}$:

$$B_{X_0,X_0'}(t) = B_{X_0,X_0'}(t+\pi/\Omega), \quad (54)$$

irrespectively of what are the distributions of g_k and Δ_k. In particular, this periodicity implies a complete loss of distinguishability, $B_{X_0,X_0'}(t_n) = 1$, at the turning points $t_n = n\pi/\Omega$, just like in the approximate analysis above. The behavior of $B_{X_0,X_0'}(t)$ can be approximated using the law of large numbers in the following way:

$$\log B_{X_0,X_0'} = \sum_{k\in\mathrm{mac}}\log B^{(k)}_{X_0,X_0'} \approx N_{\mathrm{mac}}\left\langle\log B^1_{X_0,X_0'}\right\rangle$$
$$\leqslant N_{\mathrm{mac}}\log\langle B^1_{X_0,X_0'}\rangle, \quad (55)$$

where we used the concavity of the logarithm. Furthermore,

$$\left\langle B^1_{X_0,X_0'}\right\rangle = 1 - \langle E(\beta)^2\rangle\langle\sin^2(\delta\xi\ \sin\tau)\rangle.$$

We are interested in the last average as it determines the time dependence. For simplicity we will assume a uniform distribution of g over some interval $[0,\bar{g}]$. This corresponds to a spectral density with a sharp cutoff at $\bar{g}$. Elementary integration gives

$$\langle\sin^2(\delta\xi\ \sin\tau)\rangle = \frac{1}{\bar{g}}\int_0^{\bar{g}} dg\,\sin^2\left[\frac{g\delta X_0}{\Omega}\sin\tau\right] \quad (56)$$
$$= \frac{1}{2}\left[1 - \mathrm{sinc}\left(\frac{2\bar{g}\delta X_0}{\Omega}\sin\tau\right)\right], \quad (57)$$

where $\mathrm{sinc}x \equiv \sin x/x$, which leads to

$$B_{X_0,X_0'} \approx \left\{1 - \frac{1}{2}\langle E(\beta)^2\rangle\left[1 - \mathrm{sinc}\left(\frac{2\bar{g}\delta X_0}{\Omega}\sin\tau\right)\right]\right\}^{N_{\mathrm{mac}}}. \quad (58)$$

Since $1 - \mathrm{sinc}x \approx x^2/6 + O(x^4)$ rises(decays) around the turning points $t_n = n\pi/\Omega$ are given by the small-amplitude approximation (50), with λ_{dist} rescaled by an unimportant factor $\sqrt{3}$. A sample plot of (58) is presented in Fig. 3 for different values of $\delta\bar{\xi} = \bar{g}\delta X_0/\Omega$. The decoherence factor can be analyzed in the same steps (55)–(58) with the complication that it is composed of both slow- and fast-moving parts. We need the average of (43). For simplicity we will assume

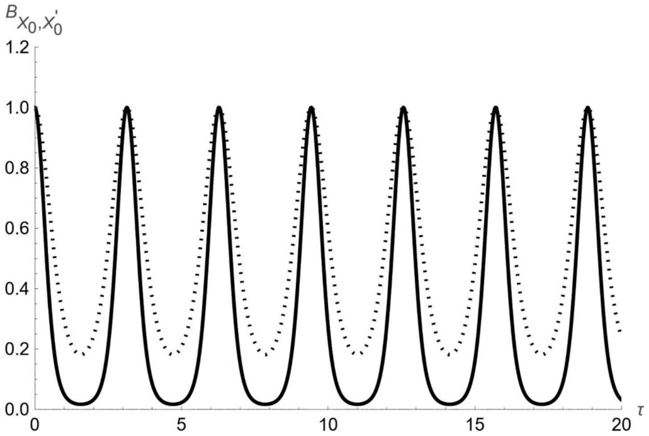

FIG. 3. Generalized overlap for $N_{\text{mac}} = 100$ spin environments with different interaction couplings ξ (the solid and dashed lines stand for $\bar{\xi} = 0.9$ and 0.6, respectively). ($\tilde{\Delta} = \frac{1}{6}$, $\Omega = 3$, $\beta\Delta = 1$, $\bar{\xi}' = 0.1$) are chosen.

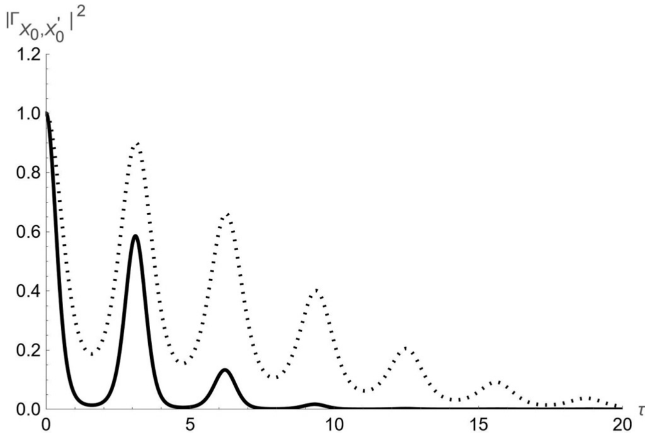

FIG. 4. Decoherence factors for $N_u = 20$ spin environments with different interaction couplings ξ (the solid and dashed lines stand for $\xi = 0.9$ and 0.6, respectively). ($\tilde{\Delta} = \frac{1}{6}$, $\Omega = 3$, $\beta\Delta = 1$, $\xi' = 0.1$ are chosen.)

$\Delta_k = \Delta$ so the only randomness is in g_k:

$$\left\langle \left|\Gamma_{X_0,X_0'}\right|^2 \right\rangle = \langle \cos^2(\delta\xi \ \sin\tau)\rangle - \frac{\langle \sin^2[\tilde{\Delta}(\xi^2 - \xi'^2)\tau] \cos^2(\delta\xi \ \sin\tau)\rangle}{\cosh^2(\beta\Delta/2)}$$

$$= \frac{\cosh(\beta\Delta)}{\cosh(\beta\Delta)+1}\langle \cos^2(\delta\xi \ \sin\tau)\rangle \tag{59}$$

$$+ \frac{1}{2\cosh(\beta\Delta)+2}\{\langle \cos[2\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\rangle \tag{60}$$

$$+ \langle \cos[2\tilde{\Delta}(\xi^2 - \xi'^2)\tau] \cos(2\delta\xi \ \sin\tau)\rangle\} \tag{61}$$

$$\equiv \Gamma_{\text{fast}}(\tau) + \Gamma_{\text{slow}}(\tau), \tag{62}$$

where we used trigonometric and hyperbolic identities to simplify the expressions. The first term (59) is the fast oscillating part which is equal to

$$\Gamma_{\text{fast}}(\tau) = \frac{\cosh(\beta\Delta)}{2\cosh(\beta\Delta)+2}\left[1 + \text{sinc}\left(\frac{2\bar{g}\delta X_0}{\Omega}\sin\tau\right)\right], \tag{63}$$

where we used above the same averaging as in (56) and (57). It is clearly time periodic due to the periodic frequency $\sin\tau$, just like (58), but it is multiplied by a temperature-dependent factor that is always smaller than 1. The behavior around the turning points $t_n = n\pi/\Omega$ is again given by the Gaussian law (49) with the λ_{dec} rescaled by $\sqrt{3}$. The terms (60) and (61) are the slow oscillating parts, contributing for nonzero temperatures. They can be calculated explicitly for a uniform distribution by using Fresnel integrals as they contain g^2 under the cosine. For example, the term (60) is proportional to

$$f(\tau) \equiv \langle \cos[2\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\rangle = \frac{1}{\sqrt{2\tilde{\Delta}(\bar{\xi}^2 - \bar{\xi}'^2)\tau}} C(\sqrt{2\tilde{\Delta}(\bar{\xi}^2 - \bar{\xi}'^2)\tau}), \tag{64}$$

where $C(x) \equiv \int_0^x du \ \cos u^2$ is the cosine Fresnel integral and $\bar{\xi} \equiv \bar{g}X_0/\Omega$. The long-time behavior of this term is determined by an asymptotic expansion for large x, $C(x) \approx \sqrt{\pi/8} + \sin x^2/2x + O(x^{-3})$ [33], which gives $f(\tau) \sim 1/\sqrt{\tau}$ for long times. The last term, (61), is a bit more complicated but can be manipulated using basic trigonometric identities [refer to (B2)]. As a result, from (64) and (B2) it follows that for large times $\Gamma_{\text{slow}}(\tau) \sim 1/\sqrt{\tau}$. And finally:

$$|\Gamma_{X_0,X_0'}|^2 \approx [\Gamma_{\text{fast}}(\tau) + \Gamma_{\text{slow}}(\tau)]^{N_u} \tag{65}$$

$$= [\Gamma_{\text{fast}}(\tau)]^{N_u} + O\left(\frac{1}{\sqrt{\tau}}\right) \tag{66}$$

$$= \left[\frac{\cosh(\beta\Delta)}{\cosh(\beta\Delta)+1}\right]^{N_u}\left[\frac{1}{2} + \frac{1}{2}\text{sinc}\left(\frac{2\bar{g}\delta X_0}{\Omega}\sin\tau\right)\right]^{N_u} + O\left(\frac{1}{\sqrt{\tau}}\right). \tag{67}$$

Despite the presence of a time-periodic term, unlike the generalized overlap (58) this function can effectively decay with time, meaning decoherence can take place. This is due to the temperature-dependent prefactor in (67), which multiplies the sinc term and which decays with temperature and for high temperatures (small β) is of the order $\sim 2^{-N_u}$. Thus, although the sinc term is equal to one at the turning points $t_n = n\pi/\Omega$, its contribution is damped by the temperature-dependent term. A sample plot of the exact expression (65) using (63), (64), and (B2) is presented in Fig. 4.

We conclude that for cosine trajectories (8), although the central oscillator can be effectively decohered, the environment, however, is unable to reliably store the amplitude information for all times as there are periodic and complete losses of distinguishability at the turning points. Thus, objective states can form only between the turning points. We suspect these perfect revivals are caused by the recoil-less approximation, which completely neglects the damping of the central oscillator. They are also in contrast with the oscillator environment, where for the same trajectory (8) and under the same approximation (5) no such revivals were observed, but rather a steady decay [14]. The revivals, in turn, appeared in the QBM model for sine trajectories, corresponding to initial position squeezing, which was in agreement with the earlier works [34]. It is therefore interesting to study more general trajectories in the current model too.

A. Arbitrary trajectory

An arbitrary trajectory of the central oscillator is obtained by adding a constant phase ϕ to (8):

$$X(t) = X_0 \cos(\tau + \phi). \tag{68}$$

It is then interesting to investigate how this phase can affect the objectivity, especially whether there is a possibility to overcome the asymmetry between the decaying decoherence factor and the monotonously oscillating generalized overlap found above. The phase changes the Fourier components $V^{(1)}$ and $V^{(-1)}$ in high-frequency expansion in (14),

$$V^{(\pm 1)} = \frac{gX_0}{2}\sigma_z \to \frac{gX_0}{2}\sigma_z e^{\pm i\phi}. \tag{69}$$

Consequently, as seen in (15) and (16), $K(t)$ gets a phase change (18):

$$K(t) = \xi\sigma_z \sin(\tau + \phi), \tag{70}$$

while the Floquet Hamiltonian H_F remains the same as in (17). In (13) $\phi \neq 0$ contributes to $U_{X_0,X_0'}$ due to the nontrivial initial kicks $K(0)$. Explicitly, from (13) and (37), we obtain $U_{X_0,X_0'} = u_0 + i\vec{u}\cdot\vec{\sigma}$ with [cf. (38)]

$$\begin{aligned}
u_0 &= \cos[\delta\xi\ \sin\phi]\cos[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\cos[\delta\xi\ \sin(\tau+\phi)] + \sin[\delta\xi\ \sin\phi]\cos[\tilde{\Delta}(\xi^2 + \xi'^2 - 2)\tau]\sin[\delta\xi\ \sin(\tau+\phi)],\\
u_1 &= -\cos[(\xi+\xi')\sin\phi]\sin[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\cos[\delta\xi\ \sin(\tau+\phi)] + \sin[(\xi+\xi')\sin\phi]\\
&\quad\times \sin[\tilde{\Delta}(\xi^2 + \xi'^2 - 2)\tau]\sin[\delta\xi\ \sin(\tau+\phi)],\\
u_2 &= -\sin[(\xi+\xi')\sin\phi]\sin[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\cos[\delta\xi\ \sin(\tau+\phi)] - \cos[(\xi+\xi')\sin\phi]\\
&\quad\times \sin[\tilde{\Delta}(\xi^2 + \xi'^2 - 2)\tau]\sin[\delta\xi\ \sin(\tau+\phi)],\\
u_3 &= \sin[\delta\xi\ \sin\phi]\cos[\tilde{\Delta}(\xi^2 - \xi'^2)\tau]\cos[\delta\xi\ \sin(\tau+\phi)] - \cos[\delta\xi\ \sin\phi]\cos[\tilde{\Delta}(\xi^2 + \xi'^2 - 2)\tau]\sin[\delta\xi\ \sin(\tau+\phi)].
\end{aligned} \tag{71}$$

The only relevant components for a decoherence factor (34) and a generalized overlap (35) associated with the thermal state (39), i.e., with $\vec{a} = (E(\beta), 0, 0)$, are u_0 and u_1, which follows from (32). We first analyze the small-ξ approximation as it is easier. The factors u_0^2 and u_1^2 then read as

$$\begin{aligned}
u_0^2 &= 1 - \delta\xi^2[\sin^2\phi + \sin^2(\tau+\phi) - 2\sin\phi\ \sin(\tau+\phi)\cos(2\tilde{\Delta}\tau)] + O[\xi^4],\\
u_1^2 &= O[\xi^4],
\end{aligned} \tag{72}$$

which from (34) and (35) lead to the following single-spin expressions:

$$\begin{aligned}
\left|\Gamma^1_{X_0,X_0'}\right|^2 &= 1 - \delta\xi^2[\sin^2\phi + \sin^2(\tau+\phi) - 2\sin\phi\ \sin(\tau+\phi)\cos(2\tilde{\Delta}\tau)] + O[\xi^4],\\
&= \exp[-\delta\xi^2|\sin\phi + e^{i2\tilde{\Delta}\tau}\sin(\tau+\phi)|^2] + O(\xi^4),\\
B^1_{X_0,X_0'} &= 1 - E(\beta)^2\delta\xi^2[\sin^2\phi + \sin^2(\tau+\phi) - 2\sin\phi\ \sin(\tau+\phi)\cos(2\tilde{\Delta}\tau)] + O[\xi^4]\\
&= \exp[-\delta\xi^2 E(\beta)^2|\sin\phi + e^{i2\tilde{\Delta}\tau}\sin(\tau+\phi)|^2] + O(\xi^4).
\end{aligned} \tag{73}$$

Figures 5 and 6 show sample plots of decoherence factors and fidelity, respectively. For multiple spins, the law of large numbers gain can be used implies that the exponents above are rescaled by the fraction sizes, similarly to (49) and (50):

$$|\Gamma_{X_0,X_0'}|^2 \approx \exp\left[-N_u\frac{\langle g^2\rangle\delta X_0^2}{\Omega^2}|\sin\phi + e^{i2\tilde{\Delta}\tau}\sin(\tau+\phi)|^2\right], \tag{74}$$

$$B_{X_0,X_0'} \approx \exp\left[-N_{\mathrm{mac}}E(\beta)^2\frac{\langle g^2\rangle\delta X_0^2}{\Omega^2}|\sin\phi + e^{i2\tilde{\Delta}\tau}\sin(\tau+\phi)|^2\right]. \tag{75}$$

We see that the decays are governed by the same length scales (51) and (52) and the functions are doubly periodic with the periods given by Ω^{-1} and Δ^{-1}. Apart from this, the behavior in the small-ξ approximation is essentially the same as for $\phi = 0$ case (49) and (50). We will see that this will change for a general ϕ. Some remarks are in order. For the purpose of this work, we are assuming Δ_k being the same for all spins. This avoids complicated averages of the type $\int d\Delta\ \tanh(\beta\Delta/2)\cos(2\Delta t)$ although obviously different Δ_k can introduce dephasing, helping to counter the periodicity. This possibility will be investigated elsewhere. Here, we concentrate on randomized coupling constants g_k.

We now estimate the objectvity markers for arbitrary parameters. According to our procedure [cf. (55)], we need the averages of the single-spin functions. The detailed calculations are presented in Appendix B. As before, we separate between the fast and slow oscillating parts:

$$\left\langle\left|\Gamma^1_{X_0,X_0'}\right|^2\right\rangle = \Gamma_{\mathrm{fast}}(\tau) + \Gamma_{\mathrm{slow}}(\tau), \tag{76}$$

where oscillating parts are

$$\Gamma_{\text{fast}}(\tau) \equiv \frac{1}{8}[2 + \text{sinc}\{\delta\bar{\xi}[\sin\phi + \sin(\tau+\phi)]\} + \text{sinc}\{\delta\bar{\xi}[\sin\phi - \sin(\tau+\phi)]\}] + \frac{E(\beta)^2}{8}[2 + \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi + \delta\bar{\xi}\sin(\tau+\phi)]\} + \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi - \delta\bar{\xi}\sin(\tau+\phi)]\}] \tag{77}$$

and decaying parts behaving asymptotically as $1/\sqrt{\tau}$ are

$$\Gamma_{\text{slow}}(\tau) \equiv \sum_{a,b,c} d^{\Gamma}_{abc} F^{\Gamma}[a,b,c] = O\left(\frac{1}{\sqrt{\tau}}\right) \tag{78}$$

with $F^{\Gamma}[a,b,c]$ defined in (B5) and d^{Γ}_{abc} can be identified in (B3) and (B4). Similarly,

$$\langle B_{X_0,X_0'}\rangle = B_{\text{fast}}(\tau) + B_{\text{slow}}(\tau), \tag{79}$$

where

$$B_{\text{fast}}(\tau) \equiv 1 - \frac{E(\beta)^2}{8}[4 - \text{sinc}\{\delta\bar{\xi}[\sin\phi + \sin(\tau+\phi)]\} - \text{sinc}\{\delta\bar{\xi}[\sin\phi - \sin(\tau+\phi)]\} - \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi + \delta\bar{\xi}\sin(\tau+\phi)]\} - \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi - \delta\bar{\xi}\sin(\tau+\phi)]\}] \tag{80}$$

and

$$B_{\text{slow}}(\tau) \equiv \sum_{a,b,c} d^{B}_{abc} F^{B}[a,b,c] = O\left(\frac{1}{\sqrt{\tau}}\right) \tag{81}$$

with $F^B[a,b,c]$ defined in (B5) and d^B_{abc} can be identified in (B3) and (B4). As $\tau \to \infty$, Γ_{slow} and B_{slow} die out as $1/\sqrt{\tau}$ and only Γ_{fast} and B_{fast} remain. Thus, as $\tau \to \infty$ a total decoherence factor and a total generalized overlap are given by the fast movers only:

$$|\Gamma_{X_0,X_0'}|^2 = [\Gamma_{\text{fast}}(\tau) + \Gamma_{\text{slow}}(\tau)]^{N_u} = [\Gamma_{\text{fast}}(\tau)]^{N_u} + O\left(\frac{1}{\sqrt{\tau}}\right), \tag{82}$$

$$B_{X_0,X_0'} = [B_{\text{fast}}(\tau) + B_{\text{slow}}(\tau)]^{N_{\text{mac}}} = [B_{\text{fast}}(\tau)]^{N_{\text{mac}}} + O\left(\frac{1}{\sqrt{\tau}}\right). \tag{83}$$

The trajectory with $\phi = 0$, i.e., $X = X_0 \cos\tau$ studied before, is rather particular in the structure of $U_{X_0,X_0'}$ [Eq. (38)] in that the slow-moving part with the frequency $\tilde{\Delta}(\xi^2 - \xi'^2)$ completely vanishes, leaving only the fast-moving part in $B_{X_0,X_0'}$. In this oscillating case, it is not possible to have a decay $B_{X_0,X_0'}$ regardless of the number of spins. However, as we see for $\phi \neq 0$ the situation is different as even a small ϕ leads to the dephasing of the sinc functions in (77) and (81), which in turn lead to a decay of both functions for many spin environments as we demonstrate below.

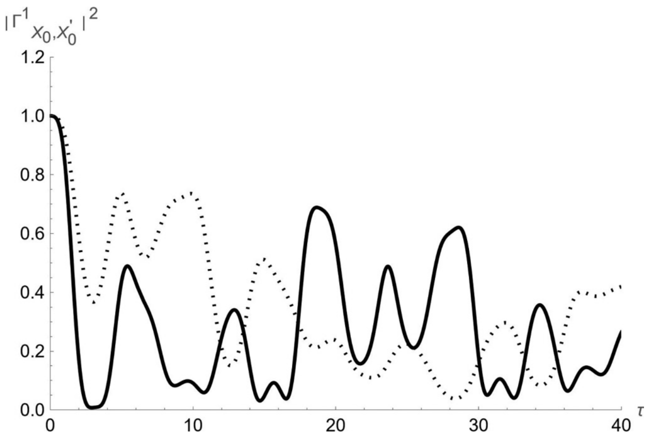

FIG. 5. Time dependence of decoherence factor for a single-spin environment with $\phi = \pm\pi/2$ [Eq. (43)]. The solid and dashed lines correspond to $\xi = 0.9$ and 0.6, respectively, while $\xi' = 0.1$ is fixed. The rest of the parameters are $\tilde{\Delta} = \frac{1}{6}$, $\beta\Delta = 1$. The separation between a slow and fast oscillation is less clear due to involvement of another frequency than for $\phi = 0$ in Fig. 1.

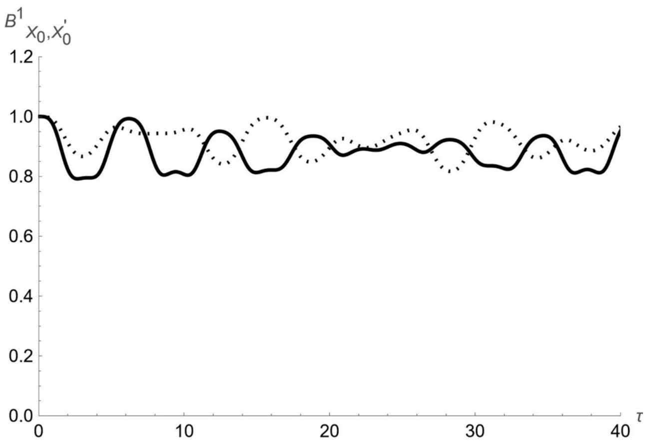

FIG. 6. Time dependence of the generalized overlap for a single-spin environment with $\phi = \pm\pi/2$ [Eq. (43)]. The solid and dashed lines correspond to $\xi = 0.9$ and 0.6, respectively, while $\xi' = 0.1$ is fixed. The rest of the parameters are $\tilde{\Delta} = \frac{1}{6}$, $\beta\Delta = 1$. The separation between a slow and fast oscillation is less clear due to involvement of another frequency than for $\phi = 0$ in Fig. 2.

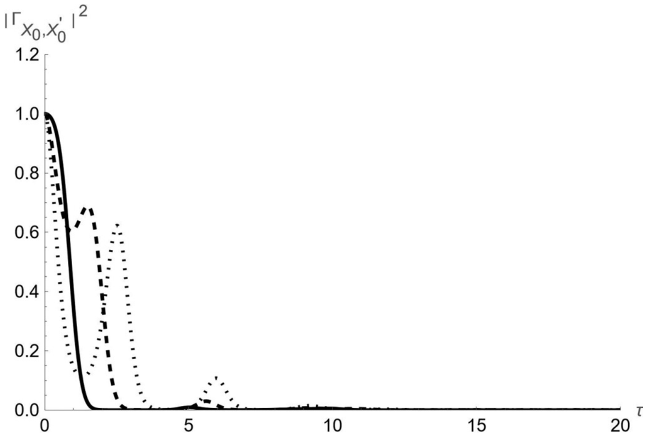

FIG. 7. Decoherence factor for large-spin environments as a function of time and different phases ϕ. The dotted, dashed, and thick lines correspond to $\phi = \pi/10, \pi/4, \pi/2$, respectively. ($\bar{\xi} = 0.9,\ \beta\Delta = 1,\ \bar{\xi}' = 0.1, N_u = 20$ are chosen.)

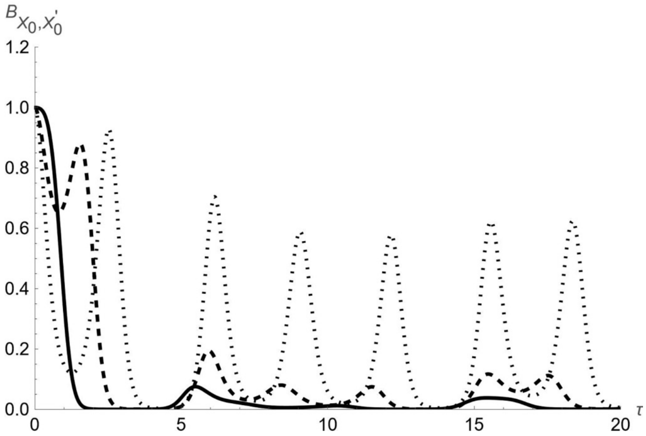

FIG. 8. Generalized overlap for large-spin environments as a function of time and different phases ϕ. The dotted, dashed, and thick lines correspond to $\phi = \pi/10, \pi/4, \pi/2$, respectively. ($\bar{\xi} = 0.9,\ \beta\Delta = 1,\ \bar{\xi}' = 0.1, N_{\rm mac} = 100$ are chosen.)

We show that $\max[\Gamma_{\rm fast}(\tau)] < 1, \max[B_{\rm fast}(\tau)] < 1$ for any $\tau > 0$, which means both functions are exponentially damped as multiple spins are considered [cf. (82) and (83)]. It will be convenient to introduce the following functions:

$$g_-(\tau) \equiv \text{sinc}\{\delta\bar{\xi}[\sin\phi + \sin(\tau+\phi)]\} + \text{sinc}\{\delta\bar{\xi}[\sin\phi - \sin(\tau+\phi)]\}, \tag{84}$$

$$g_+(\tau) \equiv \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi + \delta\xi\,\sin(\tau+\phi)]\} + \text{sinc}\{(\bar{\xi}+\bar{\xi}')\sin\phi - \delta\xi\,\sin(\tau+\phi)]\}. \tag{85}$$

Their extrema for $\xi, \xi' < 1$, which guarantees the high-frequency expansion and the positivity of sinc functions, are given by

$$\sin(\tau + \phi + \pi/2) = 0, \tag{86}$$

$$\sin(\tau + \phi) = 0. \tag{87}$$

Since at $\phi = 0$ the second condition indicates the maxima, $g_\pm(\tau)$ continues being shifted to the left by $\phi \neq 0$ as $\sin(\tau + \phi) = 0$ moves to the left. So it can be recognized that the first condition gives the minima while the other one the maxima, which read as, for $t > 0$,

$$\max(\Gamma_{\rm fast}) = \frac{1}{4}\{1 + \text{sinc}(\delta\bar{\xi}\sin\phi) + E(\beta)^2(1 + \text{sinc}[(\bar{\xi}+\bar{\xi}')\sin\phi])\} \tag{88}$$

$$= \frac{1+E(\beta)^2}{2} - \frac{\phi^2}{24}[\delta\xi^2 + E(\beta)^2(\bar{\xi}+\bar{\xi}')^2] + O(\phi^4) \tag{89}$$

and

$$\max(B_{\rm fast}) = 1 - \frac{E(\beta)^2}{4}\{2 - \text{sinc}(\delta\bar{\xi}\sin\phi) - \text{sinc}[(\bar{\xi}+\bar{\xi}')\sin\phi)]\} \tag{90}$$

$$= 1 - \frac{\phi^2}{12}E(\beta)^2(\bar{\xi}^2 + \bar{\xi}'^2) + O(\phi^4). \tag{91}$$

Note that these values depend on $(\bar{\xi}, \bar{\xi}')$ and $\beta\Delta$ but not directly on the tunneling energy $\tilde{\Delta}$, which is nevertheless necessary to damp the slow-moving parts as we have shown earlier. We see from the above expressions that

(1) $\max(B_{\rm fast}) < 1$ for any nonzero ϕ, provided $E(\beta) > 0$, i.e., the temperature is finite $\beta > 0$;

(2) $\max(\Gamma_{\rm fast}) < 1$ for any nonzero ϕ, provided $E(\beta) < 1$, i.e., the temperature is nonzero $\beta < \infty$.

This, in turn, implies via (82) and (83) that outside the temperature extremes, both markers $|\Gamma_{X_0,X_0'}|^2$ and $B_{X_0,X_0'}$ are asymptotically damped for $N_u, N_{\rm mac} \gg 1$ and the state (23) approaches the objective state. The amount of damping, and hence the quality of the objectivity in the state, depends on N_u, $N_{\rm mac}$ and the temperature. For small ϕ, we obtain from (82) and (83) the following bounds:

$$\left|\Gamma_{X_0,X_0'}\right|^2 \lessapprox \left[\frac{1+E(\beta)^2}{2} - \frac{\phi^2}{24}[\delta\xi^2 + E(\beta)^2(\bar{\xi}+\bar{\xi}')^2]\right]^{N_u}$$
$$\approx \left[\frac{1+E(\beta)^2}{2}\right]^{N_u} \times \exp\left[-N_u\phi^2\frac{\delta\xi^2 + E(\beta)^2(\bar{\xi}+\bar{\xi}')^2}{12[1+E(\beta)^2]}\right], \tag{92}$$

$$B_{X_0,X_0'} \lessapprox \left[1 - \frac{E(\beta)^2\phi^2}{12}(\bar{\xi}^2 + \bar{\xi}'^2)\right]^{N_{\rm mac}}$$
$$\approx \exp\left[-N_{\rm mac}\frac{E(\beta)^2\phi^2}{12}(\bar{\xi}^2 + \bar{\xi}'^2)\right]. \tag{93}$$

This situation is to be contrasted with the previous section, where we showed that the cosine trajectory (8) does not lead to the permanent damping of the generalized overlap for any amount of spins in the macrofraction and hence no permanent objective states can be formed. Sample plots of both markers, using the exact expressions calculated in Appendix B, are presented in Figs. 7 and 8. We see in particular that it is more difficult to damp the generalized overlap as it takes about 5× more spins than needed to induce the decoherence. This is to be expected as the spins are encoding the continuous variable: the oscillation amplitude X_0.

V. CONCLUDING REMARKS

We analyzed the emergence of objectivity in the last canonical models of decoherence, a boson-spin model in SBS state formation that so far has not been used in the rest. Due to the complicated dynamics, we used the recoil-less limit where the influence of the environment on the central oscillator is assumed to be negligible. This is an opposite limit to the usual Born-Markov one, but the most appropriate for studying information transfer to the environment during the decoherence process. The recoil-less limit can be viewed as a version of the Born-Oppenheimer approximation, where the central system evolves unperturbed and affects the environmental spins via coupling to its classical trajectory, which acts as an external time-dependent force. The resulting effective dynamics of the environment allows for the use of the Floquet theory. We perform the analysis in the first order of the high-frequency expansion and demonstrate and find, in particular, an interesting fact: fast-moving parts of the motion are detrimental to the emergence of objectivity while the slow-moving parts enable it. Another interesting aspect of the model, not present in other canonical models, is a mismatch between the encoded variable, which is a continuous positionlike variable (the oscillation amplitude), and the encoding system, which is finite dimensional (a collection of spins). In this respect, we show two facts. First, we derive two characteristic wavelengths: one corresponding to the decoherence scale on the side of the central system and another governing the resolution, with which collections of environmental spins encode the continuous variable. The lengths are different, in particular, the encoding one depends on the temperature and is larger than the decoherence one, which shows the phenomenon of "bound information" in the environment: the resolution of a possible read-out from the environment is lower than the scale on which coherences are destroyed. Both length scales depend on the fraction size, i.e., the bigger the size, the lower the length scales, which is quite intuitive. However, the presence of the length scales does not guarantee a stable formation of objectivity. We show that the latter depends on the type of motion of the central system. In particular, for initial states with a well-defined momentum, there can be only a momentary formation of objectivity, while even a small departure from this specific initial condition leads to an asymptotic formation of objective states. This is exactly opposite to what one finds in QBM, where initially well-defined momentum states lead to a stable appearance of objectivity, and one of the examples showing how spin and oscillator environments differ.

Our analysis can be applied for the quantum measurement theory in the following points. First, our result is an example of how a continuous variable can be measured by finite-dimensional systems in a realistic scenario of an open quantum dynamics. Second, from a broader perspective, it could be used to have an exemplary answer to the fundamental question of when the measurement is completed. One may postulate that it is completed when the system+measuring apparatus are close enough to the SBS state, which guarantees an objective character of the measurement result. This remark of course applies to the whole of the SBS and quantum Darwinism program.

ACKNOWLEDGMENT

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APPENDIX A: GENERALIZED OVERLAP IN BLOCH REPRESENTATION

We will get a geometrical expression for a generalized overlap $B^2(\rho, \rho')$ for a qubit when ρ and ρ' are unitarily related and its relation with a decoherence factor $|\Gamma_{X_0,X_0'}|^2$. $B(\rho, \rho')$ is defined in Eq. (26) as

$$B(\rho, \rho') = \mathrm{Tr} M + 2\sqrt{\det M}, \tag{A1}$$

where

$$M = \sqrt{\rho}\rho'\sqrt{\rho} \tag{A2}$$

with

$$\rho \equiv U\rho_0 U^\dagger,\ \rho' \equiv U'\rho_0 U'^\dagger.$$

Using the cyclic property of a trace and determinant, $B(\rho, \rho')$ in Eq. (A1) is rewritten as

$$B(\rho, \rho') = \mathrm{Tr}\tilde{M} + 2\sqrt{\det \tilde{M}}, \tag{A3}$$

where $W = U^\dagger U' = U^\dagger_{X_0,X_0'}$ and

$$\tilde{M} = W\rho_0 W^\dagger \rho_0. \tag{A4}$$

With the notations $W = u_0 - i\vec{u}\cdot\vec{\sigma}$ and $\rho_0 = (1 + \vec{a}\cdot\vec{\sigma})/2$, we express $W\rho_0$ as

$$W\rho_0 = \tfrac{1}{2}(v_0 + \vec{v}\cdot\vec{\sigma}), \tag{A5}$$

where

$$\begin{aligned} v_0 &\equiv u_0 - i\vec{a}\cdot\vec{u}, \\ v_i &\equiv u_0 a_i - iu_i + (\vec{u}\times\vec{a})_i. \end{aligned} \tag{A6}$$

With $\rho_0 W = (W^\dagger\rho_0)^\dagger$ and $W^\dagger(\vec{u}) = W(-\vec{u})$ we obtain $\rho_0 W$:

$$\rho_0 W = \tfrac{1}{2}(v_0 + \vec{\bar{v}}\cdot\vec{\sigma}), \tag{A7}$$

where

$$\bar{v}_i \equiv u_0 a_i - iu_i - (\vec{u}\times\vec{a})_i. \tag{A8}$$

Using Eqs. (A4), (A5), and (A7), we obtain

$$\tilde{M} = \tilde{m}_0 + \sum_i \tilde{m}_i\sigma_i, \tag{A9}$$

where

$$\begin{aligned} \tilde{m}_0 &\equiv \tfrac{1}{4}(|v_0|^2 + \vec{v}\cdot\vec{\bar{v}}^*), \\ \tilde{m}_i &\equiv \tfrac{1}{4}[v_i v_0^* + \bar{v}_i^* v_0 + i(\vec{v}\times\vec{\bar{v}}^*)_i], \end{aligned}$$

and hence $\mathrm{Tr}M$ is expressed as

$$\mathrm{Tr}M = \mathrm{Tr}\tilde{M} = \tfrac{1}{2}(|v_0|^2 + \vec{v}\cdot\vec{\bar{v}}^*). \tag{A10}$$

3.3 Holevo bound and objectivity in the boson-spin model

3.3.1 Summary

The purpose of this article is to formulate and study a different description of objectivization process from what we used before, by using the language of quantum channels. Both quantum Darwinism and SBS approaches rely on the information transfer from a central system to the environment. This information transfer can be described as a quantum channel [1]. In this article we use the Holevo quantity [36], which is the upper bound on the classical capacity of a quantum channel, to study information transfer in the boson-spin model.

We work in the simplest case that only a single spin-$\frac{1}{2}$ environment remains under the observation. What is novel in our analysis is the continuous character of the information and the use of the continuous version of the Holevo theorem [51]. The continuous information from a harmonic oscillator stored into a finite dimensional spin system is understood at some given resolution scale.

We confirm the following results which were found in the previous work using the generalized overlap. The zero phase in the trajectory of oscillator does not lead to an asymptotic but only a periodic behaviour in the Holevo quantity as the generalized overlap does not vanish for large time. The self-Hamiltonian plays a significant role in stabilizing the asymptotic patterns in the Holevo quantity, breaking oscillatory behaviours. The interaction coupling is found to increase the Holevo quantity.

In addition, we found how a squeezing parameter and a displacement parameter in the initial state of the oscillator affect the Holevo quantity. As the squeezing parameter increase, the initial distribution is more localized. This implies that a harmonic oscillator initially had the lower information, which leads to the lower Holevo quantity. On the other hand, it turns out that as a displacement parameter goes to zero, the Holevo quantity also becomes lower. Noticing that the effective quantum state of the boson-spin model does not have a translational symmetry, a displacement parameter not only control the position distribution of an oscillator but also the strength of interaction. For instance, while a vanishing displacement parameter makes the probability distribution the absence of the interaction dominant, a non-zero displacement not only makes the quantum state localized at the corresponding position but also still keeps the interaction strong accordingly.

3.3.2 Work contribution

My contribution to the preprint is following:

- Taking a part in discussions and formulations of the initial idea.
- Performing all the the analytical and numerical calculations and analysis.
- Generating all the plots.
- Writing a large part of the manuscript.

Holevo bound and objectivity in the boson-spin model

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(Dated: September 2, 2024)

Emergence of objective, classical properties in quantum systems can be described in the modern language of quantum information theory. In this work, we present an example of such an analysis. We apply the quantum channel theory to a boson-spin model of open quantum systems and calculate, using recoilless approximation and the Floquet theory, the Hoevo quantity, which bounds the capacity of the channel, broadcasting information about the central system into its environment. We analyze both the short-time regime, showing quadratic in time initial growth of the capacity, and the asymptotic regime. Complicated dependence on the model parameters, such as temperature, tunneling energy for the environment, etc is also analyzed, showing e.g. regimes where the Holevo bound reaches its maximum.

Keywords: decoherence, spectrum broadcast structure, quantum Darwinism, Holevo quantity

I. INTRODUCTION

Most modern studies of how the objectivity of the macroscopic world appears during the quantum-to-classical transition rely on the quantum Darwinism idea [1, 2]. It is a more advanced form of a decoherence process and states that information about the system of interest in order to become objective has to be broadcasted in multiple copies into the environment during the decoherence process. This rises a natural question: What is the capacity of such a broadcasting channel? This question has been studied in a series of works in spin-spin model, e.g. in a models where a central spin interacts with a spin environment [3–9]. In this work, we complement those studies by considering a boson-spin model, where the central system is a harmonic oscillator. We will use the results obtained in our earlier studies of the model [10], where we considered a stronger form of objectivity known as the spectrum broadcast structures (SBS) [11, 12]. These are specific multipartite state structures, which encode an operational form of objectivity and explicitly show objectification as a form of a broadcasting process. The novel point of our studies is that now the central system is infite-dimensional. Thus, we will investigate how well continuous-variable information is broadcast into the finite dimensional spin environments. Since the infinite amount of continuous information cannot be stored into any range of finite degrees of freedom, encoding continuous information into finite environment should be understood as recorded information in finite bits with some finite resolution length [10]. As a result, the mutual information between the system with infinite degrees of freedom and one with a finite degrees of freedom is understood as how many bits are used to encode continuous information with a given resolution.

More precisely, we will be interested in the capacity of channels, broadcasting system-related information into the environment during the open evolution and decoherence. Our main tool will be the well-known Holevo quantity $\chi(\rho)$, which bounds the mutual information $I(X:Y)$, accessible in the environment [13, 14]:

$$I(X:Y) \leq S\left(\sum_X p_X \rho_X\right) - \sum_X p_X S(\rho_X) \equiv \chi(\rho), \tag{1}$$

where $S(\cdot)$ is the von Neumann entropy, ρ_X are states encoding classical parameter X, distributed with some probability p_X. We will calculate the Holevo bound for a boson-spin model in a so-called recoilless limit, where a central oscillator is not much affected by the presence of the environment. This is the opposite limit to the commonly studied Born approximation, see e.g. [15], but is more suitable for objectivity studies where we are interested in how information flows from the system to the environment. We will analyze both the short time and the asymptotic behavior, identifying how quickly the system-to-environment channel capacity growth and at what level it is stabilized, depending on the model parameters.

II. DYNAMICS OF THE SYSTEM

We start describing the system and its approximate dynamics, based on [10]. The Hamiltonian H for the joint system of a harmonic oscillator of a mass M and an angular frequency Ω and interacting qubits with a couplings g_i and a self-energies Δ_i is given by

$$H = H_S + \sum_i H_E^{(i)} + \sum_i H_{\text{int}}^{(i)}, \tag{2}$$

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where

$$H_S = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\Omega^2\hat{X}^2, \quad (3)$$

$$H_E^{(i)} = -\frac{\Delta_i}{2}\sigma_x^{(i)}, \quad (4)$$

$$H_{\rm int}^{(i)} = g_i\hat{X}\otimes\sigma_z^{(i)}. \quad (5)$$

Despite the apparent simplicity, the above dynamics is complicated and for the purpose of this work we will use a series of approximations, which nevertheless are enough to demonstrate the main features of the dynamics. First of all, since we are considering information transfer into environment, it is suitable to assume that the state of environment is allowed to change while the dynamics of the harmonic oscillator is not significantly influenced by interaction with the environment. This is called recoilless approximation and is the opposite to the broadly studied Born approximation. Negligence of the environmental recoil makes it possible to use the well-known Born-Oppenheimer approximation and the following ansatz can be used [16]:

$$|\Psi_{S:E}\rangle = \int dX_0\phi_0(X_0)e^{-i\hat{H}_St}|X_0\rangle U_{\rm eff}(X(t;X_0))|\psi_0\rangle. \quad (6)$$

Here $X(t;X_0)$ is an approximate trajectory followed by the central system and parameterized by the initial position X_0, $\phi_0(X_0) = \langle X_0|\phi_0\rangle$ is the wave function of the initial state of a central system, $|\psi_0\rangle$ is the initial state of the environment, and the effective evolution $U_{\rm eff}$ is governed by the corresponding effective Hamiltonian $H_{\rm eff}$:

$$H_{\rm eff} = \sum_i\left[-\frac{\Delta_i}{2}\sigma_x^{(i)} + g_iX(t;X_0)\sigma_z^{(i)}\right]. \quad (7)$$

Hence, the total unitary evolution operator is read by

$$U_{S:E}(t) = \int dX_0 e^{-iH_St}|X_0\rangle\langle X_0|\otimes U_{\rm eff}(X(t;X_0)). \quad (8)$$

Assuming a completely separable initial state,

$$\rho_{S:E}(0) = \rho_S(0)\otimes\bigotimes_i\rho_E^{(i)}(0), \quad (9)$$

the dynamics of the total density matrix $\rho_{S:E}(t)$ is written as:

$$\rho_{S:E}(t) = \int dX_0dX_0'\langle X_0|\rho_S(0)|X_0'\rangle e^{-iH_St}|X_0\rangle\langle X_0'|e^{iH_St} \otimes\bigotimes_{i=1}^N U_i(X_0,t)\rho_E^{(i)}(0)U_i^\dagger(X_0',t), \quad (10)$$

where $U_i(X_0,t)$ corresponds to the effective Hamiltonian for the ith spin, H_i, in $H_{\rm eff}$ in (7). In particular, we will be interested in local states of environment fragments, called macrofractions. We thus trace out everything from the above state except for the chosen macrofraction (arbitrary at this moment):

$$\rho_{\rm mac}(t) = {\rm Tr}_{SE\setminus{\rm mac}}\rho_{S:E}(t) = \int dX_0p(X_0)\rho_{\rm mac}(X_0), \quad (11)$$

where $p(X_0) \equiv \langle X_0|\rho_S(0)|X_0\rangle$ is the probability distribution of the initial position and:

$$\rho_{\rm mac}(X_0) \equiv \bigotimes_{i\in{\rm mac}} U_i(X_0,t)\rho_E^{(i)}(0)U_i^\dagger(X_0,t) \quad (12)$$

is the conditional state corresponding to X_0. We will be calculating the Holevo quantity, associated with the ensemble (II), (12).

A. Initial state of the system

Before we proceed, we need to specify the initial state of the system in order to obtain the approximate trajectories and the initial distribution $p(X_0)$; cf. (II). For definiteness, we choose the displaced squeezed state, which allows for an easy control which initial quadrature is well defined and which is not:

$$|\phi_0\rangle = D(\alpha)S(\zeta)|0\rangle \equiv |\alpha,\zeta,0\rangle,\ \alpha = |\alpha|e^{i\phi}, \zeta = re^{i\theta}, \quad (13)$$

where $D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a}$ and $S(\zeta) \equiv e^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)}$ are the displacement and the squeezing operators respectively. Equivalently, a squeezed coherent state $S(\zeta)D(\tilde{\alpha})|0\rangle$ can be used with a substitution

$$\alpha \to \tilde{\alpha} = \alpha\cosh r - e^{i\theta}\alpha^*\sinh r. \quad (14)$$

In the absence of the environmental decoherence, the evolution of the above state would be simply given by $|\alpha,\zeta,t\rangle = e^{-iH_St}|\alpha,\zeta,0\rangle$, leading to the well-known time-dependent Gaussian probability distribution of the position (see e.g. [17]):

$$p_t(X) = |\langle X|\alpha,\zeta,t\rangle|^2 = \left(\frac{1}{\pi\eta(t)}\right)^{1/2}\exp\left[-\frac{(X-q(t))^2}{\eta(t)}\right], \quad (15)$$

where:

$$\eta(t) \equiv \frac{1}{M\Omega}(\cosh 2r + \cos(2\Omega t + \theta)\sinh 2r) > 0, \quad (16)$$

$$q(t) \equiv \sqrt{\frac{2}{M\Omega}}|\alpha|\cos(\Omega t + \phi) \quad (17)$$

This motivates the following choice of the trajectory in the Born-Oppenheimer ansatz (6):

$$X(t;X_0) = X_0\cos(\Omega t + \phi), \quad (18)$$

and defines the initial probability distribution through $p(X_0) = p_{t=0}(X_0)$.

B. Floquet dynamics

The effective Hamiltonian for the ith spin, H_i, in H_{eff} in (7) is given by:

$$H_i = -\frac{\Delta_i}{2}\sigma_x^{(i)} + g_i X_0 \cos(\Omega t + \phi)\sigma_z^{(i)}. \quad (19)$$

The unitary evolution operator $U(X_0, t)$ on a single qubit can be calculated with the help of the Floquet theory and the high frequency expansion as it was done e.g. in [10]:

$$U(X_0, t) = e^{-iK(t)} e^{-iH_F t} e^{iK(0)}, \quad (20)$$

where

$$H_F t = -\tilde{\Delta}(1 - \xi^2)\tau\sigma_x + O(\xi^3), \quad (21)$$
$$K(t) = \xi\sigma_z \sin\tau + O(\xi^2) \quad (22)$$

and the following dimensionless parameter are introduced:

$$\tau \equiv \Omega t,\ \xi \equiv g X_0/\Omega,\ \tilde{\Delta} \equiv \Delta/2\Omega. \quad (23)$$

together with a rescaled coupling strength:

$$\tilde{g} \equiv \frac{g}{\Omega}, \quad (24)$$

Operator (20) has the following Bloch representation:

$$U(X_0, t) = U_0\mathbb{I} + iU_1\sigma_x + iU_2\sigma_y + iU_3\sigma_z, \quad (25)$$

where

$$\begin{aligned} U_0 &= \cos[\xi(\sin(\tau+\phi) - \sin\phi)]\cos[\tilde{\Delta}(1-\xi^2)\tau] \\ U_1 &= \cos[\xi(\sin(\tau+\phi) + \sin\phi)]\sin[\tilde{\Delta}(1-\xi^2)\tau] \\ U_2 &= \sin[\xi(\sin(\tau+\phi) + \sin\phi)]\sin[\tilde{\Delta}(1-\xi^2)\tau] \\ U_3 &= -\sin[\xi(\sin(\tau+\phi) - \sin\phi)]\cos[\tilde{\Delta}(1-\xi^2)\tau]. \end{aligned} \quad (26)$$

We choose an initial state of the environment $\rho_E(0)$ to be a thermal state for $H_E = -\Delta\sigma_x/2$ in (2):

$$\rho_E(0) = \frac{1}{2}\left[\mathbb{I} + E(\beta)\sigma_x\right], \quad (27)$$

where $E(\beta) \equiv \tanh(\beta\Delta/2)$ with $\beta \equiv 1/k_B T$. Applying $U(X_0, t)$ from (25) to (27), the final state $\rho_{X_0}(\tau)$ for a single environmental qubit can be written in the Bloch representation as:

$$\begin{aligned} \rho_{X_0}(\tau) &= U(X_0, t)\rho_E(0)U^\dagger(X_0, t) \\ &= \frac{1}{2}[\mathbb{I} + \vec{a}(X_0, \tau)\cdot\vec{\sigma}]. \end{aligned} \quad (28)$$

The explicit expressions $\vec{a}(X_0, \tau)$ were obtained e.g. in [10] and are shown in Appendix A2.

III. HOLEVO QUANTITY FOR A SINGLE ENVORONMENT

We consider the simplest case that the remaining spin after unobserved spin degrees of freedom traced out in the decoherence process, is an observed single spin.

According to the Holevo's theorem, the quantum mutual information $I(X:Y)$ between a preparation system X and a measurement system Y is bounded by the Holevo quantity $\chi(\rho)$:

$$I(X:Y) \le \chi(\rho) \equiv S(\bar{\rho}) - \bar{S}, \quad (29)$$

where the average quantities $\bar{\rho}$ and $\bar{S}$ are defined by

$$\begin{aligned} \bar{\rho} &\equiv \int dX p(X)\rho_X, \\ \bar{S} &\equiv \int dX p(X) S(\rho_X). \end{aligned} \quad (30)$$

Here $\{\rho_X\}$ is a set of prepared states for random variables X with the probability density distribution $\{p(X)\}$ and $S(\rho_X)$ is the entropy for a state ρ_X. Infinite dimensionality of ρ_X could lead $\chi(\rho)$ to be infinite unless physical constraints, e.g. the number of photons to be fixed, are applied [18]. In our case ρ_X has a finite dimension, so $\chi(\rho)$ is finite.

A. Entropy of the average state

Using (15) for $t = 0$ the average density matrix $\bar{\rho}(\tau)$ for a qubit as introduced in (30) can be now calculated

$$\bar{\rho}(\tau) = \int dX_0 p(X_0)\rho_{X_0}(\tau). \quad (31)$$

$\bar{\rho}(\tau)$ is again written in the Bloch representation:

$$\begin{aligned} \bar{\rho}(\tau) &= \frac{1}{2}\int dX_0 p(X_0)(\mathbb{I} + \vec{a}(X_0, \tau)\cdot\vec{\sigma}) \\ &\equiv \frac{1}{2}\left[\mathbb{I} + E(\beta)\vec{\mu}(\tau)\cdot\vec{\sigma}\right], \end{aligned} \quad (32)$$

where $\mu_i(\tau)$ are given by:

$$\begin{aligned} \mu_1(\tau) &\equiv \frac{1}{2}\text{Re}\{I[0, k_-, 0] + I[0, k_+, 0] + D_{12}(\tau)\} \\ \mu_2(\tau) &\equiv \frac{1}{2}\text{Im}\{I[0, k_-, 0] + I[0, k_+, 0] + D_{12}(\tau)\} \\ \mu_3(\tau) &\equiv \text{Re}[D_3(\tau)]. \end{aligned} \quad (33)$$

The expressions for $I[\cdot,\cdot,\cdot]$ are given in (B6) and (B7) and the decaying quantities $D_{12}(\tau)$ and $D_3(\tau)$ are defined as

$$\begin{aligned} D_{12}(\tau) &\equiv \frac{1}{2}(I[-k, k_-, -k_0] + I[k, k_-, k_0] \\ &\quad - I[-k, k_+, -k_0] - I[k, k_+, k_0]), \\ D_3(\tau) &\equiv \frac{1}{2}(I[-k, \delta k/2, -k_0] - I[k, \delta k/2, k_0]), \end{aligned} \quad (34)$$

with:

$$\begin{aligned} k_+ &= 2\tilde{g}[\sin(\tau+\phi)+\sin\phi], \\ k_- &= 2\tilde{g}[\sin(\tau+\phi)-\sin\phi], \\ \delta k &= k_+ - k_-, \\ k &= -2\tilde{g}^2\tau\tilde{\Delta}, \\ k_0 &= 2\tau\tilde{\Delta}. \end{aligned} \tag{35}$$

Note that the quantities $D_{12}(\tau)$ and $D_3(\tau)$ in (34) are a combination of $I[\pm k,\cdot,\cdot]$ with $k=-2\tilde{g}^2\tau\tilde{\Delta}\neq 0$ and hence vanish as $t\to\infty$. The eigenvalues of $\bar{\rho}(\tau)$ in (32) are $\bar{\lambda}_{1,2}=(1\pm\mu(\tau)E(\beta))/2$, $\mu(\tau)\equiv|\vec{\mu}(\tau)|$. Thus, the entropy $S(\bar{\rho})$ is expressed as

$$\begin{aligned} S(\bar{\rho}) &= -\bar{\lambda}_1\log_2\bar{\lambda}_1-\bar{\lambda}_2\log_2\bar{\lambda}_2 \\ &= -\frac{1}{2}(1+\mu(\tau)E(\beta))\log_2(1+\mu(\tau)E(\beta)) \\ &\quad -\frac{1}{2}(1-\mu(\tau)E(\beta))\log_2(1-\mu(\tau)E(\beta))+1, \end{aligned} \tag{36}$$

Since $\rho_{X_0}(\tau)$ and $\rho_E(0)$ in (28) are only unitarily related, $\bar{S}(\rho)=S(\rho_E(0))$. With the eigenvalues of $\rho_E(0)$, $(([1+E(\beta)]/2,[1-E(\beta)]/2))$, $\bar{S}$ is time-independent:

$$\begin{aligned} \bar{S} &= S(\rho_E(0)) \\ &= -\frac{1}{2}(1+E(\beta))\log_2(1+E(\beta)) \\ &\quad -\frac{1}{2}(1-E(\beta))\log_2(1-E(\beta))+1. \end{aligned} \tag{37}$$

The Holevo quantity $\chi(\rho)$ is

$$\begin{aligned} \chi(\rho) &= S(\bar{\rho})-\bar{S} \\ &= -\frac{1}{2}(1+\mu(\tau)E(\beta))\log_2(1+\mu(\tau)E(\beta)) \\ &\quad -\frac{1}{2}(1-\mu(\tau)E(\beta))\log_2(1-\mu(\tau)E(\beta)) \\ &\quad +\frac{1}{2}(1+E(\beta))\log_2(1+E(\beta)) \\ &\quad +\frac{1}{2}(1-E(\beta))\log_2(1-E(\beta)). \end{aligned} \tag{38}$$

$\chi(\rho)$ in (38) can be viewed as a difference between Shannon entropies, $H[(1+\mu(\tau)E(\beta))/2]$ and $H[(1+E(\beta))/2]$ for a qubit. All the relevant parameters are contained in $\mu(\tau)$ and a temperature dependence in $E(\beta)$. The Shannon entropy $H(p)$ for $p\geq 1/2$ is a decreasing function of p and its slop is steeper as $p\to 1$. As $\mu(\tau)$ gets smaller $\chi(\rho)$ gets bigger. Since $|dH(p)/dp|$ approaches the maximum as $E(\beta)\to 1$ $(T\to 0)$, with $\mu(\tau)$ being fixed, $\chi(\rho)$ gets bigger as $T\to 0$, i.e. as a temperature decreases, information is better transferred. This is intuitively clear and consistent with the results in [10, 19] that the distinguishability decreases as temperature increases.

B. Short time behaviour

As seen in Fig. 1, $\chi(\rho)$ quickly grows and gets stabilized from the beginning. To verify this behaviour, we investigate a short time behaviour of $\chi(\rho)$. For this purpose, as $\chi(\rho)$ is determined only by $\mu(\tau)$ and $E(\beta)$, we expand $\mu(\tau)$ in τ and $\tilde{g}$ up to $O(\tau^2)$ and $O(\tilde{g}^2)$ according to the small coupling approximation, $\tilde{g}\ll 1$. Following the details given in Appendix C, $\mu^2(\tau)$ in (C5) is expressed as

$$\begin{aligned} &\mu^2(\tau) \\ &= 1-2\tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)] \\ &+O(\tau^3)+O(\tilde{g}^3) \end{aligned} \tag{39}$$

This expression clearly shows that $\mu^2(\tau)$ is a decreasing function in τ and hence $\chi(\rho)$ is an increasing function in τ since as seen in (38) $\chi(\rho)$ is a decreasing function in $\mu(\tau)$. Expanding $\chi(\rho)$ in τ up to $O(\tau^2)$ and $O(\tilde{g}^2)$ from (C7),

$$\begin{aligned} \chi(\rho) &= \frac{E^2(\beta)}{4}[\log(1-E(\beta))-\log(1+E(\beta))]\mu''(0)\tau^2 \\ &\quad +O(\tau^3)+O(\tilde{g}^3) \\ &= \frac{E^2(\beta)}{2}\log\frac{1+E(\beta)}{1-E(\beta)} \\ &\quad \times\tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)] \\ &\quad +O(\tau^3)+O(\tilde{g}^3) \\ &= \Lambda\tau^2+O(\tau^3)+O(\tilde{g}^3), \end{aligned} \tag{40}$$

where

$$\Lambda\equiv\frac{E^2(\beta)}{2}\log\frac{1+E(\beta)}{1-E(\beta)}[1-(1-4\tilde{\Delta}^2)\sin^2\phi]\eta^2\tilde{g}^2. \tag{41}$$

and from (16)

$$\eta\equiv\eta(0)=\frac{1}{M\Omega}(\cosh 2r+\cos\theta\sinh 2r), \tag{42}$$

$$q\equiv q(0)=\sqrt{\frac{2}{M\Omega}}|\alpha|\cos\phi. \tag{43}$$

(40) is one of our main results. It shows that $\chi(\rho)$ grows quadratically in time for short times. The speed of the growth is given by the factor Λ from (41). Interestingly, the larger ϕ is, the larger the initial growth of $\chi(\rho)$ is, which is opposite to behaviour for $\tau\to\infty$. This is shown in Fig.2. Fig.3 clearly shows those short time behaviours at different values of ϕ.

C. Asymptotic Holevo quantity

Let us introduce the following asymptotic quantity $\chi_\infty(\rho)$. Noticing that $\chi(\rho)$ quickly approaches $\chi_\infty(\rho)$ as it evolves, $\chi_\infty(\rho)$ is not only simpler to analyze but also more relevant to the time scale in our focus on classicality.

$$\chi_\infty(\rho)=\lim_{\tau\to\infty}\chi(\rho). \tag{44}$$

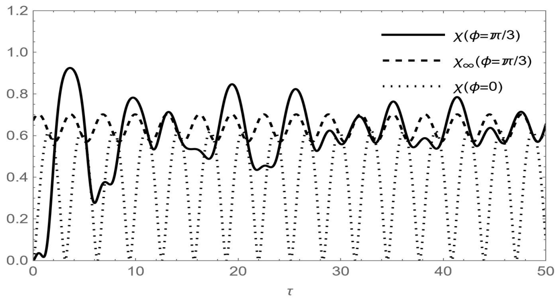

FIG. 1: Comparison of the Holevo quantity $\chi(\rho)$ with the asymptotic form $\chi_\infty(\rho)$ ($M = 1$. $\Omega = 5$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, $\phi = \frac{\pi}{3}$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$, $\beta = 10$).

It turns out that the contribution of $D_{12}(\tau)$ and $D_3(\tau)$ in (34) to the Holevo quantity $\chi(\rho)$ in (38) with μ_i in (33) becomes negligible for large τ. The difference between the exact expression $\chi(\rho)$ and the asymptotic quantity $\chi_\infty(\rho)$ in (52) is negligible after some initial stabilization period and it is easier to analyze $\chi_\infty(\rho)$ than $\chi(\rho)$. As $\tau \to \infty$ in (B7),

$$\lim_{\tau\to\infty} I[-k, k_-, -k_0] = \lim_{\tau\to\infty} I[k, k_-, k_0] \qquad (45)$$
$$= \lim_{\tau\to\infty} I[-k, k_+, -k_0] = \lim_{\tau\to\infty} I[k, k_+, k_0] = 0.$$

Especially, it is easy to notice in (34) that $D_{12}(\tau)$ and $\mathrm{Re}[D_3(\tau)]$ at $\phi = 0$ identically vanish, without making any contribution to $\mu_i(\tau)$ in (33), i.e. there is no difference between $\chi(\rho)$ and $\chi_\infty(\rho)$ at $\phi = 0$,

$$\chi(\rho) = \chi_\infty(\rho) \text{ at } \phi = 0. \qquad (46)$$

This implies that at $\phi = 0$ $\chi(\rho)$ has only a oscillatory pattern without the asymptotic behaviour, shown in Fig.2. In (34) $D_{12}(\tau)$ and $D_3(\tau)$ asymptotically vanish as $\tau \to \infty$,

$$\lim_{\tau\to\infty} D_{12}(\tau) = \lim_{\tau\to\infty} D_3(\tau) = 0. \qquad (47)$$

The reason that $D_{12}(\tau)$ and $D_3(\tau)$ vanish for large time scale is due to the self-Hamiltonian $-\Delta\sigma_x/2$ in (3), i.e. non-zero $k = -2\tilde{g}^2\tau\tilde{\Delta}$, which is proportional to τ. For $\phi = 0$ there is no $\tilde{\Delta}$ dependence for $\chi(\rho)$, so $\chi(\rho)$ shows a monotonous periodic pattern without information growth. This verifies that $\phi = 0$ case does not show the distinguishability in the objectivity shown in [10]. Due to (47) $\mu_i(\tau)$ tends to approach as

$$\lim_{\tau\to\infty} \mu_1(\tau) = \frac{1}{2}\mathrm{Re}(I[0, k_-, 0] + I[0, k_+, 0])$$
$$\lim_{\tau\to\infty} \mu_2(\tau) = \frac{1}{2}\mathrm{Im}(I[0, k_-, 0] + I[0, k_+, 0]) \qquad (48)$$
$$\lim_{\tau\to\infty} \mu_3(\tau) = 0.$$

Thus, the asymptotic quantity μ_∞ is expressed as

$$\begin{aligned}\mu_\infty &\equiv \lim_{\tau\to\infty} \sqrt{\mu_1^2(\tau) + \mu_2^2(\tau) + \mu_3^2(\tau)} \\ &= \frac{1}{2}\left| e^{-\frac{\eta k_-^2}{4}} + e^{-\frac{\eta k_+^2}{4}} e^{iq\delta k_\pm} \right| \\ &= \frac{1}{2}\Big\{ e^{-\frac{\eta k_-^2}{2}} + e^{-\frac{\eta k_+^2}{2}} \qquad (49) \\ &+ 2e^{-\frac{\eta(k_-^2+k_+^2)}{4}} \cos[2\tilde{g}q_0 \sin 2\phi] \Big\}^{1/2} \\ &\le \cos[\tilde{g}q_0 \sin 2\phi],\end{aligned}$$

where $q_0 \equiv |\alpha|\sqrt{\frac{2}{M\Omega}}$, η and q were defined in (42). Fig.1 shows how quickly $\chi(\rho)$ approaches the corresponding asymptotic limit $\chi_\infty(\rho)$.

For large τ, the eigenvalues of $\bar{\rho}$, $\bar{\lambda}_{1,2}$, approach

$$\bar{\lambda}_1 \to \frac{1}{2}(1 + \mu_\infty E(\beta)),$$
$$\bar{\lambda}_2 \to \frac{1}{2}(1 - \mu_\infty E(\beta)) \qquad (50)$$

and $S(\bar{\rho})$ approaches $S_\infty(\bar{\rho})$,

$$\begin{aligned}S_\infty(\bar{\rho}) = &-\frac{1}{2}(1 + \mu_\infty E(\beta)) \log_2(1 + \mu_\infty E(\beta)) \\ &- \frac{1}{2}(1 - \mu_\infty E(\beta)) \log_2(1 - \mu_\infty E(\beta)) + 1. \qquad (51)\end{aligned}$$

The asymptotic Holevo quantity $\chi_\infty(\rho)$ is

$$\begin{aligned}\chi_\infty(\rho) &= S_\infty(\bar{\rho}) - \bar{S} \\ &= -\frac{1}{2}(1 + \mu_\infty) \log_2(1 + \mu_\infty E(\beta)) \\ &- \frac{1}{2}(1 - \mu_\infty E(\beta)) \log_2(1 - \mu_\infty E(\beta)) \\ &+ \frac{1}{2}(1 + E(\beta)) \log_2(1 + E(\beta)) \qquad (52) \\ &+ \frac{1}{2}(1 - E(\beta)) \log_2(1 - E(\beta)).\end{aligned}$$

D. Parameter dependence

$\chi(\rho)$ depends on parameters from the Hamiltonian, (Δ, g, Ω, M) and the initial conditions, (β, ϕ, η, q_0). For our interest in large time behaviours, we analyze $\chi_\infty(\rho)$ instead of $\chi(\rho)$. $\chi_\infty(\rho)$ is parameterized by two parameters, μ_∞ and $E(\beta)$. It increases as μ_∞ as well as $E(\beta)$ decreases. μ_∞ in (49) is a function of $|\vec{\gamma}|$, where $\vec{\gamma} \equiv \vec{\alpha} + \vec{\beta}$, whose magnitudes and the relative angle ψ are given by

$$|\vec{\alpha}|(\tilde{g}, \eta, \phi, \tau) \equiv e^{-\frac{\eta k_-^2}{4}} = e^{-\eta\tilde{g}^2\cos^2(\tau/2+\phi)\sin^2(\tau/2)}, \quad (53)$$
$$|\vec{\beta}|(\tilde{g}, \eta, \phi, \tau) \equiv e^{-\frac{\eta k_+^2}{4}} = e^{-\eta\tilde{g}^2\sin^2(\tau/2+\phi)\cos^2(\tau/2)}, \quad (54)$$
$$\psi(\tilde{g}, q_0, \phi) \equiv 2\tilde{g}q_0 \sin 2\phi. \qquad (55)$$

$|\vec{\alpha}|$ and $|\vec{\beta}|$ depend on a squeezing parameter η, ϕ and τ contained in $k_\pm$ in (35). This geometrical description is useful to analyze $\mu_\infty(|\vec{\gamma}|)$ and hence $\chi_\infty(\rho)$. It is immediately noticeable that for large η, $\chi_\infty(\rho)$ is high. We list possible cases when $\chi_\infty(\rho)$ is minimized and maximized with the configuration of $(|\vec{\alpha}|, |\vec{\beta}|, \psi)$ in (53) for fixed temperature first.
Maximizing $\chi(\rho)$ ($\mu_\infty \to 0$):

1. $\eta \to \infty$ ($\vec{\alpha} = \vec{\beta} = 0$): This corresponds to the uniform distribution of $p(X_0)$. It leads to $S(\bar{\rho}) \to 1$ and $\chi(\rho) = \chi_M(\beta)$.

2. $|\vec{\alpha}| = |\vec{\beta}| \to \tau+\phi = (2n+1)\pi$ ($n \in \mathbb{Z}$) and $\psi = (2n+1)\pi$: As long as $\psi = (2n+1)\pi$ is satisfied, $\chi_\infty(\rho)$ arrives at the local maxima at $\tau + \phi = (2n+1)\pi$.

3. $\tilde{g} \uparrow \to |\vec{\alpha}| \downarrow, |\vec{\beta}| \downarrow$ and $\psi \uparrow$: Increasing $\tilde{g}$ increases $\chi_\infty(\rho)$

4. $\phi \uparrow$: Apart from oscillatory parts, as ϕ increases $\chi_\infty(\rho)$ increases.

Fig.2 shows that the larger ϕ is the more information is encoded in the spins. On the other hand, Fig.4 shows that when $\vec{\alpha}$ and $\vec{\beta}$ are anti-parallel, $\chi_\infty(\rho)$ is maximized, i.e. for the parameters $\tilde{g}$, q and ϕ satisfying

$$2\tilde{g}|\alpha|\sqrt{\frac{2}{M\Omega}}\sin 2\phi = \pi, \tag{56}$$

all $\chi(\rho)$ quickly get maximized regardless of values of ϕ. $\chi(\rho)$ with this condition is the maximum rather than at $\phi = \pi/2$. Fig.5 shows that $\chi(\rho)$ under the maximization condition (56) asymptotically has a higher value larger than those without it. Also, it shows that non-zero ϕ is crucial to have an asymptotic behavior stabilizing the maximum.
Minimizing $\chi(\rho)$ ($\mu_\infty \to 1$):

1. $\eta \to 0$ ($\vec{\alpha} = \vec{\beta} \to 1$) and $\psi = 0$: This corresponds to $p(X_0) \to \delta(X_0 - q)$, i.e. the initial density matrix is a localized pure state, which leads to $\chi_\infty(\rho) = 0$. There are two cases for $\psi = 0$.

2. $\sin 2\phi = 0$: As seen in (53) at $\phi = 0$ $|\vec{\alpha}| = |\vec{\beta}|$ and $\chi(\rho)$ vanishes periodically while $\phi = \pi/2$ does not since $|\vec{\alpha}| \neq |\vec{\beta}|$ $\phi = 0$ is consistent with the result that $\sin\phi = 0$ does not lead to a vanishing generalized overlap (distinguishability) [10].

Now we verify the intuition that reducing temperature increases distinguishability. As mentioned below (38), $\chi_\infty(\rho)$ is a difference between the Shannon entropies $H[(1+\mu_\infty E(\beta))/2]$ and $H[(1+E(\beta))/2]$. This difference gets larger as $E(\beta)$ is closer to 1. Recognizing that for $p > 1/2$ $dH(p)/dp < 0$ and $|dH(p)/dp|$ increases as p

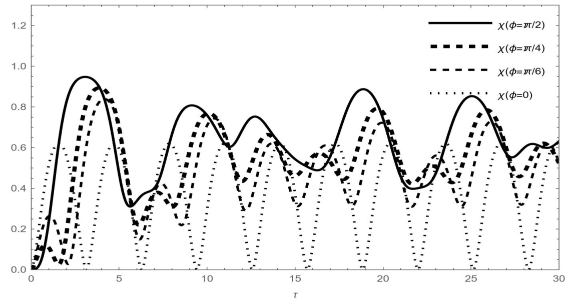

FIG. 2: The Holevo quantities at different initial conditions, $\phi = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}, 0)$. ($M = 1$. $\Omega = 5$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$, $\beta = 10$).

increase. For $\beta' > \beta$ ($T > T'$),

$$\begin{aligned}&\chi_\infty(\beta') - \chi_\infty(\beta)\\&= H[(1+\mu_\infty E(\beta'))/2] - H[(1+\mu_\infty E(\beta))/2]\\&- \{H[(1+E(\beta'))/2] - H[(1+E(\beta))/2]\}\\&\approx \Delta E\left[\mu_\infty\frac{\partial H(p')}{\partial p'} - \frac{\partial H(p)}{\partial p}\right] > 0,\end{aligned} \tag{57}$$

where $p' = (1+\mu_\infty E(\beta))/2$ and $p = (1+E(\beta))/2$. This means that as a temperature gets lower $\chi_\infty(\rho)$ gets larger. This is consistent with the fact that lowering temperature enhances the distiguishability [10]. All theses observations are consistent with effects on the objectivity in [10]. The temperature dependence of the Holevo quantity is shown in Fig.6.

Finally, we wish to mention the maximum $\chi(\rho)$. Define

$$\begin{aligned}&g[\mu(\tau)E(\beta)]\\&\equiv -\frac{1}{2}(1+\mu(\tau)E(\beta))\log_2(1+\mu(\tau)E(\beta))\\&-\frac{1}{2}(1-\mu(\tau)E(\beta))\log_2(1-\mu(\tau)E(\beta)) \le 0.\end{aligned} \tag{58}$$

Since $dg(\mu)/d\mu \le 0$, at $\mu = 0$, $g[\mu(\tau)E(\beta)] = 0$, which is the maximum. As $\chi(\rho) = g(\mu E(\beta)) - g(E(\beta))$, $\chi(\rho)$ is the maximum $\chi_M(\beta)$ at $\mu = 0$ with β fixed.

$$\begin{aligned}\chi_M(\beta) &\equiv \frac{1}{2}(1+E(\beta))\log_2(1+E(\beta))\\&+\frac{1}{2}(1-E(\beta))\log_2(1-E(\beta))\\&= -g(E(\beta)) \ge 0.\end{aligned} \tag{59}$$

Since $-dg(E(\beta))/dE(\beta) \ge 0$, $\chi_M(\beta)$ is 1, the largest, at $E(\beta) = 1$, i.e. $\beta \to \infty$ ($T = 0$).

$$0 \le \chi(\rho) \le \chi_M(\beta) \le 1. \tag{60}$$

IV. CONCLUSION

This article examined how the information of continuous degrees of freedom is encoded into a system of fi-

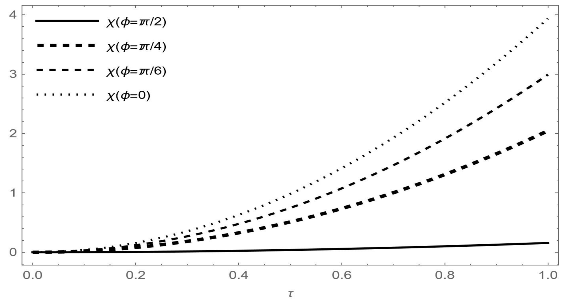

FIG. 3: Short time behaviours of the Holevo quantities at different initial conditions $\phi = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}, 0)$. ($M = 1$. $\Omega = 5$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$, $\beta = 10$).

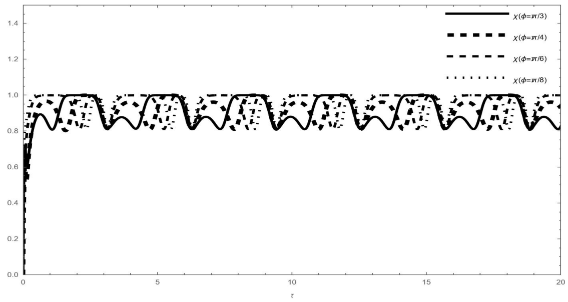

FIG. 4: The Holevo quantities reaching the maxima with the squeezing parameter η fixed and q varied at $\phi = (\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}, 0)$. ($M = 1$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, Ω under the condition $\psi = 2\tilde{g}q_0 \sin 2\phi = \pi$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$, $\beta = 10$).

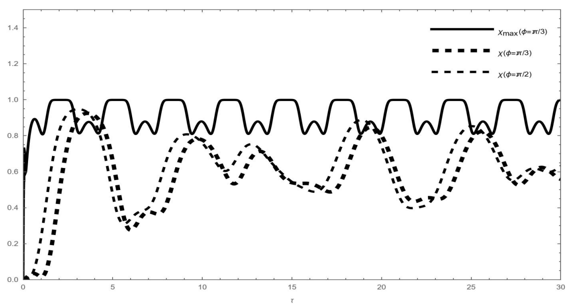

FIG. 5: Comparison among the Holevo quantities at $\psi = 2\tilde{g}q_0 \sin 2\pi/3 = \pi$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ with $\Omega = 5$. ($M = 1$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$, $\beta = 10$).

nite degrees of freedom with a simple system, a boson-spin model. On the dualism of the quantum Darwinism between objectification and information transfer, we wanted to verify the objectivity that was investigated with the decoherence factor for decoherence and the generalized overlap for distiguishability in the last work [10] for the boson-spin system, especially distinguishability, with a point of view of information transfer by calculating the mutual information bound, the Holevo quantity.

We considered the simplest situation, in which a cen-

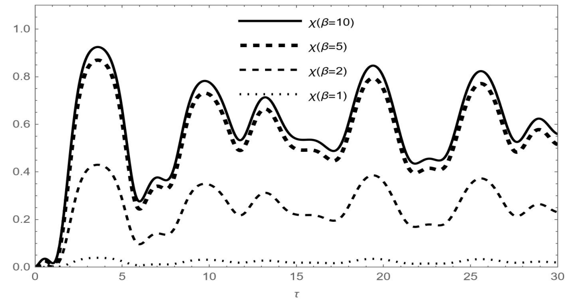

FIG. 6: Temperature dependence of the Holevo quantities at $\beta = (10, 5, 2, 1)$. ($M = 1$. $\Omega = 5$, $r = 1$, $\theta = 0$, $|\alpha| = 1$, $\phi = \frac{\pi}{3}$, $\tilde{g} = \frac{1}{2}$, $\Delta = 1$).

tral harmonic oscillator and a single spin system remain after unobserved spins traced out. In order to apply for the Holevo theorem, we assumed that the system is in a perfect decoherence in a position basis up to a local unitary transformation. An initial probability distribution is chosen from a displacement and squeezed state and an initial state of a spin is chosen to be a thermal state. In this setting we investigated how the Holevo quantity for our model is shown to be consistent with distinguishability measured by the generalized overlap. We verified that as expected, the high temperature is against the precision of encoding, which is consistent with the result [10, 19].

We established the role of a self-Hamiltonian of a spin environment, that it is necessary to make sure asymptotic stabilization in the Holevo quantity. We found the relation between the Holevo quantity and the parameters in the system. We verified that the intuitive relations of the Holevo quantity with a squeezing parameter, temperature and a coupling constant, that increasing a squeezing parameter and a coupling constant and decreasing temperature increase the Holevo quantity. Increasing displacement parameter q also increases the Holevo quantity, which can be understood as q is the size of position space in statistical distribution. We explained on the dependence of the Holevo quantity on ϕ why a vanishing initial phase ϕ in the oscillator trajectory does not provide the distinguishability. This verifies that $\phi = 0$ does not provide the objectivity on a position basis.

ACKNOWLEDGMENTS

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Appendix A: Bloch Representation

Before integrating to get $\mu_i(\tau)$ in (33), we first need to express $\rho_{X_0}(\tau)$ defined in (28) in the Bloch representation. Given the expression for $U(X_0, t)$ in (25) and an initial thermal state $\rho_E(0) = [\mathbb{I} + E(\beta)\sigma_x]/2$ in (27), $\rho_{X_0}(\tau)$ is written in the Bloch form,

$$\begin{aligned}\rho_{X_0}(\tau) &= U(X_0,t)\rho_E(0)U^\dagger(X_0,t)\\ &= \frac{1}{2}[\mathbb{I} + \vec{a}(X_0,\tau)\cdot\vec{\sigma}],\end{aligned} \tag{A1}$$

where

$$\begin{aligned}a_1(X_0,\tau) &= E(\beta)(2U_0^2 + 2U_1^2 - 1),\\ a_2(X_0,\tau) &= 2E(\beta)(U_1U_2 - U_0U_3),\\ a_3(X_0,\tau) &= 2E(\beta)(U_0U_2 + U_1U_3)\end{aligned} \tag{A2}$$

and from (26)

$$\begin{aligned}U_0 &= \cos[\xi(\sin(\tau+\phi) - \sin\phi)]\cos[\tilde{\Delta}(1-\xi^2)\tau]\\ U_1 &= \cos[\xi(\sin(\tau+\phi) + \sin\phi)]\sin[\tilde{\Delta}(1-\xi^2)\tau]\\ U_2 &= \sin[\xi(\sin(\tau+\phi) + \sin\phi)]\sin[\tilde{\Delta}(1-\xi^2)\tau]\\ U_3 &= -\sin[\xi(\sin(\tau+\phi) - \sin\phi)]\cos[\tilde{\Delta}(1-\xi^2)\tau].\end{aligned} \tag{A3}$$

It would be more conveniently to express $a_i(X_0,\tau)$ in (A2) as an exponential series for the Gaussian integration,

$$\begin{aligned}a_1(X_0,\tau) &= E(\beta)\mathrm{Re}[c_{12}(X_0,\tau)],\\ a_2(X_0,\tau) &= E(\beta)\mathrm{Im}[c_{12}(X_0,\tau)],\\ a_3(X_0,\tau) &= E(\beta)\mathrm{Re}[c_3(X_0,\tau)],\end{aligned} \tag{A4}$$

where

$$\begin{aligned}c_{12}(X_0,\tau) \equiv\ & \frac{1}{4}[2e^{ik_-X_0} + 2e^{ik_+X_0}\\ &+ e^{i(k_-X_0 - kX_0^2 - k_0)} + e^{i(k_-X_0 + kX_0^2 + k_0)}\\ &- e^{i(k_+X_0 - kX_0^2 - k_0)} - e^{i(k_+X_0 + kX_0^2 + k_0)}],\\ c_3(X_0,\tau) \equiv\ & \frac{1}{2}[e^{i[-(k_+ - k_-)X_0/2 + kX_0^2 + k_0]}\\ &- e^{i[(k_+ - k_-)X_0/2 + kX_0^2 + k_0]}],\end{aligned} \tag{A5}$$

with $\tilde{g} \equiv g/\Omega$ and

$$\begin{aligned}k_- &= 2\tilde{g}[\sin(\tau+\phi) - \sin\phi],\\ k_+ &= 2\tilde{g}[\sin(\tau+\phi) + \sin\phi],\\ k &= -2\tilde{g}^2\tau\tilde{\Delta},\\ k_0 &= 2\tau\tilde{\Delta}.\end{aligned} \tag{A6}$$

Appendix B: Gaussian integrals

The Holevo quantity $\chi(\rho)$ consists of two parts $S(\bar{\rho})$ and $\bar{S}$ in (29). Computing $S(\bar{\rho})$ requires the Gaussian integral with the initial probability distribution in (31). The Gaussian distribution $p(X_0)$ is given in (15),

$$p(X_0) = \left(\frac{1}{\pi\eta}\right)^{1/2}\exp\left[-\frac{(X_0-q)^2}{\eta}\right], \tag{B1}$$

where (η, q) were defined in (42). Since $p(X_0)$ in (15) and (c_{12}, c_3) are Gaussian in complex space, from (A4) and (A5) the integrals $\mu_i(\tau)$ in (33) can be written as Gaussian integrals

$$\begin{aligned}\mu_1(\tau) &= \mathrm{Re}\int dX_0 p(X_0)c_{12}(X_0,\tau),\\ \mu_2(\tau) &= \mathrm{Im}\int dX_0 p(X_0)c_{12}(X_0,\tau),\\ \mu_3(\tau) &= \mathrm{Re}\int dX_0 p(X_0)c_3(X_0,\tau).\end{aligned} \tag{B2}$$

Using the following integral formula

$$\begin{aligned}I[a,b,c] &= \sqrt{\frac{1}{\pi\eta}}\int_{-\infty}^{\infty} dy\, e^{-(y-y_0)^2/\eta + iay^2 + iby + ic}\\ &= \sqrt{\frac{1}{1-ia\eta}}\exp\left[\frac{ia}{1-ia\eta}\left(y_0^2 + \frac{b}{a}y_0 + i\frac{b^2\eta}{4a}\right) + ic\right],\end{aligned} \tag{B3}$$

$\mu_i(\tau)$ in (B2) are obtained as

$$\begin{aligned}&\int dX_0 p(X_0)c_{12}(X_0,\tau)\\ &= \frac{1}{4}[2I[0,k_-,0] + 2I[0,k_+,0]\\ &+ I[-k,k_-,-k_0] + I[k,k_-,k_0]\\ &- I[-k,k_+,-k_0] - I[k,k_+,k_0],\end{aligned} \tag{B4}$$

and

$$\begin{aligned}&\int dX_0 p(X_0,\tau)c_3(X_0,\tau)\\ &= \frac{1}{2}(I[k,-\delta k/2,k_0] - I[k,\delta k/2,k_0]),\end{aligned} \tag{B5}$$

where

$$\begin{aligned}I[0,k_-,0] &= e^{-\frac{\eta k_-^2}{4}}e^{ik_-q},\\ I[0,k_+,0] &= e^{-\frac{\eta k_+^2}{4}}e^{ik_+q},\\ I[-k,k_-,-k_0] &= \sqrt{\frac{1}{1+ik\eta}}\,e^{\frac{-ik}{1+ik\eta}\left(q^2 - \frac{k_-}{k}q - i\frac{k_-^2\eta}{4k}\right) - ik_0},\\ I[k,k_-,k_0] &= \sqrt{\frac{1}{1-ik\eta}}\,e^{\frac{ik}{1-ik\eta}\left(q^2 + \frac{k_-}{k}q + i\frac{k_-^2\eta}{4k}\right) + ik_0},\\ I[-k,k_+,-k_0] &= \sqrt{\frac{1}{1+ik\eta}}\,e^{\frac{-ik}{1+ik\eta}\left(q^2 - \frac{k_+}{k}q - i\frac{k_+^2\eta}{4k}\right) - ik_0},\\ I[k,k_+,k_0] &= \sqrt{\frac{1}{1-ik\eta}}\,e^{\frac{ik}{1-ik\eta}\left(q^2 + \frac{k_+}{k}q + i\frac{k_+^2\eta}{4k}\right) + ik_0},\end{aligned} \tag{B6}$$

and with $\delta k \equiv k_+ - k_-$

$$\begin{aligned}I[-k,\delta k/2,-k_0] &= \sqrt{\frac{1}{1+ik\eta}}\,e^{\frac{-ik}{1+ik\eta}\left(q^2 - \frac{\delta k_\pm}{2k}q - i\frac{\delta k_\pm^2\eta}{16k}\right) - ik_0},\\ I[k,\delta k/2,k_0] &= \sqrt{\frac{1}{1-ik\eta}}\,e^{\frac{ik}{1-ik\eta}\left(q^2 + \frac{\delta k}{2k}q + i\frac{\delta k^2\eta}{16k}\right) + ik_0}.\end{aligned} \tag{B7}$$

The decaying quantities $D_{12}(\tau)$ and $D_3(\tau)$ as $t\to\infty$ are defined as

$$\begin{aligned} D_{12}(\tau) &\equiv \frac{1}{2}(I[-k,k_-,-k_0]+I[k,k_-,k_0] \\ &\quad -I[-k,k_+,-k_0]-I[k,k_+,k_0]), \\ D_3(\tau) &\equiv \frac{1}{2}(I[-k,\delta k/2,-k_0]-I[k,\delta k/2,k_0]). \end{aligned} \tag{B8}$$

Appendix C: Short time expansion

We expand $I[\cdot,\cdot,\cdot]$ defined in (B6) and (B8) up to $O(\tau^2)$ and $O(\tilde{g}^2)$. Expanding (k_+,k_-,k,k_0) in (A6) up to $O(\tau^2)$ with the corresponding coefficients (c_0,c_1,c_2,d_0,d),

$$\begin{aligned} k_+ &= c_0+c_1\tau+c_2\tau^2 \\ &= 4\tilde{g}\sin\phi+[2\tilde{g}\cos\phi]\tau-[\tilde{g}\sin\phi]\tau^2+O(\tau^3) \\ k_- &= c_1\tau+c_2\tau^2 \\ &= [2\tilde{g}\cos\phi]\tau-[\tilde{g}\sin\phi]\tau^2+O(\tau^3) \\ k &= d\tau=-2\tilde{g}^2\tilde{\Delta}\tau \\ k_0 &= d_0\tau=2\tilde{\Delta}\tau \end{aligned} \tag{C1}$$

and the square root

$$\sqrt{\frac{1}{1-ik\eta}}=\sqrt{\frac{1+ik\eta}{1+k^2\eta^2}}\approx 1+\frac{1}{2}ik\eta. \tag{C2}$$

With those series, the exponent of $I[k,k_+,k_0]$ in (B6) is expanded

$$\begin{aligned} &\frac{ik}{1-ik\eta}\left(q^2+\frac{k_+}{k}q+i\frac{k_+^2\eta}{4k}\right)+ik_0 \\ &= -\frac{1}{4}\eta c_0^2+iqc_0 \\ &+ i\left(-\frac{1}{2}\eta c_0c_1+qc_1+q^2d+d_0\right)\tau \\ &+ \left(-\frac{1}{4}\eta c_1^2-\frac{1}{2}\eta c_0c_2+iqc_2\right)\tau^2+O(\tilde{g}^3) \end{aligned} \tag{C3}$$

Similarly, all the other $I[\cdot,\cdot,\cdot]$ can be expanded by replacing (k,k_+,k_0) with the relevant parameters. The Holevo quantity $\chi(\rho)$ in (38) is determined by μ and $E(\beta)$, so what we need is to compute $\mu(\tau)$ in (33). By substituting the (C1), (C2) and (C3) into (B6) and (B7), $\mu(\tau)$ is expanded up to $O(\tau^2)$ and $O(\tilde{g}^2)$,

$$\begin{aligned} &\mu^2(\tau) \\ &= 1-\frac{1}{2}\tau^2\eta c_1^2-\frac{1}{8}\tau^2\eta d_0^2c_0^2\cos^2(qc_0/2) \\ &+ O(\tau^3)+O(\tilde{g}^3) \\ &= 1-2\tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)] \end{aligned} \tag{C4}$$

and hence

$$\begin{aligned} \mu(\tau) &= \sqrt{1-2\tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)]} \\ &= 1-\tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)] \\ &+ O(\tau^4). \end{aligned} \tag{C5}$$

Keeping in mind $\mu(0)=1$, $\mu'(0)=0$,

$$\begin{aligned} &\chi(\mu=1)=0 \\ &\left.\frac{\partial\chi}{\partial\mu}\right|_{\mu=1}=0 \\ &\left.\frac{\partial^2\chi}{\partial^2\mu}\right|_{\mu=1}=\frac{E^2(\beta)}{2}[\log(1-E(\beta))-\log(1+E(\beta))] \end{aligned} \tag{C6}$$

$$\begin{aligned} \chi(\rho) &\approx \frac{E^2(\beta)}{4}[\log(1-E(\beta))-\log(1+E(\beta))]\mu''(0) \\ &\approx \frac{E^2(\beta)}{2}\log\frac{1+E(\beta)}{1-E(\beta)} \\ &\times \tau^2\eta^2\tilde{g}^2[\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi\cos^2(2q\tilde{g}\sin\phi)] \\ &\approx \frac{E^2(\beta)}{2}\log\frac{1+E(\beta)}{1-E(\beta)}(\cos^2\phi+4\tilde{\Delta}^2\sin^2\phi)\eta^2\tilde{g}^2\tau^2. \end{aligned} \tag{C7}$$

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Chapter 4

Conclusion and outlook

4.1 Conclusion

Using two simple but important models, where a central system is a harmonic oscillator and environmental systems are either a collection of harmonic oscillators or spins, i.e. QBM and the boson-spin model, respectively, we investigated how our classical world can emerge from quantum mechanics based on the recent approach called spectrum broadcast structure (SBS), which is a stronger version of quantum Darwinism. Specifically, we looked for one of major classical properties, "objectivity" emerging from pure quantum nature. The objectivity can be characterized by two quantities, which we call the objectivity markers, i.e. the decoherence factor and the generalized overlap. SBS and quantum Darwinism explain how classicality is approached by interactions with the environment, through spreading of the information about the system into the environment with a large redundancy. In both models, we used the thermal environment and considered the situation that a central system state is hardly affected by the environment (Born-Oppenheimer or recoilless approximation). This approximation is especially suited to see information transfer to the environment.

In our study of QBM model, we showed that the objectivity is identified by two length scales, the decoherence length and the novel distinguishability length characterizing the decoherence factor and the environmental distinguishability, respectively. Under a hot thermal initial state of the oscillator environment, we found that the decoherence length is much smaller than the distinguishability length. This implies that decoherence is easier to achieve than distinguishability. They are also in the complementary relation.

In the boson-spin model, we analyzed the dynamics of the system in order to compute the objectivity markers by using the Floquet theory and the high-frequency expansion in an interesting way. First, it is confirmed that the most necessary condition for the objectivity is a large number of environments. Second, we found that the initial condition of trajectories for the central harmonic oscillator in position space strongly affects the objectivity. The smaller initial amplitude of a harmonic oscillator associated with the initial phase is given, the larger decoherence is created. Third, we found that achieving objectivity implies a breaking periodicity in the objectivity markers. The phase of a classical trajectory for a central harmonic oscillator and a self energy in a spin system play an important role in breaking a periodicity and hence achieving objectivity.

Finally, moving to the quantum information point of view, we analyzed the continuous Holevo quantity, the upper bound for a channel capacity formed by interaction in the boson-spin model. The behaviour of the Holevo quantity in time shows how the maximum information can be transferred from the continuous system, an oscillator to the spin-$\frac{1}{2}$ system. After the initial growth, the Holevo quantity is saturated to the maximum with some fluctuations. This maximum can be surprisingly high even for a single spin. This is mainly due to the existence of a spin self-Hamiltonian. Also, the dependence of the initial trajectory verifies the minimum initial amplitude of a harmonic oscillator maximizes the distinguishability by showing the maximum Holevo quantity.

4.2 Outlook

Quantum Darwinism and SBS were proposed to derive a classical limit from quantum mechanics by emphasizing the role of environment. First, the main difficulty in obtaining general solutions is the complexity in open quantum systems. Most studies in open quantum systems adopt various approximations focusing only on particular physical regimes. On the other hand, there are still untouched parts. First, the physical origin of and legitimacy of the partial tracing are not clear, though it is an important tool in open quantum systems. Finally, there is no mechanism to explain how quantum evolution selects one particular quantum state in measurement, which is “the big problem” in quantum measurement theory.

Besides those well-known issues addressed above, problems to be tackled in simple open systems is to investigate how information in physical space propagates. In our systems we have considered only interactions in one spacetime point for simplicity, although in reality each spacetime point can be an interaction point. In more complicated models of environments, e.g. where there is an onion-like structure of interactions between different parts. In this view it is known that the propagation of quantum information is naturally limited with a speed, called the Lieb-Robinson bound [52]. This bound was derived in lattice systems but the existence of speed limit is the result in general lattice systems. I believe that research in this direction will open new areas in open quantum systems in both dynamical and relativistic point of view.

4.2.1 Open questions

This thesis is a step towards hopefully more general theory of the quantum-to-classical transition. We wish to address some open problems that we have not solved yet:

1. To compute higher order corrections in the influence functional in QBM since the exact action for coupled oscillators is well-known in an integral form.

2. To search for a general complementarity relation between the decoherence factor and the generalized overlap in QBM and boson-spin model;
   We found the complementary relation between the decoherence and the distinguishability lengths under certain conditions in QBM but it seems that there might exist a more general relation.

3. To compute effects from non-bi-linear interaction to the objectivity markers;
   Computations would be more difficult since the convenient multiplicative property in the objectivity markers cannot be used.

4. To compute higher order corrections to the Born-Oppenheimer ansatz for the effective Hamiltonian.

5. To include a more general phase;
   In more general consideration, a classical trajectory in the effective Hamiltonian for environment can have its own phase for each different amplitude.

6. To compute higher order corrections for the decoherence factor and the generalized overlap in the high frequency expansion in the boson-spin model.

7. To compute the Holevo bound in the boson-spin model with a multi-spin system.

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