Bi-local geodesic operators as a tool of investigating the optical properties of spacetimes

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Dedicated to my grandmother

Abstract

In my thesis, I present one particular example of the formalism capable of describing the propagation of a family of light rays in a curved spacetime. It is based on the resolvent operator of the geodesic deviation equation for null geodesics which is known as the *bilocal geodesic operator* (BGO) formalism. The BGO formalism generalizes the standard treatment of light ray bundles by allowing observations extended in time or performed by a family of neighbouring observers. Furthermore, it provides a more unified picture of relativistic geometrical optics and imposes a number of consistency requirements between the optical observables.

The thesis begins with a brief introduction of the transfer matrix and its relativistic versions known as the Jacobi propagators and the bilocal geodesic operators. A brief literature review is given illustrating various interpretations of bilocal operators in contexts of extended objects, gravitational waves, and seismology.

The second chapter is dedicated to the basics of differential geometry with an emphasis on the geometry of the tangent bundle, which will later provide a foundation for the BGO formalism. We start from the coordinate systems on the base manifold and its tangent bundle and then study coordinate-dependent and independent representations of induced higher-dimensional vectors. Next, we discuss the notion of the geodesic flow and define the BGOs in terms of this flow. Finally, we display how certain results obtained on the tangent bundle lead to the differential equations for BGOs.

In the third chapter, I present my original work on a fully analytical derivation of the BGOs for static spherically-symmetric spacetimes. Firstly, I summarize two different techniques to obtain an exact solution, both resting upon symmetries of the spacetime and integrability of geodesic (deviation) equations. The methods are then applied to derive the solution both in coordinate and parallel-transported frames. Finally, the results are used to study optical distance measures in Schwarzschild spacetime.

In the fourth chapter, I present several theorems about the inequality concerning optical distance measures. The result is valid irrespective of spacetime symmetries or lack thereof and depends on the validity of General Relativity together with rather standard assumptions about the matter content and propagation of light in the Universe. The chapter concludes with a short discussion about the possibility of experimental verification or rejection of the mathematical result.

In the last chapter, I summarize the content of the thesis and ponder its possible extensions.

Streszczenie

W pracy prezentuję formalizm opisujący propagację wiązek promieni światła, nazywany formalizmem *bi-lokalnych operatorów geodezyjnych* (BGO). Jego podstawą jest rezolwenta równania dewiacji geodezyjnych dla geodezyjnych zerowych. Formalizm BGO uogólnia standardowy opis wiązek światła, pozwalając na obserwacje rozciągające się w czasie lub wykonane przez rodzinę obserwatorów znajdujących się blisko siebie. Ponadto wprowadza on ujednolicony opis relatywistycznej optyki geometrycznej, co pozwala udowodnić ścisłe relacje między obserwablami i krzywizną czasoprzestrzeni.

Dysertacja rozpoczyna się krótkim wprowadzeniem do tematyki macierzy przejścia (transfer matrix) i jej relatywistycznych uogólnień, zwanych czasami propagatorami Jacobiego lub bi-lokalnymi operatorami geodezyjnymi. Podaję też krótki przegląd literatury ilustujący rozmaite zastosowania tych opertatrów w kontekście równań ruchu rozciągłych ciał w ogólnej teorii względności, fal grawitacyjnych i sejsmologii.

Drugi rozdział poświęcony jest matematycznym postawom geometrii różniczkowej, przede wszystkim geometrii wiązki stycznej. Materiał ten będzie potem podstawą formalizmu BGO. Rozpoczynam od przypomnienia pojęcia układu współrzędnych na czasoprzestrzeni i na wiązce stycznej oraz opisuję zależne i niezależne od układu współrzędnych metody rozkładu wektorów stycznych do wiązki stycznej. Natępnie opisuję analog kongruencji geodezyjnych na wiązce stycznej, zwany geodezyjnym przepływem (*geodesic flow*) i definiuję przy jego pomocy bilokalne operatory geodezyjne. Na koniec wyprowadzam równania różniczkowe na te obiekty korzystając z geometrii wiązki stycznej.

W trzecim rozdziale prezentuję swoją oryginalną pracę, w której wyprowadzam dokładne wyrażnenia na BGO dla statycznych, sferycznie symetrycznych czasoprzestrzeni. Opisuję najpierw dwie techniki otrzymywania dokładnych rozwiązań, obie korzystające z wielkości zachowanych i całkowalności równań geodezyjnych w sferycznie symetrycznych czasoprzestrzeniach. Metody te stosuję potem do wyprowadzenia rozwiązań wyrażonych w reperze współrzędnościowym i transportowanym równolegle. Na koniec, korzystając z tych wyników, badam optyczne miary odległości (odległość paralaktyczną i rozmiaru kątowego) na czasoprzestrzeni Schwarzschilda.

W czwartym rozdziale prezentuję dwa ogólne twierdzenia dotycząnie nierówności między optycznymi miarami odległości. Są one prawdziwe bez względu na to, czy czasoprzestrzeń jest symetryczna, czy nie. Zakładają one prawdziwość ogólnej teorii względności, standardowe warunki na rozkład materii oraz przybliżenie optyki geometrycznej dla propagacji światła. Na końcu rozdziału dystkutuję pokrótce możliwość eksperymentalnej weryfikacji tej nierówności.

W ostatnim rozdziale podsumowuję treść dysertacji i rozważam rozmaite możliwe uogólnienia wyników.

Declaration

The work presented in this thesis was completed between October 2017 and November 2022 while the author was a research student under the supervision of Prof. Mikołaj Korzyński at the Center for Theoretical Physics, Polish Academy of Sciences. The coursework was completed between October 2017 and July 2022 at the Institute of Physics, Polish Academy of Sciences, and Warsaw University. No part of this thesis has been submitted for any other degree at the Center for Theoretical Physics, Polish Academy of Sciences, or any other scientific institution.

The thesis is based on the following papers:

1. Chapter 3: **J. Serbenta** and M. Korzyński, *Bilocal geodesic operators in static spherically-symmetric spacetimes*, Class. Quantum Grav., vol 39, 155002, 2022 [117],

2. Chapter 4: M. Korzyński and **J. Serbenta**, *esting the null energy condition with precise distance measurements*, Phys. Rev. D, 105, 084017, 2022 [62].

Descriptions of the contributions of the authors are given separately before each publication in the respective chapters.

In addition to the work presented in this thesis, the author has also contributed to the following closely related articles:

3. M. Grasso, M. Korzyński, and **J. Serbenta**, *Geometric optics in general relativity using bilocal operators*, Phys. Rev. D, vol. 99, 064038, 2019 [61],

4. M. Korzyński, J. Miśkiewicz, and **J. Serbenta**, *Weighing the spacetime along the line of sight using times of arrival of electromagnetic signals*, Phys. Rev. D, vol. 104, 024026, 2021 [41].

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Acronyms and notations

List of acronyms

Acronym	Meaning
BGO	Bilocal Geodesic Operator
GDE	Geodesic Deviation Equation
GE	Geodesic Equation
NEC	Null Energy Condition
ODE	Ordinary Differential Equation
SNT	Semi-Null Tetrad

Notations and their meanings

Notation	Description
g	metric tensor
∇	Levi-Civita connection
$\Gamma^{\alpha}_{\ \mu\nu}$	Christoffel symbol
M	semi-Riemannian manifold
T_pM	tangent space at $p \in M$
$\left(\left. \frac{\partial}{\partial x^\mu} \right\|_p \right)$	coordinate basis vector of T_pM
$\mathbf{X}_p = v^\mu \left. \frac{\partial}{\partial x^\mu} \right\|_p$	element of T_pM
TM	tangent bundle of M
$T_{(p,\mathbf{X}_p)}TM$	tangent space to the tangent bundle at $(p, \mathbf{X}_p) \in TM$
$\left(\left. \frac{\partial}{\partial x^\mu} \right\|_p , \left. \frac{\partial}{\partial v^\mu} \right\|_p \right)$	induced basis vector of $T_{(p,\mathbf{X}_p)}TM$
$\mathbf{Y}_{(p,\mathbf{X}_p)}$	element of $T_{(p,\mathbf{X}_p)}TM$
Greek indices $\alpha, \beta, \ldots$	run from 0 to 3
Latin lowercase indices $i, j, \ldots$	run from 1 to 8
$\mathbf{G}$, G^i	geodesic spray and its component
π_M	projection map from TM to M
γ, γ^μ	geodesic curve and its representation in a coordinate system
$\dot{\gamma}$	ordinary derivative of γ wrt the affine parameter
ϕ	coordinate function of a coordinate chart; geodesic flow
$\wedge$	wedge product
$\lrcorner$	interior product
$\mathcal{L}_{\mathbf{X}}$	Lie derivative along an element of tangent space
$\cong$	is isomorphic to
$\oplus$	direct sum
subscripts H, V	a part of a vector fully contained in horizontal or vertical subspace
superscript T	generalized transpose of operator

Contents

Chapter 1

General introduction

The theory of transfer matrices and propagators is a fairly common tool in theoretical physics and mathematics. In many physical systems the propagation of disturbances can be described as the propagation of waves, which at some level of approximation can be described using resolvent operators mapping some initial state of the physical system to the state at some later time. This approach has been used in many branches of physics ranging from the theory of elasticity to the quantum field theory.

In astrophysics and cosmology, most observations require a good understanding of the propagation of electromagnetic waves through a curved spacetime. In the high-frequency limit the propagation of waves, which is almost always applicable in astrophysical situations, can be well approximated by propagation along null geodesics [28, 53, 54, 79, 90, 48]. Still, for sources of a finite extent we need to consider a whole family of null geodesics, corresponding to the light rays from different points of the source's cross-section. While the general problem of light ray propagation is non-linear, it can be simplified for geodesics remaining close to a given one. Namely, if one geodesic is known, then the behaviour of neighbouring null geodesics can be described sufficiently well by the first order *geodesic deviation equation* (GDE). As a system of linear ordinary differential equations, it admits the propagation matrix or resolvent formulation, in which the solution at a later time can be obtained from the state at an earlier moment by the action of a linear operator.

In the most general sense, the geodesic deviation equation relates the relative motions of physical particles or light rays to the spacetime geometry. However, to our knowledge, there has been still no complete discussion in the literature of the transfer matrix technique to the geodesic deviation equation in the case of null geodesics, representing light rays. The goal of this thesis is to fill in this gap and provide a new perspective on geometric optics in general relativity. Following our previous work, we will refer to this framework as the *bi-local geodesic operator*, or BGO formalism.

The BGO formalism can describe all nontrivial optical effects as experienced by a source and an observer in two regions of spacetime connected by a null geodesic. The effects are considered at the lowest order in the perturbations of the positions of the source and the observer. The main starting point is the GDE around a null geodesic. Note that this equation is fully relativistic, in the sense that it makes no approximations regarding the metric tensor, such as the Newtonian or post-Newtonian approximation. On the other hand, unlike the more familiar optical scalars formalism (also known as Sachs formalism) [109, 91] it is based on a linear system of equations. We show in this work that this opens up the possibility to apply directly the machinery of linear algebra to many problems in geometrical optics.

The formalism can describe, among other things, the standard effects of gravitational lensing, in the form of the magnification and elliptical deformation of the source's image. This information is stored in a lower-dimensional operator known as the Jacobi matrix [115, 91][1]. However, in its most general form it also includes geodesic perturbations related to the variation of the observer's position or the observation time. This way the BGO formalism expands the standard set of observables by including the effects of the observer's displacements, i.e the parallax effects, as well as the time variations of the redshift and position, i.e. the drift effects. Furthermore, in the BGO formalism, it is relatively easy to show that these effects are in fact related to each other and derive precise mathematical relations between the observables. As an example, the second paper from this thesis [62] explores general relations between the magnification of a source and the parallax effects.

Let us mention that the mathematical apparatus of propagator matrices along the timelike geodesics in GR has been considered earlier in the context of timelike geodesics. It was used in the studies of the motion of massive extended [19] and charged [18] bodies, and the propagation of gravitational waves and their memory effects [31] under the name of Jacobi propagator[2]. In the null case the resolvent formalism was occasionally used as a technical tool to describe gravitational lensing in inhomogeneous Universe models [32]. However, the authors restrict the space of solutions of the GDE to those which correspond to momentary observations, excluding this way any drift effects.

A different treatment of propagation of light rests upon Synge's world function [99, 123, 128, 61]. Namely, it turns out that the first and second derivatives of the world function have a direct connection to the solutions of the geodesic equation and the GDE respectively. This approach is less useful in practice since the formula for the world function requires full knowledge of the solution to the GE, which in general is very difficult. Therefore, we do not follow this approach in this Thesis, although we note here that the BGO formalism can indeed be formulated in the language of the world function [61].

As mentioned above, the wave-like propagation of disturbances and their high-frequency limit can also be found in other contexts, for example, in deformable media which falls under the theory of seismology. In this context, the solutions of the elastodynamic equations in high-frequency regime can be approximated by seismic rays, and their propagation in an appropriate approximation is ruled by bi-local operators, quite similar to the BGOs [30, 13, 14, 129].

More recently, the BGO approach has also appeared in other works in a slightly different formulation. In [37, 16] the authors applied the perturbative expansion of the solution to the GDE in powers of the Riemann tensor, and later used it to estimate the optical scalars. In [126], the non-relativistic counterpart of BGOs was presented, together with an extensive discussion of its symplectic properties. However, the authors again limit their considerations to momentary observations and do not consider the drift effects.

The aim of the first part of this thesis is to provide a solid geometrical foundation for the BGO formalism valid for any smooth manifold equipped with a metric and its Levi-Civita connection. Then we will use this to display the relation of BGOs to various optical effects and observables.

The two papers included in this Thesis illustrate the application of the BGOs. In the first paper we consider the BGOs in static spherically symmetric spacetimes which are derived in two different ways. The first approach one to vary the solution to the GE with respect to its initial conditions. However, a simple variation leads to non-covariant results. The problem is averted by making sure the variations themselves are expressed in a covariant language. The

[1]Here by the Jacobi matrix D_{ij} we mean the resolvent operator of the GDE projected on the screen space. In the weak lensing literature there is a similar but different Jacobian matrix of the lens equation (also known as the amplification or magnification matrix [115]) defined by $A_{ij} = \frac{\partial \beta^i}{\partial \theta^j}$. The crucial difference is that the second matrix requires a clear separation of the background and the lens. In general, they are related by $D_{ij} = (D_B)_{ik} A_{kj}$ with D_B being the solution on the background, but in practice for homogeneous and isotropic spacetimes $(D_B)_{ij}$ can be replaced by an averaged area distance, i.e. $\bar{d}_A \delta_{ij}$[8].

[2]In the mathematical literature of Lorentzian manifolds it is known as the Jacobi tensor [5]

geometric interpretation of this requirement will be outlined in the mathematical introduction to the Thesis, in Sec. 2.4. The second method we propose directly involves the GDE, its conservation laws, and symplectic properties, whose geometric foundations will be laid in Secs. 2.5-2.7. Finally, the BGOs are used to describe optical distance measures in Schwarzschild spacetime.

In the second paper, we prove several distance inequalities assuming GR, propagation of light in vacuum, and reasonable conditions on the matter. Inequalities themselves are proved with the use of Sachs optical equations [109], while the BGOs allow us to relate the parallax effect to the behaviour of a special bundle of rays. This relation is crucial for the proof of inequality.

We would also like to bring attention to a complementary work on BGOs [43, 42, 40]. In these papers, the authors introduce a Mathematica package to calculate the BGOs in arbitrary evolving spacetimes. Similarly to our approach here, the authors rewrite the GDE as two systems of the first order ODEs which are then integrated backward in time. Then, to integrate the solution forward in time they apply the symplecticity of the total resolvent operator. One of the goals of this thesis is to reveal a deeper geometrical meaning of the first-order formulation and explain why the total transfer matrix is not just a formal construction.

Chapter 2

Mathematical preliminaries

2.1 Introduction

The *bilocal geodesic operators*, or BGOs, are a very useful tool in the study of light propagation between distant regions of spacetime. They relate deviations of the initial point and its tangent vector to their counterparts at the other end of the curve. Since they are defined for arbitrary initial data sets, they describe how a curved spacetime influences the evolution of any perturbed geodesic, provided that the perturbation is not too large. In the context of light propagation, the BGOs describe how a perturbed light ray is bent by the spacetime curvature, or, more broadly, how any whole infinitesimal bundle of light rays is affected by the curvature. They also describe how the results of observations performed at the observer's endpoint of the geodesic segment vary with respect to the observer's proper time. In the relativistic and cosmological literature these variations are known as drifts. Physical aspects of drift effects [60], general relativistic parallax [62], times of arrivals [61], and BGOs [41] have been discussed previously, along with some geometrical remarks.

This chapter, apart from serving as a mathematical introduction, contains also the first part of the new results of the Thesis. We show here how we can define the BGOs with the help of the geometry of the tangent bundle. We will therefore begin by briefly reviewing the basic notions of differential geometry, such as the tangent bundle, geodesics, and their lifts to the tangent bundle. Recall that the geodesics induce a special vector field on the tangent bundle known as the geodesic spray, which in turn defines the geodesic flow. This flow is analogous to a fluid flow in the mechanics of continuous media. We will show that the tangent map to this flow, or the deformation gradient tensor of the fictitious fluid related to the flow, naturally splits into 4 covariant, bilocal operators we can identify with the BGOs.

In the meantime, we also develop a geometrical formulation of the techniques used in the first paper [117] contained in this Thesis, in which we present two methods of solving the GDE exactly and determining the BGOs. One of them requires the computation of the total variation of a geodesic. However, the variation needs to be decomposed in a covariant manner into the position variation at a point and the variation of the tangent vector. As we will see, the meaning of this step is easy to understand if we express it in the language of the tangent bundle. The key construction in this case is the covariant splitting of the tangent space to the tangent bundle into the horizontal and vertical subspaces.

The geodesic flow preserves the symplectic structure of the tangent bundle [126, 59]. It turns out that this has a direct implication for the symplectic properties of BGOs and light propagation, see for example the proof of the distance inequality [62]. Therefore we will also discuss the relation between the BGOs and the symplectic geometry of the tangent bundle in the later part of this section.

As a general introduction to the topics presented below, we recommend the following references [72, 132, 6, 68, 110, 69, 52], while more specialized references are given in the text.

2.2 Charts and projections

The purpose of this section is to introduce some basic elements of differential geometry on which we will build the formalism for BGOs. We begin with the basic concepts of coordinates and tangent spaces on a manifold and then show how they apply to the tangent bundle. Since the main goal of this work is to understand the physics of light propagation, we will also discuss the geometric and physical interpretation of the elements of tangent spaces in the context of geometric optics in curved spacetimes.

We begin by setting up the model for spacetime. The spacetime will be represented by a smooth 4-dimensional Lorentzian manifold M with metric g and its Levi-Civita connection ∇. For its signature we choose $(-+++)$. We note that although we are interested in manifolds describing the spacetime geometry, the theory to be presented below can be equally applied to Riemannian or semi-Riemannian manifolds of arbitrary finite dimensionality.

Every element of M is simply a point, or more precisely, an event in spacetime. In order to assign coordinates to this point we need to introduce a *chart*. By chart we mean the structure $\left(U, \phi, \mathbb{R}^4\right)$, where $U \subset M$ is an open set and ϕ is a map from U to $\mathbb{R}^4$. If (x^μ) is our coordinate system on U, then

$$\forall p \in U : \phi\left(p\right) = \left(x^\mu\left(p\right)\right). \tag{2.1}$$

That is, ϕ allows us to assign to each point of the spacetime 4 numbers x^μ known as coordinates, in a smooth way. Note that from the point of view of differential geometry x^μ does not transform as a component of a vector. For the sake of simplicity, we assume the manifold M and the metric g to be smooth.

Tangent vectors to a manifold at a point form a vector space known as the *tangent space*. While there are several ways to introduce it, we will follow here the standard approach, resting upon linear maps $X : C^k\left(M\right) \to \mathbb{R}$ acting on functions, which possess Leibniz-like properties of differentiation and are known as *derivations* [110, 70]. At any $p \in M$ the set of derivations forms a vector space with a well-defined natural basis $\left\{ \left.\frac{\partial}{\partial x^\mu}\right|_p \right\}$, given by partial derivatives with respect to all coordinates, known as the *coordinate basis*. Any so-constructed vector space at $p \in M$ is denoted by T_pM and called *the tangent space at p*.

The formal definition using linear operators is somewhat too abstract for our purposes, therefore we will also consider its other geometric interpretations. First, consider all differentiable curves passing through a point $p \in M$. Then all their tangent vectors at p span a vector space at this point. A more physical way of thinking about tangent vectors is to depict them as all possible small displacements of the point p. We can think either of infinitesimal displacement or, alternatively, of small but finite displacements, such that the scale of the displacement is much smaller than the curvature scale or any other relevant length scale in the neighbourhood of p. The latter interpretation requires the use of Riemann's normal coordinates, see [130, 71]. In other words, a vector in T_pM can also stand for a perturbation of the position of an object in the spacetime or a more general manifold. In this Thesis we will make extensive use of this interpretation since it fits best the physical situations we want to describe.

In principle, the manifold and its tangent spaces provide a sufficient background to study problems describable by 2nd order ordinary differential equations (ODEs), such as the GDE. However, the same problems can be recast as a 1st order ODE in a higher-dimensional setting [55]. The order reduction is advantageous because it allows us to study all degrees of freedom on equal footing and apply the resolvent formalism to the equation. In the case of the geodesic equation and the geodesic deviation equation this can be achieved by considering the whole *tangent bundle* TM [107, 108, 22, 45, 7, 131], which is a disjoint union of all tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM = \left\{ (p, \mathbf{X}_p) : p \in M, \mathbf{X}_p \in T_pM \right\}. \tag{2.2}$$

It comes with a natural *projection map* π_M satisfying

$$\pi_M : TM \ni (p, \mathbf{X}_p) \to p \in M. \tag{2.3}$$

One important property of the tangent bundle is that it is also a manifold. Therefore, we can introduce on it geometrical objects such as charts, coordinates, curves and vector fields.

While there are many possible charts on TM, there is a class of natural charts induced by the charts in the base manifold M. They can be considered as follows. Let $\tilde{U}$ be an open set in TM defined by $\tilde{U} = \pi_M^{-1}(U)$, where U is an open set in M defining a chart. Suppose the coordinates of $p \in U$ read (x^μ). Suppose also that a vector $\mathbf{X}_p \in T_pM$ is tangent to M at p. A vector can be expanded with respect to the local basis as $\mathbf{X}_p = v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p$, where v^μ are the components of X_p in the coordinate basis. Let $\tilde{\phi}$ be a map from $\tilde{U}$ to $\mathbb{R}^8$ such that

$$\forall \, (p, \mathbf{X}_p) \in \tilde{U} : \tilde{\phi}\left((p, \mathbf{X}_p)\right) = \left(\tilde{x}^\mu, v^\nu\right), \tag{2.4}$$

where

$$\pi_M\left(\tilde{x}^\mu, v^\nu\right) = x^\mu, \tag{2.5}$$

or equivalently

$$\tilde{x}^\mu\left(p, \mathbf{X}_p\right) = x^\mu\left(p\right). \tag{2.6}$$

Then the structure $\left(\tilde{U}, \tilde{\phi}, \mathbb{R}^8\right)$ defines a (smooth) chart on TM and $\left(\chi^i\right) = \left(\tilde{x}^\mu, v^\nu\right)$ are the local coordinates. Note that x^μ and $\tilde{x}^\mu$ are images of different mappings and so cannot be identical, but in practice their meaning is synonymous. Therefore, from now on we will drop the tilde and denote the coordinates on TM as (x^μ, v^μ). These coordinates are known as the *adapted* [112], *induced* [132], or *canonically associated* [6] coordinates. The dimension of the tangent bundle is twice the dimension of the base manifold, so in our case it is 8.

In comparison with M, its tangent bundle admits a much greater variety of curves, simply because of its higher dimension: for every curve on the base manifold there is an entire family of curves related to the former by the projection π_M. However, among all the curves in the family corresponding to a given curve in M there is a unique one known as the *lift*, which is defined as follows. Consider a differentiable curve on the spacetime $\gamma : \mathbb{R} \supseteq I \to M$. Let $\Gamma : \mathbb{R} \supseteq I \to TM$ be a curve in the tangent bundle such that for all $\lambda \in I$ the curve Γ satisfies $\Gamma\left(\lambda\right) = \left(\gamma\left(\lambda\right), \dot{\gamma}\left(\lambda\right)\right)$, that is, which in local coordinates of TM reads

$$\begin{cases} x^\mu\left(\lambda\right) & = \gamma^\mu\left(\lambda\right) \\ v^\mu\left(\lambda\right) & = \dot{\gamma}^\mu\left(\lambda\right). \end{cases} \tag{2.7}$$

Then we call Γ the lift of γ. Obviously, $\pi_M\left(\Gamma\right) = \gamma$. The main benefit of lifting a curve to TM is that the curves passing through the same point in M but differing by their tangent vectors do not cross on TM.

Finally, since our goal is the description of the perturbations of curves, we need to introduce the tangent space to TM. Pick any point $(p, \mathbf{X}_p)$ in TM. Since TM is a manifold, at every point it possesses a tangent space that we will denote by $T_{(p,\mathbf{X}_p)}TM$. Again, as a vector space, it has basis, and the coordinates (x^μ, v^ν) induce the *associated basis* $\{\mathbf{f}_{x,\mu}, \mathbf{f}_{v,\mu}\}$, where

$$\begin{aligned} \mathbf{f}_{x,\mu} &= \left.\frac{\partial}{\partial x^\mu}\right|_{(p,\mathbf{X}_p)} \\ \mathbf{f}_{v,\mu} &= \left.\frac{\partial}{\partial v^\nu}\right|_{(p,\mathbf{X}_p)}, \end{aligned} \tag{2.8}$$

i.e. the basis vectors are given by the partial derivatives with all other coordinates fixed. In analogy with the previous case, the elements of $T_{(p,\mathbf{X}_p)}TM$ can be understood as either tangent vectors to curves in TM passing through $(p, \mathbf{X}_p)$ or as infinitesimal perturbations of the point $(p, \mathbf{X}_p)$. In the latter case we may think of it collectively as the perturbation of both the point $p \in M$ *and at the same time*, of the vector $\mathbf{X}$ tangent to M at p.

2.3 Geodesic flow

Geodesics are arguably the most important family of curves in general relativity. They represent the worldlines of massive and massless particles interacting only with the gravitational field. While geodesics themselves are curves on the base manifold, it is often useful to consider them on the level of the tangent bundle. The main advantage of this approach is that the geodesic motion in the tangent bundle can be represented by a flow of the whole tangent bundle along a vector field, called the *geodesic flow*.

The geodesic flow is the central notion of this chapter, because, as we will see later, the BGOs can be defined in terms of its properties. Namely, the BGOs are related to its action on infinitesimal volume elements, or, more precisely, to the deformation gradient of the geodesic flow. Therefore we will review in this chapter its definition and discuss some of its fundamental properties.

In general relativity, a freely falling particle follows a uniquely defined curve known as the *geodesic*. It is a smooth curve $\gamma : \mathbb{R} \supset I \to M$ such that its tangent vector is parallel transported along itself:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0. \tag{2.9}$$

In coordinates this equation reads

$$\frac{d^2\gamma^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta} \frac{d\gamma^\alpha}{d\lambda} \frac{d\gamma^\beta}{d\lambda} = 0, \tag{2.10}$$

where $\gamma^\mu = x^\mu \circ \gamma$ and $\dot{\gamma}^\mu = \dfrac{d\gamma^\mu}{d\lambda}$. Moreover, we also assume that the geodesic depends smoothly on its initial data. This property will be utilised later to find solutions of the GDE.

Usually, geodesics are treated as curves in the base manifold M. However, we can also consider closely related curves generated by lifting the geodesics to TM. It turns out that this construction is closely related to the Hamiltonian approach to the geodesic motion. More precisely, the phase space for geodesic motion is naturally modeled by the cotangent bundle T^*M. In a manifold without additional structures T^*M is unrelated to TM. However, the metric g provides a natural isomorphism between both spaces, so TM and T^*M can be considered interchangeably.

A particularly useful representation of all the geodesics on the tangent bundle is known as the geodesic flow [88]. It is a family of diffeomorphisms $\phi_\lambda : TM \times \mathbb{R} \to TM$ of the tangent bundle defined by

$$\phi_\lambda(p, \mathbf{X}_p) = \left(\gamma_{(p,\mathbf{X}_p)}(\lambda), \dot{\gamma}_{(p,\mathbf{X}_p)}(\lambda)\right) \tag{2.11}$$

where γ and $\dot{\gamma}$ is the geodesic and its derivative with respect to λ with initial conditions $\gamma(\lambda_0) = p$ and $\dot{\gamma}(\lambda_0) = \mathbf{X}_p$.

The fact that ϕ_λ is a flow can be seen by checking that is satisfies $\phi_{\lambda_1+\lambda_2} = \phi_{\lambda_1} \circ \phi_{\lambda_2}$. Also, since for geodesics the length of their tangent vectors are conserved, it follows that the geodesic flow preserves a number of subbundles of TM. Namely, let S^+TM, S^-TM and S^0TM be subbundles of TM in which the geodesics respectively satisfy the constraints $\dot{\gamma}^\mu \dot{\gamma}_\mu = 1$, $\dot{\gamma}^\mu \dot{\gamma}_\mu = -1$ and $\dot{\gamma}^\mu \dot{\gamma}_\mu = 0$. Then S^+TM, S^-TM and S^0TM are preserved by the flow ϕ_λ.

2.4 Vertical and horizontal subspaces

The dimension of the tangent bundle is twice the dimension of the spacetime. Moreover, the tangent space of the tangent bundle has an associated basis (2.8), whose form suggests the possibility of splitting any vector into two independent parts proportional to vectors $\partial/\partial x^\mu$ and $\partial/\partial v^\mu$ respectively. However, this kind of splitting is not covariant, i.e. it depends on the adapted coordinates we have chosen. Nevertheless, in the case of spacetime endowed with a non-degenerate metric and its Levi-Civita connection, we have additional structure. As we will show in this section, this leads to an invariant splitting of $T_{(p,\mathbf{X}_p)}TM$ into two 4-dimensional spaces, both isomorphic to T_pM [39]. In other words. the construction is coordinate system-independent and permits the representation of vectors in $T_{(p,\mathbf{X}_p)}TM$ as pairs of tangent vectors to the spacetime. This splitting is crucial for the definition of the BGOs, so in this section we will describe it in detail and discuss its physical interpretation as well as its relation to the observations in geometrical optics.

Previously we have shown that the tangent bundle TM has a natural coordinate system (x^μ, v^μ), while its tangent space $T_{(p,\mathbf{X}_p)}TM$ can be equipped with the basis $\{\mathbf{f}_{x,\mu}, \mathbf{f}_{v,\mu}\}$ defined by the partial derivatives with respect to the coordinates, see (2.8). In the associated basis the expansion of a vector $\mathbf{X} \in T_{(p,\mathbf{X}_p)}TM$ reads

$$\mathbf{X} = X_x^\mu \frac{\partial}{\partial x^\mu} + X_v^\mu \frac{\partial}{\partial v^\mu}. \tag{2.12}$$

Vector $\mathbf{X}$ corresponds to a perturbation of a point in TM, i.e. a perturbation of a point in M and a vector tangent at that point. In the decomposition above X_x^μ stands obviously for the variation of position in M, and X_v^μ for the variation of the vector. This decomposition, however, is not covariant. Consider a general transformation of adapted coordinates on TM, induced by coordinates on M. Then the transformation from (x^μ, v^μ) to $\left(\tilde{x}^{\tilde{\mu}}, \tilde{v}^{\tilde{\mu}}\right)$ reads

$$\begin{cases} \tilde{x}^{\tilde{\mu}} = \tilde{x}^{\tilde{\mu}}(x^\mu) \\ \tilde{v}^{\tilde{\nu}} = v^\nu \dfrac{\partial \tilde{x}^{\tilde{\nu}}}{\partial x^\nu}, \end{cases} \tag{2.13}$$

where $\dfrac{\partial \tilde{x}^{\tilde{\nu}}}{\partial x^\nu}$ is the Jacobian matrix of the coordinate transformation on M. It follows that the associated basis vectors transform according to

$$\begin{cases} \dfrac{\partial}{\partial \tilde{v}^{\tilde{\nu}}} = \dfrac{\partial x^\nu}{\partial \tilde{x}^{\tilde{\nu}}} \dfrac{\partial}{\partial v^\nu} \\ \dfrac{\partial}{\partial \tilde{x}^{\tilde{\mu}}} = \dfrac{\partial v^\nu}{\partial \tilde{x}^{\tilde{\mu}}} \dfrac{\partial}{\partial v^\nu} + \dfrac{\partial x^\mu}{\partial \tilde{x}^{\tilde{\mu}}} \dfrac{\partial}{\partial x^\mu}. \end{cases} \tag{2.14}$$

This in turn implies that the Jacobian matrix of the coordinate transform in TM reads[1]

$$\frac{\partial\left(\tilde{x}^{\tilde{\mu}},\tilde{v}^{\tilde{\mu}}\right)}{\partial\left(x^{\nu},v^{\nu}\right)}=\begin{pmatrix}\dfrac{\partial\tilde{x}^{\tilde{\mu}}}{\partial x^{\nu}} & 0\\ v^{\rho}\dfrac{\partial^{2}\tilde{x}^{\tilde{\mu}}}{\partial x^{\nu}\partial x^{\rho}} & \dfrac{\partial\tilde{x}^{\tilde{\mu}}}{\partial x^{\nu}}\end{pmatrix}. \tag{2.15}$$

From the structure of the transformations it follows that the decomposition (2.12) is not covariant. Namely, after a general coordinate transformation on TM the components proportional to $\partial/\partial\tilde{x}^{\tilde{\mu}}$ may generate terms proportional to $\partial/\partial v^{\mu}$. However, we can easily identify one invariant subspace $V_{(p,\mathbf{X}_p)}\subset T_{(p,\mathbf{X}_p)}TM$, defined by

$$V_{(p,\mathbf{X}_p)}=\left\{\mathbf{X}\in T_{(p,\mathbf{X}_p)}TM:\mathbf{X}=\operatorname{span}\left(\frac{\partial}{\partial v^{\alpha}}\right)\right\} \tag{2.16}$$

that we will call the *vertical subspace*. It is invariant in the sense that the subspace defined this way does not depend on the associated basis. The definition of V can also be stated in the language of the projection maps. Let

$$d_{(p,\mathbf{X}_p)}\pi_M:T_{(p,\mathbf{X}_p)}TM\to T_pM \tag{2.17}$$

denote the differential (or the pushforward map) of the projection π_M. Its action on the basis vectors yields

$$\begin{aligned}d_{(p,\mathbf{X}_p)}\pi_M\left(\left.\frac{\partial}{\partial x^{\mu}}\right|_{(p,\mathbf{X}_p)}\right)&=\left.\frac{\partial}{\partial x^{\mu}}\right|_{p}\\ d_{(p,\mathbf{X}_p)}\pi_M\left(\left.\frac{\partial}{\partial v^{\mu}}\right|_{(p,\mathbf{X}_p)}\right)&=0.\end{aligned} \tag{2.18}$$

Hence, $\mathbf{X}$ belongs to the vertical subspace if $d_{(p,\mathbf{X}_p)}\pi_M\left(\mathbf{X}\right)=0$, and we can write $V_{(p,\mathbf{X}_p)}=\ker\left(d_{(p,\mathbf{X}_p)}\pi_M\right)$. In other words, a vector in the tangent space to the tangent bundle belongs to the vertical subspace if the corresponding the infinitesimal variation of the base point in M vanishes.

A complementary subspace to V can be constructed for manifolds endowed with an additional structure like the metric and the connection. As we have noted before, a perturbation of the position after a change of coordinates may acquire a component along $\partial/\partial v^{\nu}$. Hence, a general perturbation of the position does not preserve the decomposition with respect to the coordinate basis on TM. However, the existence of the Levi-Civita connection suggests that a slight modification of the basis motivated by the parallel transport of vectors on T_pM might avoid mixing the terms. To check if this is the case, let us rewrite (2.12) by adding and subtracting same terms:

$$\mathbf{X}=X_x^{\mu}\frac{\partial}{\partial x^{\mu}}+X_v^{\mu}\frac{\partial}{\partial v^{\mu}}=X_x^{\mu}\left(\frac{\partial}{\partial x^{\mu}}-\Gamma^{\alpha}{}_{\mu\beta}v^{\beta}\frac{\partial}{\partial v^{\alpha}}\right)+\left(X_v^{\mu}+\Gamma^{\mu}{}_{\alpha\beta}X_x^{\alpha}v^{\beta}\right)\frac{\partial}{\partial v^{\mu}}. \tag{2.19}$$

We now introduce the following notation:

$$\mathbf{X}=X_H^{\mu}\mathbf{e}_{H\mu}+X_V^{\mu}\mathbf{e}_{V\mu}, \tag{2.20}$$

[1] We remind that for a given coordinate system on the tagent bundle x^{μ} and v^{μ} are considered independent of each other.

where

$$
\begin{aligned}
X_H^\mu &= X_x^\mu \\
X_V^\mu &= X_v^\mu + \Gamma^\mu{}_{\alpha\beta} X_x^\alpha v^\beta,
\end{aligned}
\tag{2.21}
$$

and the new basis reads:

$$
\begin{aligned}
\mathbf{e}_{H\mu} &= \frac{\partial}{\partial x^\mu} - \Gamma^\alpha{}_{\mu\beta} v^\beta \frac{\partial}{\partial v^\alpha} \\
\mathbf{e}_{V\mu} &= \frac{\partial}{\partial v^\mu}.
\end{aligned}
\tag{2.22}
$$

Again, the part proportional to $\mathbf{e}_V$ belongs to $V_{(p,\mathbf{X}_p)}$. However, it is easy to check that coordinate transformations (2.13) do not mix $\mathbf{e}_H$ and $\mathbf{e}_V$. Therefore, by setting $X_V^\mu = 0$ we obtain another invariant subspace of $T_{(p,\mathbf{X}_p)}TM$ [111, 112, 113], which we call the *horizontal subspace* and denote by $H_{(p,\mathbf{X}_p)}$:

$$
H_{(p,\mathbf{X}_p)} = \left\{ \mathbf{X} \in T_{(p,\mathbf{X}_p)}TM : \mathbf{X} = \operatorname{span}\left(\mathbf{e}_H\right) \right\}. \tag{2.23}
$$

The condition above is equivalent to the requirement that $X_v^\mu + \Gamma^\mu{}_{\alpha\beta} X_x^\alpha v^\beta = 0$. In geometric terms, the equation above means that that the tangent vector $\mathbf{X}_p$ is parallel transported when the base point p varies.

The horizontal and vertical subspaces as presented above are invariant under the choice of coordinates on the base manifold, complementary and sufficient to span the entire $T_{(p,\mathbf{X}_p)}TM$. That is, $T_{(p,\mathbf{X}_p)}TM = V_{(p,\mathbf{X}_p)} \oplus H_{(p,\mathbf{X}_p)}$ and $V_{(p,\mathbf{X}_p)} \cap H_{(p,\mathbf{X}_p)} = \{0\}$. Hence, on the tangent space of TM any vector can be decomposed uniquely into horizontal and vertical parts.

Both horizontal and vertical subspaces are isomorphic to the tangent space T_pM, i.e. there are coordinate-independent isomorphisms $H_{(p,\mathbf{X}_p)} \cong T_pM$ and $V_{(p,\mathbf{X}_p)} \cong T_pM$ [6, 110]. In terms of coordinates they are given by

$$
\begin{aligned}
\mathrm{iso}^H &: H_{(p,\mathbf{X}_p)} \ni X_H^\mu \, \mathbf{e}_{H\mu} \mapsto X_H^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p \in T_pM \\
\mathrm{iso}^V &: V_{(p,\mathbf{X}_p)} \ni X_V^\mu \, \mathbf{e}_{V\mu} \mapsto X_V^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p \in T_pM
\end{aligned}
\tag{2.24}
$$

In simple terms, we have shown that any infinitesimal variation of a point and a vector at that point can be covariantly decomposed into two components. The first one, i.e. the horizontal component, corresponds to the variation of the point together with the parallel transport of the vector along this variation. The second one, i.e. the vertical part is simply the variation of the vector without changing its base point. Both components can be parametrized by vectors from the tangent space to the spacetime T_pM. In other words, an infinitesimal variation of a point and a vector can be uniquely represented by a *pair* of vectors tangent to the spacetime, each representing a different type of variation.

The horizontal-vertical splitting presented above is particularly useful in the context of geometrical optics, because the observables, such as the position drift or parallax, are directly related to the covariant variation of the direction of light propagation between nearby points of the spacetime [41]. The covariant direction variation, on the other hand, is given by the vertical components of the variation of the observation point and the light propagation direction.

2.5 Geodesic spray

We know that for a curve to be geodesic on M, its tangent vector has to satisfy Eq. (2.9). Suppose we are given *a non-vanishing* vector field $\dot{\gamma}$ satisfying the geodesic condition and defined on the whole manifold. The 3-parameter family of its integral curves is known as the geodesic congruence. In a similar way, we can define a congruence on TM that consists of lifts of all possible geodesics on the spacetime. The vector field on TM which generates these curves (in fact, the entire geodesic flow introduced in Sec. 2.3) is known as the *geodesic spray*. In other words, the geodesic spray is an infinitesimal counterpart of the geodesic flow on the level of TM. In this section we will recall the notion of the geodesic spray and review its geometric properties. In particular, we will show that the geodesic spray preserves the symplectic structure, which allows us to study the geodesic motion in the Hamiltonian formulation. In later sections dedicated to null geodesics, these geometrical features will be related to the physical properties of light rays and their propagation.

The decomposition of $T_{(p,\mathbf{X}_p)}TM$ we have introduced previously holds for any vector field on TM. In our work we apply it to a specific vector field, namely the geodesic spray. It is defined as a vector field $\mathbf{G}_{(p,\mathbf{X}_p)} : TM \ni (p, \mathbf{X}_p) \to T_{(p,\mathbf{X}_p)}TM$ [27], which in adapted coordinates reads

$$\left(G^i_{(p,\mathbf{X}_p)}\right) = \begin{pmatrix} v^\mu \\ -\Gamma^\mu{}_{\alpha\beta}\left(x^\rho\right) v^\alpha v^\beta \end{pmatrix}. \tag{2.25}$$

Its connection to geodesics is best understood in the following way. Suppose we want to find a curve in TM whose projection in M is a geodesic. This can be stated in the language of the geodesic spray as the 1st order initial value problem:

$$\begin{cases} \left(\gamma^\mu, \dot{\gamma}^\mu\right)(\lambda_0) & = \left(x^\mu_0, v^\mu_0\right) \\ \dfrac{d}{d\lambda}\left(\gamma^\mu, \dot{\gamma}^\mu\right) & = G^i_{(\gamma,\dot{\gamma})}. \end{cases} \tag{2.26}$$

In other words, all the integral curves of $\mathbf{G}$ are lifts of geodesics in TM.

The geodesic spray has a few other special properties. In local basis we have

$$\mathbf{G}_{(p,\mathbf{X}_p)} = v^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\mu} = v^\mu \mathbf{e}_{H\mu} \tag{2.27}$$

Hence, $\mathbf{G}_{(p,\mathbf{X}_p)}$ belongs to the horizontal subspace $H_{(p,\mathbf{X}_p)}$. Furthermore, $d\pi_M \mathbf{G}_{(p,\mathbf{X}_p)} = \mathbf{X}_p$. Comparison with (2.11) reveals that $\mathbf{G}$ generates the geodesic flow:

$$\mathbf{G}_{(p,\mathbf{X}_p)} = \left.\frac{d\phi_\lambda\left((p, \mathbf{X}_p)\right)}{d\lambda}\right|_{\lambda=\lambda_0} \tag{2.28}$$

Geodesic spray is also involved in the formulation of various conservation laws on the tangent bundle [121, 93, 94]. For example, consider the mapping $g(v,v) : TM \ni (x^\mu, v^\mu) \to g_{\mu\nu}\left(x^\rho\right) v^\mu v^\nu \in \mathbb{R}$, i.e. the norm of v^μ. Then taking Lie derivative of $g(v,v)$ along $\mathbf{G}$ yields

$$\begin{aligned} \mathcal{L}_{\mathbf{G}}\left(g(v,v)\right) &= v^\mu g_{\alpha\beta,\mu} v^\alpha v^\beta - 2\Gamma^\mu{}_{\alpha\beta}(x)\, v^\alpha v^\beta g_{\mu\rho} v^\rho \\ &= g_{\alpha\beta,\mu} v^\mu v^\alpha v^\beta - (g_{\alpha\rho,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho}) v^\rho v^\alpha v^\beta \\ &= 0. \end{aligned} \tag{2.29}$$

Another important structure is the sympletic form ω on TM, whose expansion in local cobasis reads

$$\begin{aligned} \omega &= dv_\mu \wedge dx^\mu \\ &= d\left(g_{\mu\nu}(x)\, v^\nu\right) \wedge dx^\mu \\ &= v^\nu g_{\mu\nu,\rho} dx^\rho \wedge dx^\mu + g_{\mu\nu} dv^\nu \wedge dx^\mu. \end{aligned} \tag{2.30}$$

Again we are interested in its flow along $\mathbf{G}$. Lie derivative acts on differential forms by

$$\mathcal{L}_{\mathbf{G}}(\omega) = d\left(\mathbf{G} \lrcorner\, \omega\right) + \mathbf{G} \lrcorner\, d\omega. \tag{2.31}$$

The second term vanishes due to Eq. (2.30). As for the first term, we have

$$\begin{aligned}\mathbf{G} \lrcorner\, \omega &= v^{\nu} g_{\mu\nu,\rho} v^{\rho} dx^{\mu} - v^{\nu} g_{\mu\nu,\rho} dx^{\rho} v^{\mu} - g_{\mu\nu} dv^{\nu} v^{\mu} - \Gamma^{\nu}{}_{\alpha\beta} v^{\alpha} v^{\beta} g_{\mu\nu} dx^{\mu} \\ &= d\left(-\frac{1}{2} g_{\mu\nu} v^{\alpha} v^{\beta}\right)\end{aligned} \tag{2.32}$$

Altogether, $\mathcal{L}_{\mathbf{G}}(\omega) = 0$, and the symplectic form is conserved.

In semi-Riemannian geometry $g(v, v)$ and ω are always preserved along the geodesic spray. However, it may happen that the manifold has additional symmetries generating additional conservation laws. Consider a mapping $\xi : TM \ni (x^{\mu}, v^{\mu}) \to v^{\mu}\xi_{\mu}(x^{\rho}) \in \mathbb{R}$ where ξ^{μ} is so far an unspecified vector field on M. We are interested in finding such ξ^{μ} for which the Lie derivative of $v^{\mu}\xi_{\mu}$ along $\mathbf{G}$ vanishes. By a straightforward calculation we find that

$$\mathcal{L}_{\mathbf{G}}(v^{\mu}\xi_{\mu}) = v^{\alpha} v^{\beta} \xi_{(\alpha;\beta)}. \tag{2.33}$$

This will vanish for arbitrary v^{μ} iff $\xi_{(\alpha;\beta)} = 0$. In other words, Eq. (2.33) holds for all geodesics iff ξ^{μ} is a Killing vector field on M. A similar argument for higher-order contractions results in equations for Killing tensor fields. Further extensions of this argument are possible by confining oneself to a subbundle. In the case of null geodesics, i.e. the subbundle S^0TM, one can weaken the requirement by demanding vanishing Lie dragging only along lifts of null geodesics:

$$\mathcal{L}_{\mathbf{G}}(v^{\mu}\xi_{\mu}) = \phi(x^{\rho})\, g_{\mu\nu} v^{\mu} v^{\nu}, \tag{2.34}$$

where $\phi : M \to \mathbb{R}$ is an arbitrary smooth function. Then instead of Killing equations, one obtains conformal Killing equations.

2.6 Lie dragging of vector along G

With all ingredients in place, we can now define the BGOs using the geodesic spray, the geodesic flow it generates, and its deformation tensor $W^{i}{}_{j}$. We begin by showing an equivalence between the Lie dragging of a vector in $T_{(p,\mathbf{X}_p)}TM$ along the geodesic spray and the GDE. By doing so we will automatically reduce the GDE to the first order linear ODE. This in turn will allow us to express the solution with the help of the resolvent operator, that is, a linear operator encapsulating all possible propagation effects of families of particles between 2 regions. In the last step, we will limit ourselves to the problem of propagation of light and demonstrate the connection between the geodesic flow and the BGOs in agreement with [41].

Let $\gamma(\lambda)$ be a geodesic with initial conditions $\gamma(\lambda_0) = p$ and $\dot{\gamma}(\lambda_0) = \mathbf{X}_p$, and let $\phi_{\lambda}(p, \mathbf{X}_p)$ be its image under the geodesic flow, i.e. the lift of the geodesic. Let $\mathbf{Y} \in T_{(p,\mathbf{X}_p)}TM$ be a vector that we drag along the integral curve of $\mathbf{G}$ to $T_{(\gamma(\lambda),\dot{\gamma}(\lambda))}TM$. We would like to describe the geometry of neigbouring geodesic, at the lowest, linear order in terms of the perturbation vector $\mathbf{Y}$. In the language of differential geometry is equivalent to requiring that $\mathbf{Y}$ is Lie dragged in the direction of $\mathbf{G}$. Then setting $\mathcal{L}_{\mathbf{G}}(\mathbf{Y}) = 0$ yields

$$\frac{dY^{i}}{d\lambda} = G^{i}{}_{,j} Y^{j}. \tag{2.35}$$

Since $\mathbf{G}$ contains Christoffel symbols, one may expect that after differentiation, covariant structures like the Riemann curvature tensor should emerge. Indeed, from Eqs. (2.21), (2.27), the relation $\dfrac{d}{d\lambda} = G^{i}\partial_{i}$ and careful decomposition of the equation to the horizontal and vertical subspace it follows that Eq. (2.35) is equivalent to the following ODE solved on the base manifold along the geodesic γ:

$$\begin{pmatrix} \nabla_{\dot{\gamma}} Y_H^{\mu} \\ \nabla_{\dot{\gamma}} Y_V^{\nu} \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\mu}{}_{\beta} \\ R^{\nu}{}_{\dot{\gamma}\dot{\gamma}\alpha} & 0 \end{pmatrix} \begin{pmatrix} Y_H^{\alpha} \\ Y_V^{\beta} \end{pmatrix} \tag{2.36}$$

where $R^{\nu}{}_{\dot\gamma\dot\gamma\mu} = R^{\nu}{}_{\alpha\beta\mu}\dot\gamma^{\alpha}\dot\gamma^{\beta}$. This equation is linear both in Y_H^{μ} and Y_V^{μ} and defines a system of linear ODEs whose solution is expressible in terms of a resolvent operator. That is, the solution can be expressed as $Y^i(\lambda) = (W_\lambda)^i{}_j Y^j(\lambda_0)$. Motivated by this, we introduce a map

$$W_\lambda : T_{(p,\mathbf{X}_p)}TM \ni \mathbf{Y} \to W_\lambda(\mathbf{Y}) \in T_{(\gamma(\lambda),\dot\gamma(\lambda))}TM \tag{2.37}$$

satisfying the system of ODEs

$$\frac{dW^i{}_k}{d\lambda} = G^i{}_{,j}W^j{}_k \tag{2.38}$$

with initial data

$$W^i{}_k(\lambda_0) = \delta^i{}_k = \begin{pmatrix} \delta^{\mu}_{\nu} & 0 \\ 0 & \delta^{\mu}_{\nu} \end{pmatrix}. \tag{2.39}$$

On the other hand, we know that the tangent space of the tangent bundle is isomorphic to the direct sum of two tangent spaces:

$$T_{(p,\mathbf{X}_p)}TM \cong T_pM \oplus T_pM. \tag{2.40}$$

Together with the decomposition of vectors in horizontal and vertical parts, this suggests that W_λ has an inherent 2×2 block structure. In the papers to be presented later, this is used extensively, but the notation is slightly different. We associate the horizontal part of $\mathbf{Y}$ with the perturbation of the position, and the vertical part – with the perturbation of the tangent vector. Therefore, the subscripts for horizontal and vertical parts are replaced with X and L respectively, in line with the notation used in [41, 62, 117]. Also, components v^μ, and thus also the tangent vector to the geodesic, will now be denoted by l^μ. Finally, we decompose W_λ into 4 operators that we label according to their connection with the initial and final data. Then the action of W_λ can be expressed as

$$\begin{pmatrix} Y_X^{\mu} \\ Y_L^{\mu} \end{pmatrix} = \begin{pmatrix} W_{XX}{}^{\mu}{}_{\nu} & W_{XL}{}^{\mu}{}_{\nu} \\ W_{LX}{}^{\mu}{}_{\nu} & W_{LL}{}^{\mu}{}_{\nu} \end{pmatrix} \begin{pmatrix} Y_X^{\nu}(\lambda_0) \\ Y_L^{\nu}(\lambda_0) \end{pmatrix}. \tag{2.41}$$

Since both the vertical and the horizontal subspaces are isomorphic to the appropriate tangent space, the operators W_{XX}, W_{XL}, W_{LX} and W_{LL} can be identified with bitensors acting between the tangent spaces at two different points:

$$W_{**} : T_{\gamma(\lambda_0)}M \to T_{\gamma(\lambda)}M. \tag{2.42}$$

The Eq. (2.36) can be rewritten as an equation for the covariant derivative of the bitensors with respect to the point $\gamma(\lambda)$:

$$\begin{pmatrix} \nabla_l W_{XX}{}^{\mu}{}_{\nu} & \nabla_l W_{XL}{}^{\mu}{}_{\nu} \\ \nabla_l W_{LX}{}^{\mu}{}_{\nu} & \nabla_l W_{LL}{}^{\mu}{}_{\nu} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \delta^{\mu}{}_{\beta} \\ R^{\mu}{}_{ll\alpha} & 0 \end{pmatrix}}_{S} \begin{pmatrix} W_{XX}{}^{\alpha}{}_{\nu} & W_{XL}{}^{\alpha}{}_{\nu} \\ W_{LX}{}^{\beta}{}_{\nu} & W_{LL}{}^{\beta}{}_{\nu} \end{pmatrix} \tag{2.43}$$

Here we denote by S a block matrix made of the curvature tensor and the unit matrix. It will later reappear in Eq. (2.48). From the block structure we also see that in principle the propagation of variations can be studied independently by appropriately setting the initial conditions, i.e. W_{XX} and W_{LX} depend only on initial position variations while W_{XL} and W_{LL} – on initial direction variations.

We note here that Eq. (2.43) takes a particularly simple form when expressed in a parallel-transported tetrad along the geodesic γ. The covariant derivatives become ordinary derivatives and we obtain simply a matrix equation

$$\frac{d}{d\lambda}\begin{pmatrix} W_{XX}{}^{\mu}{}_{\nu} & W_{XL}{}^{\mu}{}_{\nu} \\ W_{LX}{}^{\mu}{}_{\nu} & W_{LL}{}^{\mu}{}_{\nu} \end{pmatrix} = S \begin{pmatrix} W_{XX}{}^{\alpha}{}_{\nu} & W_{XL}{}^{\alpha}{}_{\nu} \\ W_{LX}{}^{\beta}{}_{\nu} & W_{LL}{}^{\beta}{}_{\nu} \end{pmatrix}. \tag{2.44}$$

2.7 Symplectic structure and BGOs

We have previously shown that the geodesic spray $\mathbf{G}$ preserves the symplectic form ω. With the help of the horizontal-vertical splitting we can now show the symplectic properties of BGOs.

Firstly, let us look at the action of ω on two arbitrary vectors $\mathbf{Y}, \mathbf{Z} \in T_{(p,\mathbf{X}_p)}TM$ with components Y^i and Z^i. We know that on a semi-Riemannian manifold one can decompose vectors uniquely into their horizontal and vertical parts:

$$\begin{aligned} \mathbf{Y} &= Y_H^\mu \mathbf{e}_{H\mu} + Y_V^\mu \mathbf{e}_{V\mu} \\ \mathbf{Z} &= Z_H^\mu \mathbf{e}_{H\mu} + Z_V^\mu \mathbf{e}_{V\mu}. \end{aligned} \tag{2.45}$$

By a straighforward calculation one can show that

$$\begin{cases} \omega\left(\mathbf{e}_{V\mu}, \mathbf{e}_{V\nu}\right) & = \quad \omega\left(\mathbf{e}_{H\mu}, \mathbf{e}_{H_\nu}\right) = 0 \\ \omega\left(\mathbf{e}_{V\mu}, \mathbf{e}_{H\nu}\right) & = -\omega\left(\mathbf{e}_{H\mu}, \mathbf{e}_{V\nu}\right) = g_{\mu\nu} \end{cases} \tag{2.46}$$

Then for any two vectors $\mathbf{Y}, \mathbf{Z} \in T_{(p,\mathbf{X}_p)}TM$ the symplectic form yields

$$\begin{aligned} \omega\left(\mathbf{Y}, \mathbf{Z}\right) &= Y_V^\mu Z_{H\mu} - Y_H^\mu Z_{V\mu} \\ &= g_{\mu\nu} Y_V^\mu Z_H^\nu - g_{\mu\nu} Y_H^\mu Z_V^\nu \\ &= \left(Y_H^\mu, Y_V^\nu\right) \begin{pmatrix} 0 & -g_{\mu\nu} \\ g_{\mu\nu} & 0 \end{pmatrix} \begin{pmatrix} Z_H^\mu \\ Z_V^\nu \end{pmatrix} \\ &= Y^i \Omega_{ij} Z^j \end{aligned} \tag{2.47}$$

where Ω_{ij} plays the role of the skew-symmetric matrix. The explicit form of Ω_{ij} is important in deriving how the symplecticity of W should be understood at the level of the block operators of W.

We will now prove the simplecticity of W. Consider the ODE for W expressed in a parallel-transported tetrad, i.e. the matrix equation (2.44). We take it together with its transpose:

$$\begin{aligned} \frac{d}{d\lambda} W &= SW \\ \frac{d}{d\lambda}\left(W^T\right) &= \left(W^T\right)\left(S^T\right) \end{aligned}, \tag{2.48}$$

where by transpose of W and S we mean

$$\begin{pmatrix} W_{XX}{}^\mu{}_\nu & W_{XL}{}^\mu{}_\nu \\ W_{LX}{}^\mu{}_\nu & W_{LL}{}^\mu{}_\nu \end{pmatrix}^T = \begin{pmatrix} W_{XX}{}_\nu{}^\mu & W_{LX}{}_\nu{}^\mu \\ W_{XL}{}_\nu{}^\mu & W_{LL}{}_\nu{}^\mu \end{pmatrix} \tag{2.49}$$

and

$$\begin{pmatrix} 0 & \delta^\mu{}_\nu \\ R^\mu{}_{ll\nu} & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & R_{\nu ll}{}^\mu \\ \delta_\nu{}^\mu & 0 \end{pmatrix}. \tag{2.50}$$

From Eq. (2.48) it follows that

$$\frac{d}{d\lambda}\left(W^T \Omega W\right) = W^T\left(S^T \Omega + \Omega S\right) W = 0. \tag{2.51}$$

Hence, $W^T \Omega W$ must be constant along l. Finally we evaluate $W^T \Omega W$ at $\lambda = \lambda_0$ and take into account initial data from Eq. (2.39). It follows that

$$W^T \Omega W = \Omega. \tag{2.52}$$

Therefore, W is symplectic. This also implies that not all components of W are independendent. In terms of block operators Eq. (2.52) is equivalent to

$$\begin{cases} {W_{LX}}_{\mu\rho}{W_{XX}}^{\rho}{}_{\nu} - {W_{XX}}_{\mu\rho}{W_{LX}}^{\rho}{}_{\nu} & = 0 \\ {W_{LX}}_{\mu\rho}{W_{XL}}^{\rho}{}_{\nu} - {W_{XX}}_{\mu\rho}{W_{LL}}^{\rho}{}_{\nu} & = -g_{\mu\nu} \\ {W_{LL}}_{\mu\rho}{W_{XX}}^{\rho}{}_{\nu} - {W_{XL}}_{\mu\rho}{W_{LX}}^{\rho}{}_{\nu} & = g_{\mu\nu} \\ {W_{LL}}_{\mu\rho}{W_{XL}}^{\rho}{}_{\nu} - {W_{XL}}_{\mu\rho}{W_{LL}}^{\rho}{}_{\nu} & = 0. \end{cases} \tag{2.53}$$

For completeness, we mention additional algebraic properties valid for smooth semi-Riemannian manifolds [41]. From Eq. (2.36) one can see that for all $C, D \in \mathbb{R}$ the vector $\mathbf{Y}$ with the horizontal/vertical decomposition

$$\begin{pmatrix} Y_X^{\mu} \\ Y_L^{\nu} \end{pmatrix} = \begin{pmatrix} (C + D\lambda)l^{\mu} \\ Dl^{\nu} \end{pmatrix} \tag{2.54}$$

is a solution, where $l^{\mu} = \dot{\gamma}^{\mu}$ as explained in Sec.2.6. Assuming that C and D can be chosen independently, the substitution of $\mathbf{Y}$ into Eq. (2.41) yields the following simultaneous set of constraints:

$$\begin{cases} {W_{XX}}^{\mu}{}_{\nu}(\lambda)\; l^{\mu}(\lambda_0) & = l^{\mu}(\lambda) \\ {W_{XL}}^{\mu}{}_{\nu}(\lambda)\; l^{\mu}(\lambda_0) & = (\lambda - \lambda_0) l^{\mu}(\lambda) \\ {W_{LX}}^{\mu}{}_{\nu}(\lambda)\; l^{\mu}(\lambda_0) & = 0 \\ {W_{LL}}^{\mu}{}_{\nu}(\lambda)\; l^{\mu}(\lambda_0) & = l^{\mu}(\lambda) \end{cases}. \tag{2.55}$$

In addition to this, any $\mathbf{Y}$ which solves Eq. (2.36) generates two conserved quantities:

$$\begin{aligned} A &= \left(Y_X^{\mu} - \lambda Y_L^{\mu}\right) l_{\mu} \\ B &= Y_L^{\mu} l_{\mu}. \end{aligned} \tag{2.56}$$

This can be checked by taking covariant derivatives of A and B along l and using Eq. (2.36) to express $\nabla_l Y_X^{\mu}$ and $\nabla_l Y_L^{\mu}$ in terms of Y_X^{μ} and Y_L^{μ}. Evaluating them at both endpoints, applying Eq. (2.41) and requiring the equalities to hold for arbitrary initial $\mathbf{Y}$ we get

$$\begin{cases} l^{\nu}(\lambda)\; {W_{XX}}^{\mu}{}_{\nu}(\lambda) & = l_{\nu}(\lambda_0) \\ l^{\nu}(\lambda)\; {W_{XL}}^{\mu}{}_{\nu}(\lambda) & = (\lambda - \lambda_0) l_{\nu}(\lambda_0) \\ l^{\nu}(\lambda)\; {W_{LX}}^{\mu}{}_{\nu}(\lambda) & = 0 \\ l^{\nu}(\lambda)\; {W_{LL}}^{\mu}{}_{\nu}(\lambda) & = l_{\nu}(\lambda_0). \end{cases} \tag{2.57}$$

Chapter 3

Bilocal geodesic operators in static spherically-symmetric spacetimes

In the first paper we present an exact solution of the GDE in terms of BGOs along any null geodesic in static, spherically symmetric spacetimes and its application to the of the behaviour of the distance measures in the Schwarzschild spacetime.

The solution is derived in two different ways. The first approach requires a general solution of the GE, either in exact form or in terms of implicit relations. Then one has to vary it with respect to the components of the initial position and direction. After a few adjustments, which make the expressions covariant, the solution is recovered. The second approach makes direct use of the symmetries of the spacetime. Namely, Killing vectors generate the first integrals of the GDE, which effectively reduces the order and the number of ODEs we need to solve. In the end, the solution depends on various conserved quantities, initial conditions and may possibly also involve nontrivial integrals of functions of metric coefficients. Thus derived solutions, expressed in the coordinate tetrad, are valid for timelike, spacelike and null geodesics whenever the chosen local coordinates are valid.

Still, the physical interpretation of the solution requires some care due to chosen coordinate system. The situation can be improved by rewriting the result in a different tetrad with a clear physical interpretation. In this paper we choose the parallel transported semi-null tetrad (SNT). We first construct it at the point of observation and then solve the parallel propagation equation while simultaneously taking into account the constraints of the SNT. Then the solution of the GDE is projected onto this SNT, in which the solution takes a much simpler form.

Finally, we use the results to study the angular diameter and the parallax distances in the spacetime of Schwarzschild black hole. The behaviour is analyzed numerically while the most apparent properties are explained qualitatively by splitting the problem into three different regions: the initial region close to the observer, the intermediate region around the black hole, and the faraway region with the source positioned at increasingly larger distances from the black hole.

Although the paper discusses only static spherically symmetric spacetimes, the principles behind the methods of solution can be readily generalized to any sufficiently symmetric spacetime. In other words, if a spacetime possesses an adequate number of Killing vectors and tensors, both methods will produce a solution, at least in the coordinate tetrad. Furthermore, the projection of the results on the SNT requires the parallel transport of the SNT. In this paper, the result was achieved by assuming an ansatz motivated by the assumed geometry of the spacetime and then fulfilling the SNT constraints. When the form of the ansatz is not self-evident, one can still avoid solving the parallel transport equations if the spacetime admits a Killing-Yano tensor. However, this is not always guaranteed, not even in the presence of a

Killing tensor. In that case one is left with a system of linear ODEs.

Author's contribution

The results were derived in collaboration with my supervisor prof. M. Korzyński whose contribution includes:

- the suggestion of the 1st algorithm;
- the idea of the investigation of distance measures in the black hole spacetime
- some help with the refinement of sections 3.3, 4.1 and 4.4-4.6.

Meanwhile, my contribution includes:

- the suggestion and the implementation of the 2nd method;
- all the derivations;
- numerical investigation and the plots.

The draft was written and published jointly by me and M. Korzyński.

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Bilocal geodesic operators in static spherically-symmetric spacetimes

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Abstract

We present a method to compute exact expressions for optical observables for static spherically symmetric spacetimes in the framework of the bilocal geodesic operator formalism. The expressions are obtained by solving the linear geodesic deviation equations for null geodesics, using the spacetime symmetries and the associated conserved quantities. We solve the equations in two different ways: by varying the geodesics with respect to their initial data and by directly integrating the equation for the geodesic deviation. The results are very general and can be applied to a variety of spacetime models and configurations of the emitter and the observer. We illustrate some of the aspects with an example of Schwarzschild spacetime, focusing on the behaviour of the angular diameter distance, the parallax distance, and the distance slip between the observer and the emitter outside the photon sphere.

Keywords: geometric optics, distance measures, geodesic deviation, gravitational lensing, black holes, exact solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

In general relativity, the motion of particles is affected by the spacetime geometry along their paths. Although the curvature itself cannot be observed, we can measure it directly by studying relative motions of neighbouring freely falling particles. The key equation in this problem is the first-order, linear geodesic deviation equation (GDE), which relates relative accelerations of particles to the Riemann curvature tensor in their vicinity. Its range of application varies from tracking nearby satellites orbiting the Earth to the observation of light coming from faraway luminous bodies.

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The current standard framework for optical measurements rests on gravitational lensing formalism introduced by Sachs [1]. Sachs formalism was critical in theoretical GR, for example, the derivation of the Kerr metric [2, 3], with somewhat lesser importance in observational GR. It uses the GDE directly or in the form of optical scalar equations. The information about the influence of geometry on light is then encapsulated the expansion, shear and twist of infinitesimal bundles of rays [4]. These objects in turn can be related to various measures of distances like the angular diameter distance and the luminosity distance. Although being relatively successful in matching the observational data to the theory, the formalism is incomplete in the following sense: it can only accommodate fixed sources and observers, so drift effects cannot be obtained from it directly [5]. Also, it is not always clear how special relativistic effects like aberration or Doppler shift affect the observables.

Recently, a new formalism of bilocal geodesic operators (BGOs) [6] has been introduced, extending the previous formalism by allowing drift effects and other effects like the parallax. It is based on the resolvent of the first order GDE. As inputs, it requires the curvature along the line of sight (LOS) and the initial and final data at its endpoints. Due to their symplectic nature and the properties of null geodesics, these operators possess several symmetries. This suggests that the complete picture is simpler than it looks at first glance.

There also exist other ways to study the geometry of geodesics. One of them is the Synge's world function, which holds the information about pairs of points connected by unique geodesics. This information can be accessed by taking derivatives of the worldfunction with respect to the endpoints. It can be shown that the second derivatives of the world function are related to the bilocal operators [7]. If the world function can be calculated exactly, the solutions of the GDE can be obtained simply by the differentiation. However, for spherically symmetric spacetimes the exact form is rarely available [8, 9], and one is usually confined to a perturbative analysis of the world function and its derivatives [10–13].

In general, the Universe is not symmetric, which means that one has to use numerical methods to solve the propagation equations for light to obtain all optical effects. However, there are many interesting cases where the geodesic equation is integrable, and one can expect that GDE in these cases is integrable.

In this paper we address some of these questions. We first describe the connection between the GDE and the BGOs and present their symplectic properties. Then we relate the BGOs to the variations of the geodesic with respect to its initial data and list a number of general and Killing-vector-induced conservation laws for the BGOs and the solutions of the GDE. Later, we apply all this knowledge to compute the BGOs for static spherically symmetric spacetimes and isolate physical effects by projecting our results onto the parallel propagated semi-null tetrad (SNT). In the last part of the paper, we consider the propagation of light in Schwarzschild spacetime, where we numerically investigate the behaviour of the angular diameter distance, the parallax distance, and their relative difference, i.e. the distance slip [6], as we displace the emitter along the null geodesic. Finally, we reformulate these results in greater generality by studying the behaviour of BGOs in the initial, intermediate, and faraway regions.

Indeed, the problem of analytical integration of GDE is not a new one. There have been many successful attempts both for timelike [14–17] and null [17–22] geodesics, but the complete picture of the solutions is lacking. Often solutions assume particular initial conditions or types of orbits. Additionally, in the null case, the studies are usually limited to the behaviour of the light ray bundle projected onto the Sachs screen. This limitation completely neglects effects due to the motion of the emitter or observer.

The extension of the geometrical optics framework may also be important for the present and upcoming astrophysical and cosmological observations. For example, in the cosmological setting the parallax as well the position and redshift drifts provide additional data which can be used to study inhomogeneities and large-scale flows of matter and further constrain cosmological models [5, 6]. On the other hand, the observational and computational advancements recently lead to the first images of the black hole shadow [23]. The theory behind it is well-developed [24], but not entirely complete. In these problems the observer is usually considered to be static or comoving with some global flow. It would be interesting to see whether the BGO formalism could be used to make the problem fully covariant and reveal new properties of the black hole shadow. Static spherically symmetric spacetimes are good starting points for such studies because they are sufficiently simple while still being good models for various types of massive compact objects.

1.1. Applications

Due to assumed symmetries, all possible applications concerning will be limited to static spacetimes with spherical symmetries. Killing vectors allow us to integrate equations exactly, and solutions include only a handful of integrals of functions of metric coefficients along the trajectory of light. Furthermore, observer effects like stellar aberration with the arbitrary alignment of observer's four-velocity are taken into account by appropriate parallel transports. Moreover, general treatment of geodesic deviation allows us to characterize the formation of caustics in a more precise manner. Now we can state precise conditions for the formation of focal or conjugate points in terms of parameters of the null geodesic. Similarly, we can quantify how the size and the shape of an image as seen by the observer depends on the positions of emitters and observers. The formalism applied here treats all optical effects on the same footing, so this information is related to previously mentioned effects and forms a consistency requirement between all of them.

In practice this means that we can study cases when the lensing and lensed structures do not fit the traditional lensing formalism, e.g. when the impact parameter of the light or distances between emitters, observers and lensing bodies are not much larger than Schwarzschild radius. Geodesic bilocal operator formalism holds both in weak and strong lensing regimes as well as all intermediate cases. Hence, we are able to patch these results and explain transitions from one regime to the other one. It is worth mentioning that general relativity is, in general, not assumed here. We only require a four-dimensional Lorentzian metric theory of gravity. The conclusions about the spacetime we reach are purely geometric. Thus, the physical interpretation depends on the choice of the theory of gravity.

The applicability is also limited by the geometrical optics framework. The radiation wavelength must be much smaller than the size of neighbourhoods of the source and the emitter or the scale of the spacetime curvature. On top of that, compared to the curvature radius, the width of the connecting tube has to be small enough for the linear GDE approximation to be valid. Finally, both endpoints have to be positioned sufficiently far from each other so that all relations between the null tangent vector and the observed position of the source on the observer's sky or the null condition could be linearized around the fiducial geodesic.

1.2. Structure of the paper

In section 2, we begin with formulating BGOs in the geometric optics regime and restating some of their properties. Then we sketch one of the methods of calculating them, based on the variation of null geodesic with respect to initial data. The second method employs Killing conservation to reduce GDE to a system of coupled first order ordinary differential equations (ODEs), which we integrate, and is described in section 3. In section 3 we also find expressions of optical observables for the emitter and observer travelling arbitrarily and describe their behaviour, with detailed derivation given in the appendices. In section 4 we estimate effects for a Schwarzschild black hole for static observers and emitters. We state our conclusions in section 5.

1.3. Notation

Greek letters $(\alpha, \beta, \ldots)$ run from 0 to 3, and uppercase Latin indices run from 1 to 2. They all enumerate tensor components in the coordinate tetrad. In some rare cases the uppercase Latin indices are also used to label linearly independent solutions of differential equations. Boldface versions of indices cover the same range but denote components in the SNT, defined in section 3.5, as opposed to the coordinate tetrad. The dot denotes the derivative with respect to the affine parameter along the null geodesic. Prime denotes differentiation with respect to r. Subscript $\mathcal{O}$ and $\mathcal{E}$ denote evaluation of the quantity at respectively the point of observation and emission, i.e. $f_{\mathcal{O}} \equiv f(\lambda_{\mathcal{O}})$.

We introduce the following short-hand notation for integrals over a null geodesic, performed both over the affine parameter and the radial coordinate r. These integrals have common kernels which we will denote $(\ell^r)^{-2}$ or $(\ell^r)^{-3}$ as well as a varying part composed of the metric coefficients $A(r)$, $B(r)$ and $C(r)$. Namely:

$$
\begin{aligned}
I_B &= \int_0^{\lambda} \frac{\mathrm{d}\tilde{\lambda}}{B(r(\lambda))(\ell^r)^2} = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{\mathrm{d}\tilde{r}}{B(\tilde{r})(\ell^r)^3} \\
I_{AB} &= \int_0^{\lambda} \frac{\mathrm{d}\tilde{\lambda}}{A(r(\lambda))B(r(\lambda))(\ell^r)^2} = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{\mathrm{d}\tilde{r}}{A(\tilde{r})B(\tilde{r})(\ell^r)^3} \\
I_{BC} &= \int_0^{\lambda} \frac{\mathrm{d}\tilde{\lambda}}{B(r(\lambda))C(r(\lambda))(\ell^r)^2} = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{\mathrm{d}\tilde{r}}{B(\tilde{r})C(\tilde{r})(\ell^r)^3} \\
I_{ABC} &= \int_0^{\lambda} \frac{\mathrm{d}\tilde{\lambda}}{A(r(\lambda))B(r(\lambda))C(r(\lambda))(\ell^r)^2} = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{\mathrm{d}\tilde{r}}{A(\tilde{r})B(\tilde{r})C(\tilde{r})(\ell^r)^3}.
\end{aligned}
\tag{1}
$$

The slash reminds us that in the case of a turning point along the photon path, the integral over r must be split into segments with appropriately chosen signs of the integrand, see [25].

We assume the speed of light $c = 1$.

2. Formulation

Let $\mathcal{M}$ be a smooth Lorentzian manifold with a metric g of signature $(-, +, +, +)$. Let (ζ^μ) be a coordinate system. Let $\gamma : [\lambda_{\mathcal{O}}, \lambda] \to \mathcal{M}$ be a geodesic connecting two points, $x_{\mathcal{O}}$ and $x_{\mathcal{E}}$, with affine parameter values $\lambda_{\mathcal{O}}$ and λ respectively. We also introduce two tetrads for decomposing geometric objects: $\left(\partial_\mu\right)$ will denote the coordinate tetrad associated with (ζ^μ), while $\left(e_{\boldsymbol{\mu}}\right)$ will denote the tetrad, which is parallel transported along γ.

We choose a coordinate system which covers the neighborhoods of both endpoints of γ. Then the geodesic curve $x^\mu(x_{\mathcal{O}}, \ell_{\mathcal{O}}, \lambda)$ is a function of the initial point $x_{\mathcal{O}}$, the initial tangent vector $\ell_{\mathcal{O}}$, and the value of the affine parameter λ, corresponding to the geodesic with aforementioned initial conditions.

Now we perturb the initial data of the geodesic at $\lambda_{\mathcal{O}}$ according to $x^\mu_{\mathcal{O}} \to x^\mu_{\mathcal{O}} + \delta x^\mu_{\mathcal{O}}$, $\ell^\mu_{\mathcal{O}} \to \ell^\mu_{\mathcal{O}} + \delta \ell^\mu_{\mathcal{O}}$ in a coordinate tetrad. Up to the linear order in perturbation, the deviation vector $\delta x^\mu = \xi^\mu$ satisfies the following first order GDE [26, 27]:

$$\nabla_\ell \nabla_\ell \xi^\mu - \mathcal{R}^\mu{}_\nu \xi^\nu = 0. \tag{2}$$

In the literature the tensor

$$\mathcal{R}^\mu{}_\nu = R^\mu{}_{\alpha\beta\nu} \ell^\alpha \ell^\beta \tag{3}$$

is also known as the (optical) tidal matrix or optical tidal tensor.

The deviation at a different point, corresponding to a different value of λ, will take the following form:

$$\begin{aligned} \delta x^\mu &= W_{XX}{}^\mu{}_\nu \delta x^\nu_{\mathcal{O}} + W_{XL}{}^\mu{}_\nu \Delta \ell^\nu_{\mathcal{O}} \\ \Delta \ell^\mu &= W_{LX}{}^\mu{}_\nu \delta x^\nu_{\mathcal{O}} + W_{LL}{}^\mu{}_\nu \Delta \ell^\nu_{\mathcal{O}}, \end{aligned} \tag{4}$$

where $\delta x^\mu_{\mathcal{O}}$, δx^μ are the position perturbations and $\Delta \ell^\mu_{\mathcal{O}}$, $\Delta \ell^\mu$ are the covariant perturbations of the tangent vector at $\lambda_{\mathcal{O}}$ and λ respectively. The covariant perturbations of tangent vectors are defined by

$$\begin{aligned} \Delta \ell^\mu_{\mathcal{O}} &= \delta \ell^\mu_{\mathcal{O}} + \Gamma^\mu{}_{\alpha\beta}\,(x_{\mathcal{O}})\, \ell^\alpha_{\mathcal{O}} \delta x^\beta_{\mathcal{O}} \\ \Delta \ell^\mu &= \delta \ell^\mu + \Gamma^\mu{}_{\alpha\beta}\,(x)\, \ell^\alpha \delta x^\beta. \end{aligned} \tag{5}$$

The equations (2) and (4) are related by

$$\begin{aligned} \xi^\mu\,(\lambda_{\mathcal{O}}) &= \delta x^\mu_{\mathcal{O}} \\ \xi^\mu\,(\lambda_{\mathcal{E}}) &= \delta x^\mu \\ \nabla_\ell \xi^\mu\,(\lambda_{\mathcal{O}}) &= \Delta \ell^\mu_{\mathcal{O}} \\ \nabla_\ell \xi^\mu\,(\lambda_{\mathcal{E}}) &= \Delta \ell^\mu. \end{aligned} \tag{6}$$

Here $W_{XX}, W_{XL}, W_{LX}, W_{LL}$ are bitensors mapping tangent vectors from $\mathcal{O}$ to $\mathcal{E}$. Together they form the bilocal geodesic operator $\mathcal{W} : T_{\mathcal{O}}M \oplus T_{\mathcal{O}}M \mapsto T_{\mathcal{E}}M \oplus T_{\mathcal{E}}M$, defined by the linear relation

$$\begin{pmatrix} \delta x^\mu \\ \Delta \ell^\nu \end{pmatrix} = \mathcal{W} \begin{pmatrix} \delta x^\alpha_{\mathcal{O}} \\ \Delta \ell^\beta_{\mathcal{O}} \end{pmatrix}. \tag{7}$$

$\mathcal{W}$ and its four constituent bitensors may be expressed as functionals of the Riemann curvature tensor along the LOS. Namely, $\mathcal{W}$ expressed in a parallel-propagated tetrad plays the role of the resolvent of the GDE with $\mathcal{O}$ as the starting point and therefore satisfies the resolvent ODE when expressed in the parallel propagated tetrad. In the same way, four bitensors can be expressed as solutions of appropriate matrix ODEs written in any parallel propagated tetrad [6].

2.1. Bilocal geodesic operators and Killing vectors

In a spacetime admitting a Killing vector we may derive additional identities and algebraic relations for the BGO, which enormously simplify the problem of determining the components of the BGO in a given spacetime.

Assume that the spacetime admits a Killing vector K^μ, i.e. $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$. Since the flow of K^μ is an isometry, it must also map geodesics into geodesics. Therefore, the deviation of the fiducial null geodesic by K^μ must preserve the geodesic character of the curve at linear and other orders. It follows that $\xi^\mu(\lambda) = K^\mu$ must be a valid solution of the GDE [28]. This can also be proven by direct differentiation of the Killing condition and substitution into the GDE so that $\nabla_\ell \nabla_\ell K^\mu - \mathcal{R}^\mu{}_\beta K^\beta = 0$ [29]. This means that the initial data of the GDE at $\mathcal{O}$ of the form $\delta x^\mu_\mathcal{O} = K^\mu_\mathcal{O}$, $\Delta \ell^\mu_\mathcal{O} = (\nabla_\ell K^\mu)_\mathcal{O}$ must be mapped into the initial data at $\mathcal{E}$ of the form $\delta x^\mu_\mathcal{E} = K^\mu_\mathcal{E}$, $\Delta \ell^\mu_\mathcal{E} = (\nabla_\ell K^\mu)_\mathcal{E}$. From (4) applied to $\mathcal{O}$ and $\mathcal{E}$ we obtain then the following identities:

$$K^\mu_\mathcal{E} = W_{XX}{}^\mu{}_\nu K^\nu_\mathcal{O} + W_{XL}{}^\mu{}_\nu (\nabla_\kappa K^\nu)_\mathcal{O} \ell^\kappa_\mathcal{O} \tag{8}$$

$$(\nabla_\kappa K^\mu)_\mathcal{E} \ell^\kappa_\mathcal{E} = W_{LX}{}^\mu{}_\nu K^\nu_\mathcal{O} + W_{LL}{}^\mu{}_\nu (\nabla_\kappa K^\nu)_\mathcal{O} \ell^\kappa_\mathcal{O}. \tag{9}$$

A dual set of identities can be derived by invoking symplectic properties of BGOs. Consider the matrix $\mathcal{W}$ defined by

$$\mathcal{W} = \begin{pmatrix} W_{XX}{}^\mu{}_\alpha & W_{XL}{}^\mu{}_\beta \\ W_{LX}{}^\nu{}_\alpha & W_{LL}{}^\nu{}_\beta \end{pmatrix}, \tag{10}$$

its transpose

$$\mathcal{W}^\mathrm{T} = \begin{pmatrix} \left(W_{XX}{}^\mathrm{T}\right)_\alpha{}^\mu & \left(W_{LX}{}^\mathrm{T}\right)_\alpha{}^\nu \\ \left(W_{XL}{}^\mathrm{T}\right)_\beta{}^\mu & \left(W_{LL}{}^\mathrm{T}\right)_\beta{}^\nu \end{pmatrix} \tag{11}$$

with $W_{**}{}^\mu{}_\alpha = \left(W^\mathrm{T}_{**}\right)_\alpha{}^\mu$, and the nondegenerate, antisymmetric matrix

$$\Omega = \begin{pmatrix} 0 & g_{\alpha\beta} \\ -g_{\gamma\delta} & 0 \end{pmatrix} \tag{12}$$

defining the symplectic structure on the tangent bundle $T\mathcal{M}$ [30–32]. It can be shown that

$$\mathcal{W}^\mathrm{T} \Omega \mathcal{W} = \Omega. \tag{13}$$

Inverting (13) and combining the result with (8) and (9) yields:

$$K_{\mathcal{O}\mu} = K_{\mathcal{E}\nu} W_{LL}{}^\nu{}_\mu - (\nabla_\sigma K_\nu)_\mathcal{E} \ell^\sigma_\mathcal{E} W_{XL}{}^\nu{}_\mu \tag{14}$$

$$(\nabla_\sigma K_\mu)_\mathcal{O} \ell^\sigma_\mathcal{O} = -K_{\mathcal{E}\nu} W_{LX}{}^\nu{}_\mu + (\nabla_\sigma K_\nu)_\mathcal{E} \ell^\sigma_\mathcal{E} W_{XX}{}^\nu{}_\mu. \tag{15}$$

2.2. The linear GDE and its conserved quantitites

The existence of Killing vectors also affects the properties of the GDE and its solutions. Namely, suppose that ξ^μ satisfies the GDE along the geodesic, and K^μ is a Killing vector. Then the following quantity is conserved along the geodesic curve [33]:

$$\xi_\mu \nabla_\ell K^\mu - K_\mu \nabla_\ell \xi^\mu = \Sigma. \tag{16}$$

This can be proven in the following way. We have that both ξ and K satisfy the GDE:

$$\begin{aligned} \nabla_\ell \nabla_\ell \xi^\mu - \mathcal{R}^\mu{}_\nu \xi^\nu &= 0 \\ \nabla_\ell \nabla_\ell K^\mu - \mathcal{R}^\mu{}_\nu K^\nu &= 0. \end{aligned} \tag{17}$$

Now contract the first equation with K_μ and the second one with ξ_μ, and subtract one from the other. Due to the symmetries of Riemann tensor we are left with

$$K_\mu \nabla_\ell \nabla_\ell \xi^\mu - \xi_\mu \nabla_\ell \nabla_\ell K^\mu = 0. \tag{18}$$

Finally, use the linearity of the covariant derivative and cancel out similar terms to write the expression as a covariant derivative along ℓ:

$$\nabla_\ell \left(K_\mu \nabla_\ell \xi^\mu - \xi_\mu \nabla_\ell K^\mu \right) = 0. \tag{19}$$

We can also assign a physical meaning to the quantity Σ [22]. Suppose C is a conserved quantity generated by a Killing vector: $C = K^\mu \ell_\mu$. Now let us take a covariant derivative along the deviation vector ξ. Recalling that ξ is Lie dragged along ℓ and that K is a Killing vector, we can show that:

$$\nabla_\xi C = \Sigma. \tag{20}$$

Hence, Σ is simply a variation of a Killing conserved quantity along ξ.

Apart from the conservation laws connected with the Killing vectors, we automatically have two conserved quantities in GDE in any spacetime. Let the geodesic tangent vector ℓ^μ and the deviation vector ξ^μ be evaluated at the same point $x^\mu(\lambda)$ along the geodesic. Then

$$\ell^\mu \xi_\mu = \mathcal{A} + \mathcal{B}\lambda, \tag{21}$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants [6]. Thus we have the conservation of two quantities: $\mathcal{B} = \nabla_\ell \xi^\mu \ell_\mu$ and $\mathcal{A} = \ell^\mu \xi_\mu - \lambda \cdot \nabla_\ell \xi^\mu \ell_\mu$. Finally, due to the symmetries of Riemann tensor, any two solutions ξ_1, ξ_2 of the GDE generate a constant of integration:

$$\xi_{1\mu} \nabla_\ell {\xi_2}^\mu - \xi_{2\mu} \nabla_\ell {\xi_1}^\mu = \text{const.} \tag{22}$$

This expression is bilinear and antisymmetric in the solutions ξ_1 and ξ_2. It defines a conserved symplectic form in the space of solutions [30].

3. Static spherically symmetric spacetimes

3.1. Solution of the geodesic equation in arbitrary spherical coordinates

A static spherically symmetric spacetime has the metric of the following form:

$$g = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + C(r)\left(\mathrm{d}\theta^2 + \sin\,\theta^2\,\mathrm{d}\phi^2\right). \tag{23}$$

The radial coordinate may be reparametrized $r \to r(\tilde{r})$. In particular, without the loss of generality, we may choose the area radius r, defined by $C(r) = r^2$. Nevertheless, in the subsequent calculations we will keep the radial coordinate general with $A(r), B(r), C(r)$ treated as independent functions.

The spacetime has a four-parameter symmetry group generated by four Killing vectors [34]:

$$\begin{aligned} T^\mu &= (1, 0, 0, 0) \\ \Phi_x^\mu &= (0, 0, -\sin\,\phi, -\cot\,\theta\,\cos\,\phi) \\ \Phi_y^\mu &= (0, 0, \cos\,\phi, -\cot\,\theta\,\sin\,\phi) \\ \Phi_z^\mu &= (0, 0, 0, 1). \end{aligned} \tag{24}$$

We will now briefly present the derivation of the general solution of the geodesic equation in the implicit form. This is a well-known material, but we present it here for completeness and to introduce the notation for the following sections.

The Killing vectors simplify the problem of solving the geodesic equation. Namely, each Killing vector generates a conserved quantity along the geodesic:

$$E = -A\,(r)\,\ell^t \tag{25}$$

$$L_x = -C\,(r)\left(\sin\,\phi\ell^\theta + \frac{\sin\,2\theta}{2}\,\cos\,\phi\ell^\phi\right) \tag{26}$$

$$L_y = C\,(r)\left(\cos\,\phi\,\ell^\theta - \frac{\sin\,2\theta}{2}\,\sin\,\phi\ell^\phi\right) \tag{27}$$

$$L_z = C\,(r)\sin^2\theta\ell^\phi, \tag{28}$$

ℓ^μ denoting the components of the tangent vector. By convention, we consider here null geodesics parametrized backwards in time, from the observer towards the emitter. For this reason we demand $\ell^t = \frac{\mathrm{d}t}{\mathrm{d}\lambda} < 0$. For simplicity we also fix the parametrization so that the observation point corresponds to $\lambda_{\mathcal{O}} = 0$.

Since ℓ^μ is tangent to a geodesic, its length $\epsilon = g_{\mu\nu}\ell^\mu\ell^\nu$ is conserved as well:

$$\epsilon = -A\,(r)\,(\ell^t)^2 + B\,(r)\,(\ell^r)^2 + C\,(r)\left(\left(\ell^\theta\right)^2 + \sin^2\theta\left(\ell^\phi\right)^2\right). \tag{29}$$

Obviously, photon worldlines correspond to $\epsilon = 0$, but we keep ϵ here unspecified to allow for unconstrained variations of the initial data of the geodesic. Upon the substitution of (25)–(28) into (29) one gets

$$\epsilon = -\frac{E^2}{A} + B\cdot(\ell^r)^2 + \frac{L_x^2 + L_y^2 + L_z^2}{C}, \tag{30}$$

where $L_x^2 + L_y^2 + L_z^2 = L^2$. This can be solved for the radial component ℓ^r:

$$\ell^r = \pm_r\sqrt{\frac{\epsilon}{B} + \frac{E^2}{AB} - \frac{L^2}{BC}}. \tag{31}$$

$\pm_r$ here denotes the two possible sign choices for the radial component. This expression for ℓ^r in terms of the conserved quantities and r (implicitly present in the metric components $A(r)$, $B(r)$, $C(r)$) will be important later. Since $\ell^r(\lambda) = \frac{\mathrm{d}r(\lambda)}{\mathrm{d}\lambda}$, (31) can be seen as a first order ODE for $r(\lambda)$, which in turn can be solved as an integral with respect to r:

$$\lambda = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{\pm_r\mathrm{d}\tilde{r}}{\sqrt{\frac{\epsilon}{B} + \frac{E^2}{AB} - \frac{L^2}{BC}}}. \tag{32}$$

We can also solve the equations (25)–(29) for ℓ^t and integrate the resulting ODE obtaining

$$t_{\mathcal{E}} - t_{\mathcal{O}} = -\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \frac{E}{A}\,\mathrm{d}\tilde{\lambda} = \fint_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{E}{A}\,\frac{\mathrm{d}\tilde{r}}{\ell^r}. \tag{33}$$

We have changed the integration variable to r in the second expression. Recall that in (31) we have expressed ℓ^r in terms of r and conserved quantities so that the second integral can be evaluated outright.

Now we will consider angular coordinates. From (25)–(28) we observe that

$$\frac{\mathrm{d}\theta}{\sin^2\theta} = \left(\frac{L_y}{L_z}\cos\phi - \frac{L_x}{L_z}\sin\phi\right)\mathrm{d}\phi,$$

which can be integrated to

$$\cot\theta_{\mathcal{O}} - \cot\theta_{\mathcal{E}} = \frac{L_x}{L_z}\left(\cos\phi_{\mathcal{E}} - \cos\phi_{\mathcal{O}}\right) + \frac{L_y}{L_z}\left(\sin\phi_{\mathcal{E}} - \sin\phi_{\mathcal{O}}\right). \tag{34}$$

This implies that the value of θ along the geodesic is completely determined by the value of the coordinate ϕ and the constants L_x, L_y and L_z. This is an expression of the fact that the geodesic is contained in a plane orthogonal to $\vec{L}$. In the final step we solve (25)–(29) for ℓ^ϕ, integrate the resulting ODE and this way derive an implicit expression for $\phi_{\mathcal{E}}$:

$$\int_{\phi_{\mathcal{O}}}^{\phi_{\mathcal{E}}} \sin^2\theta\,\mathrm{d}\phi = \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \frac{L_z}{C}\,\mathrm{d}\tilde{\lambda} = ⨍_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \frac{L_z}{C\ell^r}\,\mathrm{d}\tilde{r}. \tag{35}$$

Equations (32)–(35), together with (25)–(29), form the general solution of the geodesic equation in an implicit form.

3.2. Solution of the geodesic equation in aligned coordinates

Before going on, we note that we can also make another use of the large symmetry group of the problem. The geodesic motion in static spherically symmetric spacetime is analogous to the central force problem in Newtonian gravity. Thanks to the $SO(3)$ symmetry, one can rotate the coordinate system to contain the geodesic motion in the $\theta = \pi/2$ plane. This corresponds to angular momentum having only L_z as a non-zero component. In such a coordinate system the geodesic equation is solved by

$$\begin{aligned}
t_{\mathcal{E}} - t_{\mathcal{O}} &= -\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \frac{E}{A}\,\mathrm{d}\tilde{\lambda} \\
\lambda_{\mathcal{E}} - \lambda_{\mathcal{O}} &= ⨍_{r_{\mathcal{O}}}^{r_{\mathcal{E}}} \pm_r \sqrt{\frac{ABC}{AC\epsilon + E^2 C - L_z^2 A}}\,\mathrm{d}\tilde{r} \\
\phi_{\mathcal{E}} - \phi_{\mathcal{O}} &= \int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \frac{L_z}{C}\,\mathrm{d}\tilde{\lambda} \\
\theta &= \frac{\pi}{2}.
\end{aligned} \tag{36}$$

In the same way, we may impose the condition $t_{\mathcal{O}} = 0$ by applying an appropriate time translation. We will call the coordinate system adapted this way to a given geodesic the *aligned coordinate system*.

3.3. Bilocal geodesic operators from the variations of the general solution of the geodesic equation

The GDE together with its initial data is a system of second order ODEs. Its solution can be found analytically only in the simplest cases. However, it turns out that it is possible to circumvent this problem if we know the general solution of the geodesic equation on our manifold in an explicit or implicit form. In that case the components of $\mathcal{W}$ can be found by simple differentiation. This approach is not new and has been considered previously [15, 35], but only in the context of the Hamilton–Jacobi equation for the geodesic motion: suppose we have the solution to the geodesic equation expressed in terms of the curve parameter and the integration constants. Suppose also that we have a complete integral of the associated Hamilton–Jacobi equation. Then the variation of this integral with respect to the coordinates of the geodesic and the geodesic constants yields the solution to the GDE.

The method we present is a bit different. It avoids the Hamilton–Jacobi equation and provides a direct path from the solution of the geodesic equation to the BGOs. The technique that we will apply has already been described partially in [36], i.e. without the operators W_{LX} and W_{LL}, and we will describe it now in full detail. It is coordinate-dependent in the sense that it requires fixing a coordinate system in which we know how to solve the geodesic equation.

Let $x^\mu\left(x^\nu_{\mathcal{O}}, \ell^\nu_{\mathcal{O}}, \lambda\right)$ be the general solution to the geodesic equation written in coordinates (ξ^μ) with the initial data $x^\mu(\lambda_{\mathcal{O}}) = x^\mu_{\mathcal{O}}$, $\ell^\mu(\lambda_{\mathcal{O}}) = \ell^\mu_{\mathcal{O}}$. Let $x^\mu_{\mathcal{E}}$ denote the coordinates of the second endpoint of the geodesic, corresponding to a fixed value of the affine parameter $\lambda = \lambda_{\mathcal{E}}$, and let $\ell^\mu_{\mathcal{E}}$ denote the tangent vector to the geodesic in $\lambda = \lambda_{\mathcal{E}}$. Let us now consider the *full covariant* variation with respect to *all* variables, including λ, of the second endpoint $\mathcal{E}$, taken at $(x^\mu_{\mathcal{O}}, \ell^\mu_{\mathcal{O}}, \lambda_{\mathcal{E}})$. It reads:

$$\delta x^\mu_{\mathcal{E}} = {W_{XX}}^\mu{}_\nu \delta x^\nu_{\mathcal{O}} + {W_{XL}}^\mu{}_\nu \Delta\ell^\nu_{\mathcal{O}} + \ell^\mu_{\mathcal{E}}\delta\lambda \tag{37}$$

$$\Delta\ell^\mu_{\mathcal{E}} = {W_{LX}}^\mu{}_\nu \delta x^\nu_{\mathcal{O}} + {W_{LL}}^\mu{}_\nu \Delta\ell^\nu_{\mathcal{O}}. \tag{38}$$

These relations generalize (4) to the situation when we allow the affine parameter of the second endpoint of the geodesic to vary as well, i.e. $\lambda = \lambda_{\mathcal{E}} + \delta\lambda$. They follow simply from (4) and the definition of a geodesic. Namely, the position variation under the variations of λ, with fixed initial data ($\delta x^\mu_{\mathcal{O}} = \Delta l^\mu_{\mathcal{O}} = 0$), is by definition given by the tangent vector $\ell^\mu_{\mathcal{E}}$, and hence the last term in (37). On the other hand, the *covariant* variations of the tangent vector along any fixed geodesic γ must vanish because the tangent vector ℓ^μ is covariantly constant ($\nabla_\ell \ell^\mu = 0$), and hence no $\delta\lambda$ term in (38).

Now, it follows from equations (37) and (38) that by taking the full solution to the geodesic equation in a given coordinate system (ξ^μ), differentiating it with respect to all the components of $x^\mu_{\mathcal{O}}$, $\ell^\mu_{\mathcal{O}}$ and λ, and expressing the results in terms of covariant differentials, component by component, one can recover all the BGOs. Their components, expressed in the coordinate tetrads of the coordinate system (ξ^μ), play simply the role of the expansion coefficients in the basis $(\delta x^\mu_{\mathcal{O}}, \Delta\ell^\mu_{\mathcal{O}}, \delta\lambda)$. With this result in hand, we are now ready to describe step by step how we can evaluate $\mathcal{W}$ from the derivatives of the general solution of the geodesic equation.

We begin with ordinary total variations of $x^\mu\left(x^\nu_{\mathcal{O}}, \ell^\nu_{\mathcal{O}}, \lambda\right)$ and the tangent vector $\ell^\mu(x^\nu_{\mathcal{O}}, \ell^\nu_{\mathcal{O}}, \lambda) = \frac{\partial x^\mu}{\partial\lambda}$, taken at $\lambda = \lambda_{\mathcal{E}}$:

$$\begin{aligned} \delta x^\mu_{\mathcal{E}} &= \left(\frac{\partial x^\mu_{\mathcal{E}}}{\partial x^\nu_{\mathcal{O}}}\right)_{\ell_{\mathcal{O}},\lambda} \delta x^\nu_{\mathcal{O}} + \left(\frac{\partial x^\mu_{\mathcal{E}}}{\partial \ell^\nu_{\mathcal{O}}}\right)_{x_{\mathcal{O}},\lambda} \delta\ell^\nu_{\mathcal{O}} + \left(\frac{\partial x^\mu_{\mathcal{E}}}{\partial\lambda}\right)_{x_{\mathcal{O}},\ell_{\mathcal{O}}} \delta\lambda \\ \delta \ell^\mu_{\mathcal{E}} &= \left(\frac{\partial \ell^\mu_{\mathcal{E}}}{\partial x^\nu_{\mathcal{O}}}\right)_{\ell_{\mathcal{O}},\lambda} \delta x^\nu_{\mathcal{O}} + \left(\frac{\partial \ell^\mu_{\mathcal{E}}}{\partial \ell^\nu_{\mathcal{O}}}\right)_{x_{\mathcal{O}},\lambda} \delta\ell^\nu_{\mathcal{O}} + \left(\frac{\partial \ell^\mu_{\mathcal{E}}}{\partial\lambda}\right)_{x_{\mathcal{O}},\ell_{\mathcal{O}}} \delta\lambda. \end{aligned} \tag{39}$$

Just like in thermodynamics, the subscripts denote here variables kept fixed during respective variations. Note also that we have used $\delta x^\mu_{\mathcal{E}}$ for the variation of x^μ and $\delta\ell^\mu_{\mathcal{E}}$ for the variation of

ℓ^μ. Now we apply (5) to change the basis of variations from $\left(\delta x^\mu_\mathcal{O}, \delta\ell^\mu_\mathcal{O}, \lambda\right)$ to $\left(\delta x^\mu_\mathcal{O}, \Delta\ell^\mu_\mathcal{O}, \lambda\right)$ and switch from $\delta l^\mu_\mathcal{E}$ to $\Delta l^\mu_\mathcal{E}$ in the second equation. Together with (37) and (38), this leads to the following relations:

$$
\begin{aligned}
{W_{XL}}^\mu{}_\nu &= \left(\frac{\partial x^\mu_\mathcal{E}}{\partial \ell^\nu_\mathcal{O}}\right)_{x_\mathcal{O},\lambda} \\
{W_{XX}}^\mu{}_\nu &= \left(\frac{\partial x^\mu_\mathcal{E}}{\partial x^\nu_\mathcal{O}}\right)_{\ell_\mathcal{O},\lambda} - {W_{XL}}^\mu{}_\beta \Gamma^\beta{}_{\alpha\nu}\left(x_\mathcal{O}\right)\ell^\alpha_\mathcal{O} \\
{W_{LL}}^\mu{}_\nu &= \left(\frac{\partial \ell^\mu_\mathcal{E}}{\partial \ell^\nu_\mathcal{O}}\right)_{x_\mathcal{O},\lambda} + \Gamma^\mu{}_{\alpha\beta}\left(x_\mathcal{E}\right)\ell^\alpha_\mathcal{E} {W_{XL}}^\beta{}_\nu \\
{W_{LX}}^\mu{}_\nu &= \left(\frac{\partial \ell^\mu_\mathcal{E}}{\partial x^\nu_\mathcal{O}}\right)_{\ell_\mathcal{O},\lambda} + \Gamma^\mu{}_{\alpha\beta}\left(x_\mathcal{E}\right)\ell^\alpha_\mathcal{E} {W_{XX}}^\beta{}_\nu - {W_{LL}}^\mu{}_\beta \Gamma^\beta{}_{\alpha\nu}\left(x_\mathcal{O}\right)\ell^\alpha_\mathcal{O} \\
&\quad + \Gamma^\mu{}_{\alpha\gamma}\left(x_\mathcal{E}\right)\ell^\alpha_\mathcal{E} {W_{XL}}^\gamma{}_\beta \Gamma^\beta{}_{\alpha\nu}\left(x_\mathcal{O}\right)\ell^\alpha_\mathcal{O}.
\end{aligned}
\tag{40}
$$

We have thus expressed the four bitensors by $\ell^\mu_\mathcal{O}$, the Christoffel symbols at $\mathcal{O}$ and $\mathcal{E}$ and the derivatives of $x^\mu(x^\nu_\mathcal{O}, \ell^\nu_\mathcal{O}, \lambda)$ (the first derivatives $\frac{\partial x^\mu_\mathcal{E}}{\partial x^\nu_\mathcal{O}}, \frac{\partial x^\mu_\mathcal{E}}{\partial \ell^\nu_\mathcal{O}}, \ell^\mu_\mathcal{E} \equiv \frac{\partial x^\mu_\mathcal{E}}{\partial \lambda}$ and the second derivatives $\frac{\partial \ell^\mu_\mathcal{E}}{\partial x^\nu_\mathcal{O}} \equiv \frac{\partial^2 x^\mu_\mathcal{E}}{\partial\lambda\partial x^\nu_\mathcal{O}}, \frac{\partial \ell^\mu_\mathcal{E}}{\partial \ell^\nu_\mathcal{O}} \equiv \frac{\partial^2 x^\mu_\mathcal{E}}{\partial\lambda\partial \ell^\nu_\mathcal{O}}$). These bitensors are sufficient to reconstruct optical observables, such as the matrix of magnification and parallax, as well as position and redshift drifts (see [6]).

When we apply the BGO formalism to light rays, we need to impose one more requirement for the variations of the endpoints. Namely, we limit the admissible variations to those which, at the leading, linear order, preserve the null character of the corresponding geodesics. This can be done either before or after the variation. In the first case $\left(\Delta\ell^\sigma_\mathcal{O}\ell_{\mathcal{O}\sigma} = 0\right)$ we obtain $\mathcal{W}$ that is restricted to a subspace of codimension 1 and thus contains less information [36]. In the second case no information is lost.

In this paper we assume that a geodesic can be found for an arbitrary causal character. We thus denote the normalization parameter by ϵ, and allow it to vary arbitrarily:

$$\Delta\ell^\sigma_\mathcal{O}\ell_{\mathcal{O}\sigma} = \delta\epsilon. \tag{41}$$

We will now apply the method to our spacetime model. We note here that this is a type of ‘brute force’ approach to the problem: it is quasi-algorithmic, but at the same time it requires a lot of algebraic manipulations, involving the differentiation of conservation laws, solving of systems of linear equations, and matrix multiplication. It may therefore turn out to be unpractical for more complicated metrics without the help of computer-assisted algebra. It is, however, applicable to any metric with a sufficient number of conservation laws.

For the spacetime of our interest, the majority of expressions for the components of the geodesic are implicit, so a few more steps have to be taken. The implicit relations defining geodesics consist of two types of equations. In the first group, i.e. (25)–(29), we have the definitions of five conserved quantities expressed in terms of the initial data $x^\mu_\mathcal{O}$, $\ell^\mu_\mathcal{O}$. We can write them symbolically as

$$J_i = f_i(x^\mu_\mathcal{O}, \ell^\mu_\mathcal{O}), \tag{42}$$

with $i = 1, \ldots, 5$. The other group of equations relates the photon's position at $\lambda = \lambda_\mathcal{E}$ with the initial position and the conserved quantities. These four equations have the form of implicit relations between the coordinates of the point $x^\mu_\mathcal{E}$ on the geodesic, the corresponding value of the affine parameter $\lambda_\mathcal{E}$, the initial point $x^\mu_\mathcal{O}$ and the conserved quantities J_i. The relations are implicit, and three of them comprise integrals. In a symbolic form we may write them down as

$$\begin{aligned} h_t(t_\mathcal{E}, x^\mu_\mathcal{O}, J_i, \lambda_\mathcal{E}) &= 0 \\ h_r(r_\mathcal{E}, x^\mu_\mathcal{O}, J_i, \lambda_\mathcal{E}) &= 0 \\ h_\theta(\theta_\mathcal{E}, x^\mu_\mathcal{O}, J_i, \lambda_\mathcal{E}) &= 0 \\ h_\phi(\phi_\mathcal{E}, x^\mu_\mathcal{O}, J_i, \lambda_\mathcal{E}) &= 0. \end{aligned} \tag{43}$$

While the total number of conserved quantities is five, we need to note that the values of the three components of the angular momentum are not entirely independent. Namely, given the vector $\vec{y}_\mathcal{O} = (r_\mathcal{O}\ \sin\ \theta_\mathcal{O}\ \cos\ \phi_\mathcal{O}, r_\mathcal{O}\ \sin\ \theta_\mathcal{O}\ \sin\ \phi_\mathcal{O}, r_\mathcal{O}\ \cos\ \theta_\mathcal{O})$, defining the photon position in quasi-Cartesian coordinates, we have

$$\vec{y}_\mathcal{O}(x^\mu_\mathcal{O}) \cdot \vec{L} = 0. \tag{44}$$

Moreover, this relation, with the same $\vec{L}$, must hold at all times along a geodesic. Equation (44) expresses the fact that all orbits in a spherically symmetric spacetime are planar, with the orbital plane perpendicular to $\vec{L}$. Now, given the initial point $x^\mu_\mathcal{O}$, we may use (44) to eliminate one of the components of $\vec{L}$ in favour of the other two. Therefore the total number of independent relations we have obtained is just 8.

We know from the previous section that we can choose our coordinate system to be aligned, meaning that $\theta_\mathcal{O} = \frac{\pi}{2}, \dot{\theta}_\mathcal{O} = 0, \phi_\mathcal{O} = 0, t_\mathcal{O} = 0$ for the fiducial null geodesic γ_0. We will impose this coordinate condition, but only after the variations are performed, to keep the geodesics' variations unconstrained.

The derivation of $\mathcal{W}$ in the coordinate tetrads goes through a sequence of algebraic manipulations of the linear relations between the variations of the initial data, the data at $\mathcal{E}$ and the conserved quantities. These linear relations, in turn, are obtained by taking total variations of the implicit equations above. Note that since all algebraic operations involved in this procedure, i.e. substituting and solving for particular variations, are performed at the level of linearized relations, the method always works, although it is, in the end, rather cumbersome.

The derivation of $\mathcal{W}$ proceeds now as follows: we begin by varying the first set of equations, i.e. (25)–(28), or symbolically (42), obtaining the relations

$$\delta J_i = \mathcal{L}(\delta x^\mu_\mathcal{O}, \delta \ell^\nu_\mathcal{O}), \tag{45}$$

with $\mathcal{L}$ denoting from now on any unspecified linear relation. We then vary the second set, i.e. (32)–(35), or (43), obtaining after simple manipulations relations of the type

$$\delta x^\mu_\mathcal{E} = \mathcal{L}(\delta x^\alpha_\mathcal{O}, \delta J_i, \delta\lambda). \tag{46}$$

After the variation we impose the condition for aligned coordinates in the sense of section 3.2 for the fiducial null geodesic. This way we simplify the algebraic expressions for the coefficients present in both linear relations. We then substitute (45) into (46), obtaining direct relations between the variations of the initial data and the variations of the final position:

$$\delta x^\mu_\mathcal{E} = \mathcal{L}(\delta x^\alpha_\mathcal{O}, \delta \ell^\beta_\mathcal{O}, \delta\lambda). \tag{47}$$

We have derived this way the first half of the linear relations we need.

The second half is the linear relations between the variations of $\ell^\mu_\mathcal{E}$ and the variations of the initial data. We can obtain them from the conservation of J_i. Note that the variations δJ_i are related to the variations of the data $(x^\mu_\mathcal{E}, \ell^\mu_\mathcal{E})$ at $\mathcal{E}$ by the same functional relations as those at the initial point, i.e. we have

$$\delta J_i = \mathcal{L}(\delta x^\mu_\mathcal{E}, \delta \ell^\nu_\mathcal{E}), \tag{48}$$

with the same coefficients of $\mathcal{L}$ as in (45), but evaluated at point $\mathcal{E}$ instead of $\mathcal{O}$. We may now combine (48) with (45) and solve the resulting linear equations for $\delta\ell^\mu_\mathcal{E}$. This yields a relation of type $\delta\ell^\mu_\mathcal{E} = \mathcal{L}(\delta x^\alpha_\mathcal{O}, \delta\ell^\beta_\mathcal{O}, \delta x^\gamma_\mathcal{E}, \delta\lambda)$. We now need to eliminate $\delta x^\gamma_\mathcal{E}$ from this relation using (47) to obtain

$$\delta\ell^\mu_\mathcal{E} = \mathcal{L}(\delta x^\alpha_\mathcal{O}, \delta\ell^\beta_\mathcal{O}, \delta\lambda). \tag{49}$$

By comparing with (39), we note that the coefficients of the linear relations in (47) and (49) must be equal to the partial derivatives of the functions $x^\mu(x^\alpha_\mathcal{O}, \ell^\beta_\mathcal{O}, \lambda)$ and $\ell^\mu(x^\alpha_\mathcal{O}, \ell^\beta_\mathcal{O}, \lambda)$. Therefore, in the final step we can use (40) to calculate the components of $\mathcal{W}$ in the coordinate tetrads directly from the coefficients of $\mathcal{L}$ in (47) and (49). The result is quite complicated, and we present it in appendix A.3, while the intermediate steps of the calculations are contained in appendix A.1. However, as we will see in section 3.6, it can be simplified by a lot with an appropriate choice of the two tetrads.

3.4. Geodesic bilocal operators from the solution of GDE by using Killing vectors

The previous method is very straightforward and could in principle be implemented as an algorithm with any computer algebra program. On the other hand, its manual implementation is extremely tedious. For this reason we present a simpler method which uses directly the GDE and its conserved quantities.

3.4.1. Overview of the method. The method of Killing conservation uses the fact that each Killing vector generates the first integral of GDE. The conserved quantities can then reduce the order of the GDE system and the new first order system of ODE's is much easier to solve. In the process, one must introduce integration constants, which can be related to the perturbations of initial data via the GDE conservation equations. Then the components of W operators can be read off one by one. Unlike the method of initial data variations, where variation had to be done in arbitrary coordinates to retain all the effects of deviation, and only in the very end a particular coordinate choice was set, here we may work in the aligned coordinates from the very beginning. GDE already contains all effects we are interested in, and we can start in the aligned coordinates without loss of generality.

3.4.2. Conservation equations. From (16) we know that Killing vectors generate the first integrals of GDE. However, equation (16) requires not only the Killing vector, but also its covariant derivatives along the geodesic. We begin with the evaluation of derivatives in the aligned coordinates:

$$\begin{aligned}
&T^\mu = (1,0,0,0) && \nabla_\ell T^\mu = \frac{A'}{2}\left(\frac{\ell^r}{A}, \frac{\ell^t}{B}, 0, 0\right)\\
&\Phi^\mu_z = (0,0,0,1) && \nabla_\ell \Phi^\mu_z = \frac{C'}{2}\left(0, -\frac{\ell^\phi}{B}, 0, \frac{\ell^r}{C}\right)\\
&\Phi^\mu_x = (0,0,-\sin\phi, 0) && \nabla_\ell \Phi^\mu_x = \left(0,0,-\cos\phi\,\ell^\phi - \frac{C'}{2C}\ell^r\sin\phi, 0\right)\\
&\Phi^\mu_y = (0,0,\cos\phi,0) && \nabla_\ell \Phi^\mu_y = \left(0,0,-\sin\phi\,\ell^\phi + \frac{C'}{2C}\ell^r\cos\phi, 0\right).
\end{aligned} \tag{50}$$

The first integrals of the GDE in the form of (2), generated by Killing vectors have the following form:

$$\begin{aligned}
\Sigma_x &= C\sin\phi\frac{\mathrm{d}\xi^\theta}{\mathrm{d}\lambda} - C\xi^\theta\cos\phi\,\ell^\phi\\
\Sigma_y &= -C\cos\phi\frac{\mathrm{d}\xi^\theta}{\mathrm{d}\lambda} - C\xi^\theta\sin\phi\ell^\phi\\
\Sigma_z &= -C\frac{\mathrm{d}\xi^\phi}{\mathrm{d}\lambda} - C'\xi^r\ell^\phi\\
\Sigma_T &= A\frac{\mathrm{d}\xi^t}{\mathrm{d}\lambda} + A'\xi^r\ell^t.
\end{aligned} \tag{51}$$

By evaluating (16) at the initial point, where $\xi^\mu = \delta x^\mu_{\mathcal{O}}$ and $\nabla_l \xi^\mu = \Delta l^\mu_{\mathcal{O}}$, we find expressions of conserved quantities in terms of initial data:

$$\begin{aligned}
\Sigma_T &= -\frac{A'_{\mathcal{O}}}{2}\ell^r\delta x^t_{\mathcal{O}} + \frac{A'_{\mathcal{O}}}{2}\ell^t\delta x^r_{\mathcal{O}} + A_{\mathcal{O}}\Delta\ell^t_{\mathcal{O}}\\
\Sigma_x &= -\left(C_{\mathcal{O}}\cos\phi_{\mathcal{O}}\ell^\phi_{\mathcal{O}} + \frac{C'_{\mathcal{O}}}{2}\ell^r\sin\phi_{\mathcal{O}}\right)\delta x^\theta_{\mathcal{O}} + C_{\mathcal{O}}\sin\phi_{\mathcal{O}}\Delta\ell^\theta_{\mathcal{O}}\\
\Sigma_y &= \left(-C_{\mathcal{O}}\sin\phi_{\mathcal{O}}\ell^\phi_{\mathcal{O}} + \frac{C'_{\mathcal{O}}}{2}\ell^r\cos\phi_{\mathcal{O}}\right)\delta x^\theta_{\mathcal{O}} - C_{\mathcal{O}}\cos\phi_{\mathcal{O}}\Delta\ell^\theta_{\mathcal{O}}\\
\Sigma_z &= -\frac{C'_{\mathcal{O}}}{2}\ell^\phi_{\mathcal{O}}\delta x^r_{\mathcal{O}} + \frac{C'_{\mathcal{O}}}{2}\ell^r_{\mathcal{O}}\delta x^\phi_{\mathcal{O}} - C_{\mathcal{O}}\Delta\ell^\phi_{\mathcal{O}}.
\end{aligned} \tag{52}$$

We keep $\phi_{\mathcal{O}}$ arbitrary, because setting it to zero at this stage complicates the evaluation of W operators. We leave this value unspecified until ξ is fully expressed in terms of initial data.

Note that the first two equations in (51) are related, i.e. by shifting $\phi \to \phi - \frac{\pi}{2}$ one can obtain the second equation from the first one. Hence, in order to have four independent first integrals we need include (21), or, to be more exact, its covariant derivative along ℓ. Then we have one more first integral:

$$\begin{aligned}
\mathcal{B} &= E\dot{\xi}^t + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(B\ell^r\xi^r\right) + L_z\dot{\xi}^\phi\\
\mathcal{B} &= E\Delta\ell^t_{\mathcal{O}} + B_{\mathcal{O}}\ell^r_{\mathcal{O}}\Delta\ell^r_{\mathcal{O}} + L_z\Delta\ell^\phi_{\mathcal{O}}.
\end{aligned} \tag{53}$$

Now we have a sufficient number of equations for integration.

3.4.3. Solving the equations. We begin by solving for ξ^θ. Although it looks like we need an explicit form of $\phi(\lambda)$, actually we can integrate it without referring to any particular solution. From (51) and (52) we have that:

$$\begin{aligned}\xi^\theta &= \kappa_1 \sin\phi - \frac{\Sigma_x}{L_z}\cos\phi \\ \nabla_\ell \xi^\theta &= \xi^\theta\left(\cot\phi\,\ell^\phi + \frac{C'}{2C}\ell^r\right) + \frac{\Sigma_x}{C\sin\phi}.\end{aligned} \tag{54}$$

Here κ_1 is an arbitrary constant of integration. We see that ξ^θ depends on λ only through $\phi(\lambda)$. This is simply a reiteration of the fact, that in static spherically symmetric spacetimes the dynamics of θ is constrained by ϕ. In the same way, the dynamics of the perturbation of θ is also constrained by ϕ.

The other three components are coupled through ξ^r. From (51) and (53) we write an equation for ξ^r:

$$\dot{\xi}^r - \frac{\dot{\ell}^r}{\ell^r}\xi^r + \frac{1}{B\ell^r}\left(\frac{E\Sigma_T}{A} - \frac{L_z\Sigma_z}{C} - \mathcal{B}\right) = 0. \tag{55}$$

Integrating it yields a solution for ξ^r, which is then used to find ξ^t and ξ^ϕ:

$$\begin{aligned}\xi^r &= \kappa_2\ell^r - \ell^r\int_{\lambda_\mathcal{O}}^{\lambda_\mathcal{E}}\left(\frac{E\Sigma_T}{A} - \frac{L_z\Sigma_z}{C} - \mathcal{B}\right)\frac{\mathrm{d}\lambda}{B{\ell^r}^2} \\ \xi^t &= \kappa_3 + \int_{\lambda_\mathcal{O}}^{\lambda_\mathcal{E}}\left(\frac{\Sigma_T}{A} + \frac{EA'}{A^2}\xi^r\right)\mathrm{d}\lambda \\ \xi^\phi &= \kappa_4 - \int_{\lambda_\mathcal{O}}^{\lambda_\mathcal{E}}\left(\frac{\Sigma_z}{C} + \frac{C'}{C^2}L_z\xi^r\right)\mathrm{d}\lambda,\end{aligned} \tag{56}$$

where $\kappa_2, \kappa_3, \kappa_4$ are arbitrary constants. The last step is to express all the constants in terms of initial data—$\delta x^\mu_\mathcal{O}$ and $\Delta\ell^\mu_\mathcal{O}$. This can be done by evaluating ξ^μ and $\nabla_\ell\xi^\mu$ at $\mathcal{O}$ together with (52) and (53). Then the W operators can be found by comparing ξ^μ and $\nabla_\ell\xi^\mu$ with equations (37) and (38). Formally, we can write

$$W_{XX}{}^\mu{}_\nu = \frac{\partial\xi^\mu}{\partial\delta x^\nu_\mathcal{O}} \qquad W_{XL}{}^\mu{}_\nu = \frac{\partial\xi^\mu}{\partial\Delta\ell^\nu_\mathcal{O}} \qquad W_{LX}{}^\mu{}_\nu = \frac{\partial\nabla_\ell\xi^\mu}{\partial\delta x^\nu_\mathcal{O}} \qquad W_{LL}{}^\mu{}_\nu = \frac{\partial\nabla_\ell\xi^\mu}{\partial\Delta\ell^\nu_\mathcal{O}}. \tag{57}$$

Explicit expressions of all these components with respect to the aligned coordinate tetrad can be found in the appendix A.

3.5. Construction of a parallel propagated tetrad

In the previous sections we described how to obtain an exact solution to the GDE for static spherically symmetric spacetimes in the coordinate tetrad. However, due to the diffeomorphism invariance of general relativity, physical aspects of geodesic deviation are obscured by the choice of coordinates. In order to mitigate this problem we will project our results onto a parallel propagated tetrad. In this paper we will use the SNT [6], while the construction itself is based on Marck [37]. For more details on the complete integrability of parallel transport please check the review in [38].

The SNT $e^{\mu}{}_{\boldsymbol{\mu}} = \left(u^{\mu}, e^{\mu}_{\mathbf{A}}, \ell^{\mu}\right)$ comprises the four-velocity u^{μ}, a null vector ℓ^{μ} and two mutually orthogonal spacelike vectors $e^{\mu}_{\mathbf{A}}$, called the transverse vectors, which are also orthogonal to ℓ^{μ} and u^{μ}. This frame is defined by the following constraints:

$$\begin{aligned}
&\ell^{\mu}\ell_{\mu} = 0\\
&e^{\mu}_{\mathbf{A}}\ell_{\mu} = 0\\
&e^{\mu}_{\mathbf{A}}e_{\mathbf{B}\mu} = \delta_{\mathbf{AB}}\\
&u^{\mu}_{\mathcal{O}}u_{\mathcal{O}\mu} = -1\\
&u^{\mu}_{\mathcal{O}}e_{\mathbf{A}\mu} = 0\\
&\ell^{\mu}u_{\mathcal{O}\mu} = Q > 0,
\end{aligned} \tag{58}$$

where Q is a constant related to the normalization of the null tangent ℓ^{μ}. The construction of the frame will be done in two steps. Firstly, we will reduce the space of tetrads to a subspace of those whose two vectors are parallel propagated. This will leave a one-parameter family of tetrads at each point. Then we will use a linear transformation together with the parallel propagation equation to parallel transport the entire tetrad.

We begin the first step with an observation that ℓ^{μ} is already parallel propagated. To obtain the second vector, we notice that the vector that is perpendicular to the plane passing through the origin of the coordinate system is parallel propagated along the whole plane. In our case, this vector is $e^{\mu}_{\mathbf{1}} = \frac{1}{\sqrt{C}}\partial_{\theta}$ which is spacelike and perpendicular to the plane $z = 0$.

Now we will seek the vector $e^{\mu}_{\mathbf{2}}$. By looking at equation (58) we see that from conditions $e_{\mathbf{1}} \cdot e_{\mathbf{2}} = 0$, $e_{\mathbf{2}} \cdot \ell = 0$ and $e_{\mathbf{2}} \cdot e_{\mathbf{2}} = 1$ we get $e_{\mathbf{2}}$ up to an additive term $c(\lambda) \cdot \ell$. Hence, from purely geometric considerations and without solving *any* ODEs we get an equivalence class of vectors $e_{\mathbf{2}}$ such that for all of them $\nabla_{\ell}e_{\mathbf{2}}$ differs only by $f(\lambda) \cdot \ell$ for some f. Next, we pick a particular instance of $e_{\mathbf{2}}$, say, $\tilde{e}_{\mathbf{2}} = \alpha(\lambda)\,\partial_t + \beta(\lambda)\,\partial_r$, which is not necessarily parallel transported. Then, up to an overall sign, $\tilde{e}_{\mathbf{2}}$ reads

$$\tilde{e}^{\mu}_{\mathbf{2}} = \left(\sqrt{\frac{BC}{A}}\frac{\ell^{r}}{L_z}, -\sqrt{\frac{C}{AB}}\frac{E}{L_z}, 0, 0\right) \tag{59}$$

while the unique associated four-velocity $\tilde{u}^{\mu}$, orthogonal to both transverse vectors, is of the form

$$\tilde{u}^{\mu} = \left(\frac{E}{2AL_z^2Q}\left(L^2 + CQ^2\right), -\frac{\ell^{r}}{2L_z^2Q}, 0, \frac{Q}{2L_z} - \frac{L}{2CQ}\right). \tag{60}$$

We will call the tetrad $(\tilde{u}^{\mu}, e^{\mu}_{\mathbf{1}}, \tilde{e}^{\mu}_{\mathbf{2}}, \ell^{\mu})$ the *intermediate* SNT.

In the second step we will look for a linear λ-dependent transformation that preserves equation (58). Let $\left(\tilde{u}^{\mu}, e^{\mu}_{\mathbf{1}}, \tilde{e}^{\mu}_{\mathbf{2}}, \ell^{\mu}\right)$ and $\left(u^{\mu}, e^{\mu}_{\mathbf{1}}, e^{\mu}_{\mathbf{2}}, \ell^{\mu}\right)$ be SNTs, with the second one being parallel transported. We assume the following ansatz:

$$\begin{pmatrix} u^{\mu}\\ e^{\mu}_{\mathbf{1}}\\ e^{\mu}_{\mathbf{2}}\\ \ell^{\mu}\end{pmatrix} = \begin{pmatrix} a_u & b_u & c_u & d_u\\ a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_{\ell} & b_{\ell} & c_{\ell} & d_{\ell}\end{pmatrix}\begin{pmatrix} \tilde{u}^{\mu}\\ e^{\mu}_{\mathbf{1}}\\ \tilde{e}^{\mu}_{\mathbf{2}}\\ \ell^{\mu}\end{pmatrix}. \tag{61}$$

Since ℓ^{μ} and $e^{\mu}_{\mathbf{1}}$ are already parallel propagated, some coefficients can be set to either zero or one. The rest of the coefficients are determined by the SNT constraints and can be shown to depend on only one function $\Psi(\lambda)$. Then, up to a sign, $e_{\mathbf{2}}^{\mu}$ is

$$e_{\mathbf{2}}^{\mu} = \tilde{e}_{\mathbf{2}}^{\mu} - \frac{\Psi}{Q}\ell^{\mu} \tag{62}$$

while u^{μ} is unique and of the form

$$u^{\mu} = \tilde{u}^{\mu} + \Psi\tilde{e}_{\mathbf{2}}^{\mu} - \frac{\Psi^2}{2Q}\ell^{\mu}. \tag{63}$$

To determine Ψ, we have to use the parallel propagation equation for any single vector of the SNT. For example, demanding $\nabla_{\ell} e_{\mathbf{2}}^{\mu} = 0$ yields

$$\dot{\Psi}\frac{\ell^{\mu}}{Q} = \nabla_{\ell}\tilde{e}_{\mathbf{2}}^{\mu}, \tag{64}$$

where dot denotes the derivative with respect to λ. This has to hold for any component. For example, the equation for component ϕ yields the following simple ODE:

$$\dot{\Psi} = -\frac{C'}{2\sqrt{ABC}}\frac{E}{L_z}Q. \tag{65}$$

The initial condition for Ψ will now fix a particular choice of the solution. The natural choice is $\tilde{e}_{\mathbf{2}}$ at the initial point, leading to $\Psi(0) = 0$, and this is what we use in the next section, but other choices are possible too.

3.6. Projections of operators onto a semi-null tetrad

In the SNT all four BGO's have the following form:

$$W_{**}{}^{\boldsymbol{\mu}}{}_{\boldsymbol{\nu}} = \left(\begin{array}{c|cc|c} \alpha & 0 & 0 & 0 \\ \hline 0 & \blacksquare & 0 & 0 \\ \square & 0 & \blacksquare & 0 \\ \hline \square & 0 & \square & \alpha \end{array}\right). \tag{66}$$

Here α denotes λ, 1, 1 and 0 for the operators W_{XL}, W_{XX}, W_{LL}, W_{LX} respectively. Due to the algebraic properties of BGOs [6], the top row and the rightmost column have a fixed form irrespective of the spacetime geometry. The 2×2 submatrix in the center corresponds to the projection of the BGOs onto the screen space. It is diagonal in the SNT we have constructed, and all of its components are independent of the choice of the observer's four-velocity u as the first component of the tetrad. For this reason, the two nonvanishing components, denoted by $\blacksquare$, do not contain Q or Ψ. This does not apply to the other three nonzero components, denoted by $\square$. Altogether, there are at most five nontrivial components per BGO, but symplectic properties impose seven constraints, which brings the total number of nontrivial independent components down to 13. We present all the nontrivial components of the BGOs and the optical tidal matrix in the aligned coordinate tetrad and the SNT in the appendix A.6.

A portion of our results has been derived earlier, but in different contexts. In [18] the authors studied the luminosity of a spherically collapsing star. By following a bundle of light coming from a surface area element of the star which reached the observer and integrating over the whole surface of the star, they managed to show the dependence of the total observed flux on the radius of the surface. In [21] the topic of the study was the spherical gravitational lens and its modification due to clumpiness of the matter within the lens or a large scale matter distribution surrounding the lens itself.

4. Optical distance measures and distance slip in Schwarzschild spacetime

Most methods of distance determination in astronomy we know use light propagation one way or another. In a flat spacetime, with no relative motions of the sources and all distance measures are perfectly equivalent. In general, however, light propagation is affected by the spacetime curvature, which makes distance measures differ from each other and their counterparts in flat spacetime. This leads to many paradoxical results, such as the finite maximal value of the angular diameter distance in many FLRW Universe models [39–41] or absence of any apparent motion of light sources in others [42]. In this section, we will to present and analyze in detail the behaviour of the distance measures in static spherically symmetric spacetimes with the help of the results of the previous sections. Although this class of metrics is quite special, it is also sufficiently broad and can be used to study the nontrivial behaviour of distance measures. Even though the final results are complicated, everything can be explained exactly.

The two distance measures we consider in this paper are the *angular diameter distance* (also known as the *area distance* [4]) D_{ang} and the *parallax distance* D_{par} [6]. Just for completeness we recall also the notion of the *luminosity distance* D_{lum} [4, 43, 44], defined with the help of the measured flux of energy from a radiating body of known luminosity. It is well-known that it is related to D_{ang} via the Etherington's duality relation [4]. D_{ang} is defined using the ratio of the solid angle an extended object takes up on the observer's celestial sphere to its physical cross-sectional area. The definition of D_{par} on the other hand makes use of the trigonometric parallax effect, i.e. the dependence of the position of the source's image on the celestial sphere on the observer's transverse displacement suitably averaged over the baseline orientation. We will also discuss the distance slip μ, introduced in [6] and studied in greater detail in [45] which is defined as

$$\mu = 1 - \sigma\frac{D_{\text{ang}}^2}{D_{\text{par}}^2}, \tag{67}$$

where for sufficiently short distances $\sigma = 1$. It is an interesting quantity because it directly measures the impact of the spacetime curvature on the light propagation in a frame-independent way. Moreover, for short distances μ is equal to an integral of the mass density along the LOS.

The key observation is that the distance slip and the two distance measures between two points connected by a null geodesic γ_0 can be expressed in terms of the BGOs and the observer's four-velocity [6]. This means that the BGO formalism can be used to investigate the dependence of the distance measures on the null geodesic γ_0 and the positions of the emission and observation point along it.

After introducing the distance measures and the infinitesimally thin ray bundle formalism, we will consider the simplest nontrivial example of static spherically symmetric spacetimes, namely, the Schwarzschild black hole. We will numerically study the behaviour of ray bundles that begin at some distance from the photon sphere, propagate towards and around the black hole, possibly, winding around it a number of times, and, finally, escape to infinity. The form of trajectories will be controlled by the initial data of the fiducial geodesic. Using the results of this paper, we will discuss the properties of D_{ang}, D_{par} and μ associated with sources positioned at different points along the null geodesic. Finally, we will prove a few more general statements regarding to the behaviour of these functions. Since distance measures can be expressed in terms of W, we have all the tools to study them in general.

4.1. Infinitesimally thin bundles

The behaviour of distance measures is much simpler to grasp if we relate it to the properties of infinitesimally thin bundles of light rays. For this reason we briefly review the basics of ray bundle formalism. We follow the definitions and conventions of Perlick [4]. Let $\lambda \mapsto x(\lambda)$ be an affinely parametrized null geodesic with a tangent vector $\ell = \dot{x}$. An *infinitesimally thin bundle of rays* is the set

$$S = \{c^I \xi^\mu_I | c^1, c^2 \in \mathbb{R}, c^I c^J \delta_{IJ} \leqslant 1\}, \tag{68}$$

where, in the parallel propagated SNT, ξ^μ_I satisfies the GDE (cf (2))

$$\ddot{\xi}^\mu{}_I = \mathcal{R}^\mu{}_\nu \xi^\nu{}_{I,} \tag{69}$$

together with the orthogonality constraint

$$g_{\mu\nu} \ell^\mu \xi^\nu{}_I = 0 \tag{70}$$

and I enumerates linearly independent solutions. By construction, its cross-section by the screen space of an observer is elliptical and spacelike. This problem setting is equivalent to the one used to prove the Sachs shadow theorem [1], where a small object in a null geodesic congruence casts a shadow on a screen in motion. In either case, the area of this cross-section is a Lorentz invariant at any given point of the geodesic, i.e. it does not depend on the observer we choose. The area can be expressed as

$$\mathcal{A} = \int_\Sigma \epsilon_{\mathbf{AB}} \xi_1^{\mathbf{A}} \xi_2^{\mathbf{B}}, \tag{71}$$

where $\epsilon_{\mathbf{AB}}$ is the area two-form, and $\xi_I^{\mathbf{A}}$ are the projections of linearly independent solutions of (69) onto the screen space, i.e. space spanned by the two transverse vectors in a SNT. It evolves according to the equation

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\lambda} = \mathcal{A}\theta, \tag{72}$$

where θ is the bundle expansion. Note that the area defined this way is a signed quantity. The sign can change every time the bundle degenerates to a line or a point.

In order to determine θ one has to make use of null Raychaudhuri equations (also known as Sachs optical equations). In this paper we will only consider the twist-free (or surface-forming) bundles, i.e. those for which the twist $\omega_{\mathbf{AB}}$ vanishes. The equations read [46]

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{\theta^2}{2} - \sigma_{\mathbf{AB}} \sigma^{\mathbf{AB}} - R_{\mu\nu} \ell^\mu \ell^\nu \tag{73}$$

$$\frac{\mathrm{d}\sigma_{\mathbf{AB}}}{\mathrm{d}\lambda} = -\theta \sigma_{\mathbf{AB}} + C_{\mathbf{A}\mu\nu\mathbf{B}} \ell^\mu \ell^\nu, \tag{74}$$

which in the BGO formalism are equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} W_{**}{}^{\mathbf{A}}{}_{\mathbf{B}} = \left(-\frac{1}{2} R_{\mu\nu} \ell^\mu \ell^\nu \delta^{\mathbf{A}}{}_{\mathbf{C}} + C^{\mathbf{A}}{}_{\mu\nu\mathbf{C}} \ell^\mu \ell^\nu \right) W_{**}{}^{\mathbf{C}}{}_{\mathbf{B}}. \tag{75}$$

For the purpose of this paper we introduce two infinitesimal ray bundles along γ_0: the *vertex bundle* and the *initially parallel bundle*. The vertex bundle is a bundle of rays crossing at $\mathcal{O}$. It is defined by the singular initial conditions for θ, $\sigma_{\mathbf{AB}}$ and $\omega_{\mathbf{AB}}$ at $\mathcal{O}$ [4]:

$$\theta(\lambda) \sim \frac{2}{\lambda - \lambda_{\mathcal{O}}} \tag{76}$$

$$\sigma_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0 \tag{77}$$

$$\omega_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0. \tag{78}$$

The initially parallel bundle, on the other hand, is strictly parallel at $\mathcal{O}$, i.e.

$$\theta(\lambda) = 0 \tag{79}$$

$$\sigma_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0 \tag{80}$$

$$\omega_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0. \tag{81}$$

Both bundles are twist-free, or surface forming, i.e. $\omega_{\mathbf{AB}} = 0$ along the whole null geodesic. They are both closely related to the transverse components of the operators W_{XX} and W_{XL}: namely, we have

$$\xi^{\mathbf{A}}(\lambda) = {W_{XL}}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)\dot{\xi}^{\mathbf{B}}(\lambda_{\mathcal{O}}) \tag{82}$$

for the vertex bundle and

$$\xi^{\mathbf{A}}(\lambda) = {W_{XX}}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)\xi^{\mathbf{B}}(\lambda_{\mathcal{O}}) \tag{83}$$

for the initially parallel bundle. Note that due to the orthogonality condition (70) ξ^{μ} has only transverse components plus a component proportional to ℓ^{μ}. The latter is irrelevant from the point of view of the geometry of cross-sections (see [1]), so it is the two transverse components of ξ^{μ} given by (82) and (83) that define the distance measures.

Having developed the ray bundle formalism, now we can utilize it to understand distance measures solely in terms of the cross-sectional areas of various ray bundles.

Let us begin with the vertex bundle. By definition of the magnification matrix [6]

$$\delta\theta^{\mathbf{A}}_{\mathcal{O}} = (l_{\mathcal{O}} \cdot u_{\mathcal{O}})^{-1}\left({W_{XL}}^{-1}\right)^{\mathbf{A}}{}_{\mathbf{B}}\delta x^{\mathbf{B}}_{\mathcal{E}}. \tag{84}$$

From (84) and (82) it follows that

$$\delta\theta^{\mathbf{A}}_{\mathcal{O}} = (\ell_{\mathcal{O}} \cdot u_{\mathcal{O}})^{-1}\dot{\xi}^{\mathbf{A}}_{\mathcal{O}}. \tag{85}$$

Integrating (84) over the angular shape of the figure on the observer's sky yields

$$\tilde{\mathcal{A}}(\lambda) = (l_{\mathcal{O}} \cdot u_{\mathcal{O}})^2(\det {W_{XL}}^{\mathbf{A}}{}_{\mathbf{B}})\tilde{\Omega}_{\mathcal{O}}, \tag{86}$$

where $\tilde{\Omega}_{\mathcal{O}}$ is the angular area of the figure as observed from $\mathcal{O}$, and $\tilde{A}(\lambda)$ is the physical area of the cross-section of the ray bundle at $\mathcal{E}$. Since the angular diameter distance reads

$$D_{\text{ang}} = (l_{\mathcal{O}} \cdot u_{\mathcal{O}})\left|\det\, {W_{XL}}^{\mathbf{A}}{}_{\mathbf{B}}\right|^{1/2}, \tag{87}$$

it simply follows that

$$D_{\text{ang}} = \sqrt{\frac{|\tilde{\mathcal{A}}(\lambda)|}{\tilde{\Omega}_{\mathcal{O}}}}. \tag{88}$$

Previously introduced initial conditions for the vertex bundle imply that its cross-sectional area at $\mathcal{O}$ exhibits the following behaviour:

$$\tilde{\mathcal{A}}\,(\lambda) = (\lambda - \lambda_{\mathcal{O}})^2 (\ell_{\mathcal{O}} \cdot u_{\mathcal{O}})^2 \tilde{\Omega}_{\mathcal{O}} + \mathcal{O}\left(\lambda^3\right). \tag{89}$$

Consider now the initially parallel bundle. Its evolution is described by the W_{XX} operator, which stands for the following mapping:

$$\xi^{\mathbf{A}}(\lambda) = W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)\xi^{\mathbf{B}}(\lambda_{\mathcal{O}}). \tag{90}$$

Suppose the cross-sectional area of this bundle at $\mathcal{O}$ is $\mathcal{A}_{\mathcal{O}}$. Integration of (90) over the initial shape of the cross-section gives [7]

$$\mathcal{A}\,(\lambda) = \left(\det\, W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}\right) \mathcal{A}_{\mathcal{O}}, \tag{91}$$

which allows us to rewrite μ as

$$\mu = 1 - \frac{\mathcal{A}\,(\lambda)}{\mathcal{A}_{\mathcal{O}}} \tag{92}$$

with $\mathcal{A}\,(\lambda_{\mathcal{O}}) = \mathcal{A}_{\mathcal{O}}$. Finally, the substitution of the results presented above into (67) enables us to write down the parallax distance:

$$D_{\text{par}} = \sqrt{\left|\frac{\tilde{\mathcal{A}}\,(\lambda)}{\mathcal{A}\,(\lambda)}\right|}\sqrt{\frac{\mathcal{A}_{\mathcal{O}}}{\tilde{\Omega}_{\mathcal{O}}}}. \tag{93}$$

To sum up, the ray bundle formalism allows us to express distance measures and their slip through cross-sectional areas in a relatively simple way. The analysis can be made even more straightforward if we apply the BGO formalism.

4.2. Special points

We fix the null geodesic γ_0 and the observer's position $\mathcal{O}$ along γ_0. We can now introduce three types of special points along a null geodesic, defined by the properties of the vertex and initially parallel ray bundle. Their importance stems from the fact that they mark points where $D_{\text{ang}}(\lambda)$, $D_{\text{par}}(\lambda)$ and $\mu(\lambda)$ take particular values. Each of these points may appear an arbitrary number of times along a null geodesic or not appear at all, depending on the spacetime geometry.

We call $\mathcal{P}$ a *conjugate point* with respect to $\mathcal{O}$ iff the vertex bundle from $\mathcal{O}$ refocuses back at $\mathcal{P}$ at least along one transverse direction. This property is equivalent to the existence of a Jacobi field along γ_0, vanishing at $\mathcal{O}$ and $\mathcal{P}$, but not identically zero. It is easy to check that this happens iff $\det\, W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = 0$ between $\mathcal{O}$ and $\mathcal{P}$. Conjugate points correspond to the intersection of the fiducial geodesic with a caustic and are points of infinite magnification of images of objects located at $\mathcal{P}$ as seen in $\mathcal{O}$. We can see that in these points we formally have $D_{\text{ang}} = 0$. Moreover, as long as $\det\, W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} \neq 0$, we also have $D_{\text{par}} = 0$. On the other hand, μ does not need to take any special value in a conjugate point because its value is unrelated to the properties of the vertex bundle.

We call $\mathcal{P}$ a *focal point* iff an infinitesimal bundle of rays running parallel at $\mathcal{O}$ refocuses at $\mathcal{P}$ along at least in one direction. This happens when $\det\, W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} = 0$. The physical interpretation of these points is vanishing parallax effect along at least one baseline: the parallax matrix must be degenerate in at least one direction. This means that the displacements of the observer in this direction result in no measurable image displacement. It is straightforward to see that at these points we have diverging parallax distance, i.e. $D_{\text{par}} \to \infty$, as long as $\mathcal{P}$ is not a conjugate point as well. Moreover, at focal points we always have $\mu = 1$. However, the value of D_{ang} can be arbitrary at a focal point.

Finally, $\mathcal{P}$ is an *equidistance point* iff $D_{\text{ang}} = D_{\text{par}}$, i.e. both methods of distance determination yield the same value. The reader may check easily from (67) that at these points we have either $\mu = 0$ or $\mu = 2$.

4.3. Numerical results

Now we will use this formalism to study light propagation and notions of distances in Schwarzschild spacetime. In this analysis we are interested in a beam of light that connects static emitters and observers placed outside the black hole's photon sphere. The trajectory of the geodesic is an arbitrary arc that (possibly) winds around the black hole a finite number of times. We fix the observer's position at $r = 100r_{\text{s}}$, where r_{s} is the Schwarzschild radius, and vary the impact parameter $b = \frac{|L_z|}{E}$. Then we follow the corresponding null geodesic as we increase the affine parameter value λ, and to each value we assign an emitter placed at the point $x^{\mu}(\lambda)$. The parametrization of the geodesic is fixed by rescaling the affine parameter while keeping the products $(u_{\mathcal{O}} \cdot \ell_{\mathcal{O}})$ equal to 1 for all instances of b. This means that λ agrees with the spatial distance measured by the observer in his or her vicinity.

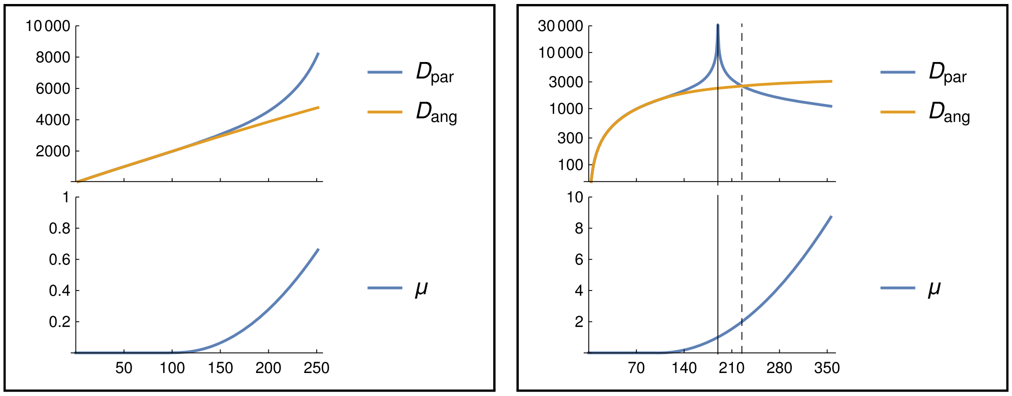
In every case the endpoint of the geodesic is placed sufficiently far away from the black hole such that the curvature effects eventually would be negligible. The plots reveal that the evolution of distance measures and μ has three distinct stages. For this reason we will discuss their behaviour in the initial, intermediate and faraway regions separately.

In principle there are two ways to investigate this problem numerically. The first approach is the numerical evaluation of the exact solutions of the geodesic equations and the GDE. The second approach is the direct numerical integration of the geodesic equation and the GDE projected onto the SNT. We choose the second approach because it provides an easier control of the problem, especially at the turning points.

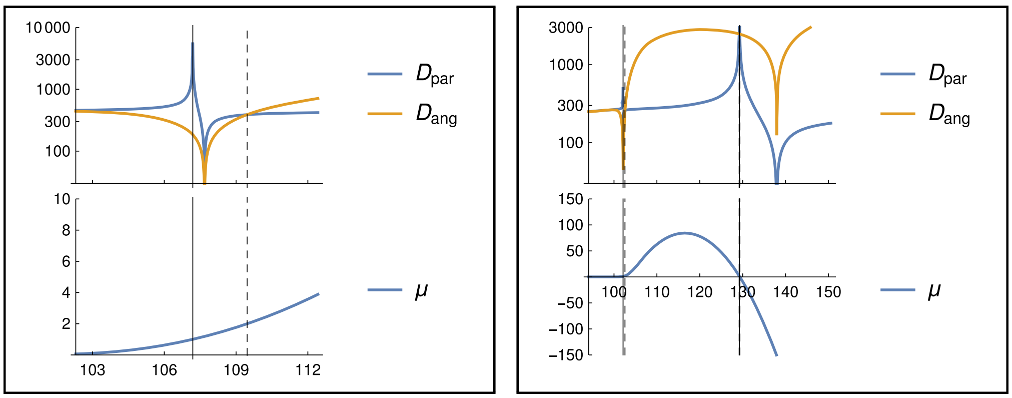
As depicted in figure 1(a), D_{par} and D_{ang} differ very slightly in the case of a large impact parameter. As λ approaches 0, distances become arbitrarily close to each other. On the other hand, the growth of λ is accompanied by a slight increase in the difference between both distance measures, with D_{par} being the larger one. Similarly, μ is slowly monotonically increasing.

A slight decrease of the impact parameter (figure 1(b)) results in the appearance of the first nontrivial effect. At first, D_{par} is practically identical to D_{ang}, but later D_{par} starts to grow faster and diverges upon reaching the focal point. Afterwards, it becomes monotonically decreasing, with D_{ang} eventually overtaking at the equidistance point. All this time both D_{ang} and μ are monotonically increasing. The positions of the focal and equidistance points, which correspond to $\mu = 1$ and $\mu = 2$, are marked respectively by the solid and dashed lines.

Decreasing the impact parameter even more (figure 1(c)) reveals several more interesting effects. Again D_{par} is initially growing faster than D_{ang} and diverges at the focal point. However, D_{ang} is not monotonic anymore and, together with D_{par}, vanishes at the conjugate point. From now on, both distances grow monotonically, with D_{par} growing faster at first but later approaching a finite value.



(a) $r_{\mathcal{O}} = 100\ r_s$, $b = 20\ r_s$ (b) $r_{\mathcal{O}} = 100\ r_s$, $b = 14.14\ r_s$



(c) $r_{\mathcal{O}} = 100\ r_s$, $b = 4.47\ r_s$ (d) $r_{\mathcal{O}} = 100\ r_s$, $b = 2.67\ r_s$

Figure 1. Dependence of D_{par}, D_{ang} and μ on initial conditions. The horizontal axis shows the value of the affine parameter along the geodesic. Parametrization is the same for all cases and is fixed by rescaling the affine parameter in a way that makes the product $(u_{\mathcal{O}} \cdot \ell_{\mathcal{O}}) = 1$. All distance measures are measured in Schwarzschild radii. Vertical solid lines denote $D_{\text{par}} = \infty$ $(\mu = 1)$, dashed lines—$D_{\text{par}} = D_{\text{ang}}$ $(\mu = 1 \pm 1)$. In the last picture the region between the first pair of lines (which almost appears as a single line) shows a behaviour similar to $b = 4.47 r_{\text{s}}$ case. The second group comprises a solid line surrounded by two dashed lines. The distances are plotted in a linear scale in plot (a) and in logarithmic scale in (b)–(d).

A further decrease of the impact parameter (figure 1(d)) exposes only one new feature: the nonmonotonicity of μ. We observe that there is a special point between two focal points, upon reaching which μ begins to decrease. Such behaviour is suggested by (67), by which the value of μ at any focal point is equal to one. Therefore, at some point in between it has to become decreasing. Apart from that, we see a higher number of appearances of previously described features. D_{ang} has an overall growing tendency, but it decreases to zero every time a conjugate point is passed. Analogously, D_{par} diverges at focal points and vanishes at conjugate points.

In the faraway region the behaviour of distance measures becomes relatively simple. D_{ang} grows without bounds, while D_{par} approaches a constant value. μ also grows indefinitely, but its sign depends on the number of focal points passed.

Although these results represent only a few selected realisations of the problem, the qualitative properties survive in the general setting. We now present these properties in three different regimes (figures 2 and 3).

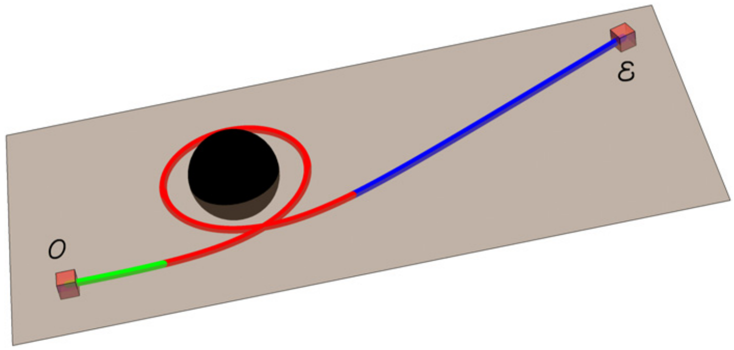


Figure 2. An illustration of the problem setting. The line depicts the null geodesic. The green, red and blue parts represent the initial, intermediate and faraway regions. The boxes at both endpoints stand for locally flat neighbourhoods. The black sphere marks the event horizon of the black hole.

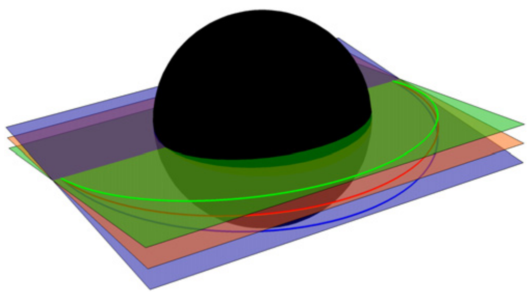


Figure 3. Due to the symmetries of Schwarzschild spacetime every geodesic is completely contained in a plane passing through the center of the black hole. Geodesics that emerge from the same point but differ in their vertical alignments belong to different planes which share a line passing through the initial point and the center. All points conjugate to the initial one lie on this line.

4.4. Initial region

We begin with two types of bundles of rays: a vertex bundle and an initially parallel bundle. They can be understood by studying W_{XX} and W_{XL}. In order to explain their behaviour in the initial region, we have to estimate the leading order behaviour. In the parallel propagated frame, expressing these operators as Taylor series around $\lambda_{\mathcal{O}} = 0$ and applying their ODEs and initial conditions in the matrix form [6] yields:

$$W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = \lambda\delta^{\mathbf{A}}{}_{\mathbf{B}} + \frac{\lambda^3}{3!}\mathcal{R}^{\mathbf{A}}{}_{\mathbf{B}} + \frac{\lambda^4}{4!}\left(2\dot{\mathcal{R}}^{\mathbf{A}}{}_{\mathbf{B}}\right) + \frac{\lambda^5}{5!}\left(3\ddot{\mathcal{R}}^{\mathbf{A}}{}_{\mathbf{B}} + \mathcal{R}^{\mathbf{A}}{}_{\mathbf{C}}\mathcal{R}^{\mathbf{C}}{}_{\mathbf{B}}\right) + \mathcal{O}\left(\lambda^6\right)$$
$$W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} = \delta^{\mathbf{A}}{}_{\mathbf{B}} + \frac{\lambda^2}{2!}\mathcal{R}^{\mathbf{A}}{}_{\mathbf{B}} + \frac{\lambda^3}{3!}\dot{\mathcal{R}}^{\mathbf{A}}{}_{\mathbf{B}} + \frac{\lambda^4}{4!}\left(\ddot{\mathcal{R}}^{\mathbf{A}}{}_{\mathbf{B}} + \mathcal{R}^{\mathbf{A}}{}_{\mathbf{C}}\mathcal{R}^{\mathbf{C}}{}_{\mathbf{B}}\right) + \mathcal{O}\left(\lambda^5\right), \tag{94}$$

where $\dot{\mathcal{R}}$ denotes the derivative of $\mathcal{R}$ with respect to the affine parameter, and every curvature term is evaluated at $\lambda = 0$.

Spatial projections of these operators live in the two-dimensional Euclidean space. Hence, one can apply the Cayley–Hamilton theorem to express the determinants in terms of traces:

$$\det{}_2\,(I+A) = 1 + \mathrm{Tr}\,A + \frac{(\mathrm{Tr}\,A)^2 - \mathrm{Tr}\left(A^2\right)}{2}. \tag{95}$$

This is particularly useful when one has to organize power expansions to higher orders. Also, it is instructive to decompose the optical tidal matrix as a sum of pure Ricci and Weyl terms, i.e. $\mathcal{R}^{\mathbf{A}}{}_{\mathbf{B}} = -\frac{1}{2}R_{\ell\ell}\delta^{\mathbf{A}}{}_{\mathbf{B}} + C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}$. Then at $\mathcal{O}$ the Taylor series expansion of determinants and μ has the following form:

$$\begin{aligned}
\det\,W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} &= \lambda^2\left(1 - \frac{\lambda^2}{3!}R_{\ell\ell} - \frac{\lambda^3}{4!}2\dot{R}_{\ell\ell} + \frac{\lambda^4}{5!}\left[\frac{5}{6}R_{\ell\ell}^2 - 3\ddot{R}_{\ell\ell}\right] - \frac{\lambda^4}{5!}\frac{5}{3}C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}C^{\mathbf{B}}{}_{\ell\ell\mathbf{A}}\right) + \mathcal{O}\left(\lambda^7\right)\\
\det\,W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} &= 1 - \frac{\lambda^2}{2!}R_{\ell\ell} - \frac{\lambda^3}{3!}\dot{R}_{\ell\ell} + \frac{\lambda^4}{4!}\left(2R_{\ell\ell}^2 - \ddot{R}_{\ell\ell}\right) - 2\frac{\lambda^4}{4!}C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}C^{\mathbf{B}}{}_{\ell\ell\mathbf{A}} + \mathcal{O}\left(\lambda^5\right)\\
\mu &= \frac{\lambda^2}{2!}R_{\ell\ell} + \frac{\lambda^3}{3!}\dot{R}_{\ell\ell} - \frac{\lambda^4}{4!}\left(2R_{\ell\ell}^2 + \ddot{R}_{\ell\ell}\right) + 2\frac{\lambda^4}{4!}C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}C^{\mathbf{B}}{}_{\ell\ell\mathbf{A}} + \mathcal{O}\left(\lambda^5\right).
\end{aligned} \tag{96}$$

Finally, we substitute these results into (87) and the parallax distance formula

$$D_{\mathrm{par}} = (l_{\mathcal{O}}\cdot u_{\mathcal{O}})\left|\det\,W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}\right|^{1/2}\left|\det\,W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}\right|^{-1/2} \tag{97}$$

and obtain the leading order behaviour for the distance measures:

$$\begin{aligned}
\frac{D_{\mathrm{ang}}}{(\ell_{\mathcal{O}}\cdot u_{\mathcal{O}})} &= \lambda - \frac{\lambda^3}{3!}\frac{R_{\ell\ell}}{2} - \frac{\lambda^4}{4!}\dot{R}_{\ell\ell} - \frac{\lambda^5}{5!}\frac{3}{2}\ddot{R}_{\ell\ell} - \frac{\lambda^5}{5!}\frac{5}{6}C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}C^{\mathbf{B}}{}_{\ell\ell\mathbf{A}} + \mathcal{O}\left(\lambda^6\right)\\
\frac{D_{\mathrm{par}}}{(\ell_{\mathcal{O}}\cdot u_{\mathcal{O}})} &= \lambda + \frac{\lambda^3}{3!}R_{\ell\ell} + \frac{\lambda^4}{4!}\dot{R}_{\ell\ell} + \frac{\lambda^5}{5!}\left(\frac{15}{4}R_{\ell\ell}^2 + \ddot{R}_{\ell\ell}\right) + \frac{\lambda^5}{5!}\frac{25}{6}C^{\mathbf{A}}{}_{\ell\ell\mathbf{B}}C^{\mathbf{B}}{}_{\ell\ell\mathbf{A}} + \mathcal{O}\left(\lambda^6\right).
\end{aligned} \tag{98}$$

From the above relations we conclude that D_{ang}, D_{par} and μ are all regular for sufficiently short distances. Furthermore, whenever $R_{\ell\ell}$ is identically zero, e.g. in vacuum or in the presence of the cosmological constant, the difference between the operators, their determinants and derived distances is observable only at a relatively high order. This explains why for the Schwarzschild spacetime in the beginning both vertex and initially parallel bundles are almost unaffected. In particular, Weyl contribution appears at the fifth order for distance measures and at the fourth order for μ. On the other hand, when $R_{\ell\ell} > 0$, the difference is already visible at the third order for distances and at the second order for μ. Moreover, in the leading order, D_{par} is larger than D_{ang}, and μ is positive. In fact, it is possible to present a more general, non-perturbative statement, valid in any spacetime. In the companion paper [45] we prove the following result:

Theorem 4.1. *Let $\mathcal{O}$ and $\mathcal{E}$ be two points along a null geodesic γ such that $\mathcal{O}$ lies in the causal future of $\mathcal{E}$ and let the NEC hold along γ_0 between $\mathcal{O}$ and $\mathcal{E}$. Assume also that between $\mathcal{O}$ and $\mathcal{E}$ there are no singular points of the infinitesimal bundle of rays parallel at $\mathcal{O}$. Then we have*

$$\mu \geqslant 0. \tag{99}$$

Moreover, $\mu = 0$ iff the transverse optical tidal tensor $R^{A}{}_{\mu\nu B}\ell^{\mu}\ell^{\nu}$ vanishes along γ_0 between $\mathcal{O}$ and $\mathcal{E}$.

4.5. Intermediate region

In the intermediate region geodesics may circle the black hole, but eventually they escape to infinity unless they fall onto the photon sphere or the black hole. In any case, both the initially parallel and vertex bundles start to converge due to Weyl focusing, as can be seen from (73) and (74). The parallel bundle is the first one to be focused because already at $\lambda_{\mathcal{O}}$ because its expansion is zero, and later it can only decrease. The position of the corresponding focal point depends on the parameters of the null geodesic, but it is reached sooner than the corresponding conjugate point. At the focal point D_{par} diverges and $\mu = 1$ while D_{ang} is regular.

If, after passing the focal point, the geodesic is still sufficiently close to the black hole, it will encounter the conjugate point where the vertex bundle will converge back to a point. Even though the initial conditions determine the position of this point, the angular coordinate ϕ is actually independent of them and always equals a multiple of π (assuming that initially $\phi = 0$). At the conjugate points both D_{par} and D_{ang} vanish, but μ is regular.

By looking at the expressions for BGOs in the parallel propagated SNT one can notice that the dependence on ϕ is periodic with the period of 2π, while the angular coordinates of focal and conjugate points differ by a multiple of π. This can be easily understood from the symmetry of the problem. In a spherically symmetric spacetime every perturbed geodesic is contained in a plane. All these planes are tilted with respect to each other, but they share one common line. All the points from which geodesics emanate or to which they converge lie on this line. Furthermore, the geodesic equation depends on the square of angular momentum. Tilting a plane implies a perturbation of the angular momentum, i.e. $L_x^2 \to L_x^2 + 2L_x\delta L_x + \delta L_x^2$. However, in the domain of the linear geodesic deviation, only linear terms should be considered. In addition to this, both L_x and L_y vanish in the aligned coordinates, which implies that the linear perturbation is also absent. Therefore, every ray of the null congruence satisfies the same geodesic equation. Provided we choose the correct affine parameter gauge, they meet at the same point.

After the focal point one can almost always expect an equidistant point where $D_{\text{par}} = D_{\text{ang}}$. This point may come either before or after the conjugate point. At this point μ is either 0 or 2, and all observables are regular.

One has to note that the number of occurrences of these special points depends on the total azimuthal angle swept by the geodesic, measured by ϕ. It is determined by the parameters of the geodesic, i.e. b and $r_{\mathcal{O}}$. Even if we take λ from $\lambda_{\mathcal{O}}$ up to infinity, the range of ϕ is bounded for geodesics that are not trapped by the black hole. It may therefore happen that several sequences of focal, conjugate or equidistant points will be traversed, or that in the end the counts of each type of point will be different. However, the qualitative behaviour at these points and in between is the same.

4.6. Faraway region

In the faraway region the curvature is becoming arbitrarily small, and the geodesic approaches a radial null line. This results in the Weyl focusing becoming negligible. Thus, effectively, nearby light rays propagate as if they were in the Minkowski spacetime. From (75) we have in that case:

$$\begin{aligned} W_{XX} &\sim A_{XX} + \lambda B_{XX} \\ W_{XL} &\sim A_{XL} + \lambda B_{XL} \end{aligned}, \tag{100}$$

where $A_{XX}, A_{XL}, B_{XX}, B_{XL}$ are constant matrices. Their precise form depends on the whole history of the null geodesics from the observation point up to the faraway region. Taking their determinants yields the asymptotic behaviour for $\lambda \to \infty$:

$$\begin{aligned} \det\, {W_{XX}}^{\mathbf{A}}{}_{\mathbf{B}} &\sim \left(\det\, {B_{XX}}^{\mathbf{A}}{}_{\mathbf{B}}\right) \lambda^2 \\ \det\, {W_{XL}}^{\mathbf{A}}{}_{\mathbf{B}} &\sim \left(\det\, {B_{XL}}^{\mathbf{A}}{}_{\mathbf{B}}\right) \lambda^2. \end{aligned} \tag{101}$$

In a generic situation we may assume that these determinants do not vanish. Then, according to (87) and (97), D_{ang} and D_{par} have the following asymptotic behaviour:

$$\begin{aligned} D_{\text{ang}} &\sim \lambda \\ D_{\text{par}} &\sim \text{const.} \end{aligned} \tag{102}$$

In other words, sufficiently far away D_{ang} is almost a linear function, while D_{par} approaches a constant value. From the intermediate region analysis we know that D_{par} is initially larger than D_{ang}. Therefore, in order to shrink to a fixed value, it must at some point match D_{ang}. For this reason the existence of the equidistant point is guaranteed for a generic geodesic.

The constancy of D_{par} is surprising, but at the same time it is actually a rather generic feature of asymptotically flat spacetimes. In Minkowski spacetime, for a baseline of fixed length, the parallax angle depends only on the position of the source. The further lies the source, the smaller the parallax angle, and this angle is close to zero for infinitely distant objects. In the general case, however, the trajectory of light will pass through a curved region and will be deflected. Then the total parallax is the sum of parallax in relatively flat regions and the contribution of light deflection in-between. Even if the parallax in the outer region can be made arbitrarily small, the passing of light through a curved region leaves its imprint that does not go away. The gravitational lensing best illustrates this: the position of an apparent image is very sensitive to its proximity to the two-dimensional projection of the lensing body and its parallax effect. The parallax of a nearby lens, such as the Schwarzschild black hole in our case, combined with light deflection provides this way an additional parallax effect for very distant objects. This in turn limits the effective parallax distance to these objects, as measured by the observer.

It may happen that for some initial conditions that matrices B_{XX} and B_{XL} are degenerate. In that case the asymptotic analysis of the behaviour of D_{ang} and D_{par} does not apply. For example, in the Schwarzschild spacetime, radial null geodesics correspond to the principal null directions and are shear free. The behaviour of distances then is analogous to the one in flat space where both D_{ang} and D_{par} grow linearly and are both equal all along the geodesic.

However, these situations require extreme fine-tuning of the initial data and do not represent the generic behaviour.

5. Conclusion

In this paper, we have presented two exact methods of solving the GDE and deriving the BGOs. The first method is based on the linear variation of the solution to the geodesic equation with respect to initial data. It requires an expression for the general solution of the geodesic equation in an explicit form or at least a sufficient number of implicit relations defining the geodesic. The second method makes use of the conserved quantities generated by Killing vectors. Every such quantity generates a conservation law for the GDE. In both cases, BGO's can be read off by taking partial derivatives of exact solutions or variations with respect to covariant perturbations of the initial data. Then we apply both methods to obtain the BGOs in a four-dimensional static spherically symmetric spacetime. In these spacetimes, a generic null geodesic is always contained in a plane passing through the origin. This allows us to reduce the dimensionality of the problem. Finally, to isolate physical effects, we project the BGOs onto a parallel-transported SNT.

In the second part of the paper, we investigated the behaviour of distance measures such as the angular diameter distance, the parallax distance and the distance slip in the Schwarzschild spacetime. We considered cases where both the source and the observer are located outside of the photon sphere. In the numerical study, we considered trajectories with a static observer and four different impact parameters. We have noticed that as the impact parameter approaches the value corresponding to the photon sphere, the occurrence and the strength of various nontrivial optical effects increases. One can observe the formation of various special points where the parallax distance diverges, parallax and angular distances become zero or equal to each other. Another interesting feature of this spacetime is that the parallax distance of a source positioned infinitely far away is finite.

In the last part, we provide a more general explanation for the observed phenomena. In the absence of matter, the curvature effects appear at a relatively high order, which explains why initially the distances are almost the same. In the intermediate region, the light is refocused to a line, and the number of such events depends on the total deflection angle. All points where this focusing happens lie on a line in the geodesic plane, which passes through the center of the black hole. In the faraway region, light rays propagate in effectively flat spacetime. However, at the same time, they carry the imprint of the previous regimes. We then show that in the faraway region the generic behaviour of distance measures in the leading order is linear for the angular diameter distance and constant for the parallax distance.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A

A.1. Variations of the implicit solutions of the geodesic equation

A.1.1. Variation of conserved quantities. We want to find how variations of initial position and direction affect (25)–(30). Due to the assumed symmetry, we can always choose aligned coordinates, where L_x and L_y are zero. However, this does not have to hold for their variations. The results are:

$$\begin{aligned}
\delta L_x|_{\mathcal{O}} &= C_{\mathcal{O}}\ell^{\phi}_{\mathcal{O}}\delta\theta_{\mathcal{O}} \\
\delta L_y|_{\mathcal{O}} &= C_{\mathcal{O}}\delta\ell^{\theta}_{\mathcal{O}} \\
\delta L_z|_{\mathcal{O}} &= \frac{C'_{\mathcal{O}}}{C_{\mathcal{O}}}L_z\delta r_{\mathcal{O}} + C_{\mathcal{O}}\delta\ell^{\phi}_{\mathcal{O}} \\
\delta E|_{\mathcal{O}} &= -A_{\mathcal{O}}\delta\ell^{t}_{\mathcal{O}} + E\frac{A'_{\mathcal{O}}}{A_{\mathcal{O}}}\delta r_{\mathcal{O}} \\
\delta\epsilon|_{\mathcal{O}} &= \left[-A'_{\mathcal{O}}\frac{E^2}{A^2_{\mathcal{O}}} + B'_{\mathcal{O}}\left(\ell^{r}_{\mathcal{O}}\right)^2 + C'_{\mathcal{O}}\frac{L^2}{C^2_{\mathcal{O}}}\right]\delta r_{\mathcal{O}} + 2E\delta\ell^{t}_{\mathcal{O}} + 2B_{\mathcal{O}}\ell^{r}_{\mathcal{O}}\delta\ell^{r}_{\mathcal{O}} \\
&\quad + 2L_z\delta\ell^{\phi}_{\mathcal{O}}.
\end{aligned} \tag{103}$$

Note that the variation of L_x is simply proportional to the variation of $\theta_{\mathcal{O}}$, and we may safely substitute it everywhere by $\delta\theta_{\mathcal{O}}$. The remaining four variations of conserved quantities can be used to parametrize the variations of the four components of the initial four-momentum $\ell^{\mu}_{\mathcal{O}}$. We also point out that even though ϵ here is arbitrary, eventually, we will set $\epsilon = 0$ to limit ourselves to null geodesics.

A.1.2. Variation of the implicit equations. Now we switch to the variation of solutions to the geodesic equations. We begin with (34) which we vary and then evaluate in the aligned coordinates:

$$\delta\theta = \cos\,\phi\delta\theta_{\mathcal{O}} + \sin\,\phi\frac{\delta\ell^{\theta}_{\mathcal{O}}}{\ell^{\phi}_{\mathcal{O}}}. \tag{104}$$

Next we use (26) and (27) to obtain ℓ^{θ}:

$$\ell^\theta = \frac{L_y \cos\phi - L_x \sin\phi}{C}. \tag{105}$$

In the aligned coordinates its variation yields:

$$\delta\ell^\theta = \frac{C_\mathcal{O}}{C}\left(\cos\phi\delta\ell^\theta_\mathcal{O} - \sin\phi\ell^\phi_\mathcal{O}\delta\theta_\mathcal{O}\right). \tag{106}$$

Note that variations of θ and ℓ^θ decouple from variations of other components of the geodesic. This is a consequence of the existence of a plane containing the geodesic.

Next we will consider the variation of r. We choose λ to be our dependent variable to avoid working with the formal solution $r(\lambda)$. From the normalization condition (30) we get:

$$\lambda - \lambda_\mathcal{O} = \int_{r_\mathcal{O}}^{r_\mathcal{E}} \pm_r \sqrt{\frac{ABC}{AC\epsilon + E^2C - L^2A}}\,\mathrm{d}\tilde{r}. \tag{107}$$

From now on we set $\lambda_\mathcal{O} = 0$ for convenience. We have that:

$$\delta\lambda = \frac{\delta r}{\ell^r} - \frac{\delta r_\mathcal{O}}{\ell^r_\mathcal{O}} + LI_{BC}\delta L - EI_{AB}\delta E - \frac{I_B}{2}\delta\epsilon. \tag{108}$$

This can be easily solved for δr:

$$\delta r = \ell^r\left(\delta\lambda + \frac{\delta r_\mathcal{O}}{\ell^r_\mathcal{O}} - I_{BC}L\delta L + I_{AB}E\delta E + \frac{I_B}{2}\delta\epsilon\right). \tag{109}$$

Variation of ℓ^r is straightforward:

$$\delta\ell^r = \frac{1}{\ell^r}\left(\frac{\delta\epsilon}{2B} + \frac{E\delta E}{AB} - \frac{L\delta L}{BC}\right) - \frac{1}{\ell^r}\left(\frac{\epsilon B'}{B^2} + \frac{E^2}{AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{L^2}{BC}\left(\frac{B'}{B} + \frac{C'}{C}\right)\right)\delta r,$$

the prime denoting here $A'(r) = A_{,r}$ etc.

Similarly, for t and ℓ^t we have:

$$\begin{aligned}\delta t &= \delta t_\mathcal{O} + E\left(\frac{\delta r_\mathcal{O}}{A_\mathcal{O}\ell^r_\mathcal{O}} - \frac{\delta r}{A\ell^r}\right) + I_{ABC}L(L\delta E - E\delta L) + I_{AB}\left(\frac{E}{2}\delta\epsilon - \epsilon\delta E\right)\\ \delta\ell^t &= -\frac{\delta E}{A} + \frac{E}{A^2}A'\delta r.\end{aligned} \tag{111}$$

In order to vary ϕ and ℓ^ϕ we start from (35) and (28). Here we have to recall that by (34), θ depends on ϕ as well. However, in the aligned coordinates, θ variations simply drop out, and we are left with the standard result:

$$\delta\phi = \delta\phi_{\mathcal{O}} + L_z\left(\frac{\delta r}{C\ell^r} - \frac{\delta r_{\mathcal{O}}}{C_{\mathcal{O}}\ell^r_{\mathcal{O}}}\right) + I_{ABC}E\left(E\delta L_z - L_z\delta E\right) + I_{BC}\left(\epsilon\delta L_z - \frac{L_z\delta\epsilon}{2}\right)$$
$$\delta\ell^\phi = \frac{\delta L_z}{C} - \frac{L_z}{C^2}C'\delta r. \tag{112}$$

In order to find $\mathcal{W}$ operators, we use variations we have obtained so far together with (40). Firstly, by reading off components for each variation, we obtain functions corresponding to partial derivatives in (40). Then, we calculate Christoffel symbols and vectors tangent to the geodesic and evaluate them at one of the endpoints as prescribed by (40).

A.2. Solution of GDE

Here we write down the solution of GDE using the method of conserved quantities.

$$\xi^\theta = \kappa_1 \sin\phi + \frac{\Sigma_x}{L_z}\cos\phi$$
$$\nabla_\ell\xi^\theta = \xi^\theta\left(\cot\phi\,\ell^\phi + \frac{C'}{2C}\ell^r\right) - \frac{\Sigma_x}{C\sin\phi}$$
$$\xi^r = \ell^r\left(\kappa_2 - E\Sigma_T I_{AB} + L_z\Sigma_z I_{BC} + \mathcal{B}I_B\right)$$
$$\nabla_\ell\xi^r = \frac{\mathcal{B} - E\nabla_\ell\xi^t - L_z\nabla_\ell\xi^\phi}{B\ell^r}$$
$$\xi^t = \kappa_3 + \kappa_2 E\left(\frac{1}{A_{\mathcal{O}}} - \frac{1}{A}\right) + \Sigma_T\left(E^2\frac{I_{AB}}{A} - L_z^2 I_{ABC} + \epsilon I_{AB}\right) + L_z\Sigma_z E\left(I_{ABC} - \frac{I_{BC}}{A}\right) + E\mathcal{B}\left(I_{AB} - \frac{I_B}{A}\right)$$
$$\nabla_\ell\xi^t = \frac{A'\ell^r}{2A}\left(\kappa_3 + \kappa_2\frac{E}{A_{\mathcal{O}}} + L_z I_{ABC}\left(\Sigma_z E - \Sigma_T L_z\right) + I_{AB}\left(\Sigma_T\epsilon + E\mathcal{B}\right)\right) + \frac{\Sigma_T}{A}$$
$$\xi^\phi = \kappa_4 + \kappa_2 L_z\left(\frac{1}{C} - \frac{1}{C_{\mathcal{O}}}\right) + \Sigma_z\left(L_z^2\frac{I_{BC}}{C} - E^2 I_{ABC} - \epsilon I_{BC}\right) + E\Sigma_T L_z\left(I_{ABC} - \frac{I_{AB}}{C}\right) + L_z\mathcal{B}\left(\frac{I_B}{C} - I_{BC}\right)$$
$$\nabla_\ell\xi^\phi = \frac{C'\ell^r}{2C}\left(EI_{ABC}\left(\Sigma_T L_z - \Sigma_z E\right) - I_{BC}\left(\Sigma_z\epsilon + L_z\mathcal{B}\right) + \kappa_4 - \kappa_2\frac{L_z}{C_{\mathcal{O}}}\right) - \frac{\Sigma_z}{C}.$$

A.3. BGO's expressed in the coordinate tetrad

$$
\begin{aligned}
{W_{XX}}^{t}{}_{t} &= 1 + \frac{1}{2} A'_{\mathcal{O}} \ell^r_{\mathcal{O}} \left(L_z^2 I_{ABC} - \frac{E^2}{A} I_{AB} - \epsilon I_{AB} \right) \\
{W_{XX}}^{t}{}_{\phi} &= \frac{C'_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2} E L_z \left(I_{ABC} - \frac{I_{BC}}{A} \right) \\
{W_{XX}}^{t}{}_{r} &= \frac{E}{\ell^r_{\mathcal{O}}} \left(\frac{1}{A_{\mathcal{O}}} - \frac{1}{A} \right) + \frac{C'_{\mathcal{O}} L_z^2 E}{2 C_{\mathcal{O}}} \left(\frac{I_{BC}}{A} - I_{ABC} \right) \\
&\quad + \frac{A'_{\mathcal{O}} E}{2 A_{\mathcal{O}}} \left(L_z^2 I_{ABC} - \frac{E^2}{A} I_{AB} - \epsilon I_{AB} \right) \\
{W_{XX}}^{r}{}_{t} &= \frac{A'_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2} E \ell^r I_{AB} \quad {W_{XX}}^{r}{}_{\phi} = \frac{C'_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2} L_z \ell^r I_{BC} \\
{W_{XX}}^{r}{}_{r} &= \frac{\ell^{\,r}}{\ell^r_{\mathcal{O}}} + \frac{\ell^r}{2} \left(E^2 \frac{A'_{\mathcal{O}}}{A_{\mathcal{O}}} I_{AB} - L_z^2 \frac{C'_{\mathcal{O}}}{C_{\mathcal{O}}} I_{BC} \right) \\
{W_{XX}}^{\theta}{}_{\theta} &= \cos\phi - C'_{\mathcal{O}} \ell^r_{\mathcal{O}} \frac{\sin\phi}{2 L_z} \\
{W_{XX}}^{\phi}{}_{t} &= \frac{A'_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2} L_z E \left(\frac{I_{AB}}{C} - I_{ABC} \right) \\
{W_{XX}}^{\phi}{}_{\phi} &= 1 + \frac{C'_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2} \left(\frac{L_z^2}{C} I_{BC} - E^2 I_{ABC} - \epsilon I_{BC} \right) \\
{W_{XX}}^{\phi}{}_{r} &= \frac{L_z}{\ell^r_{\mathcal{O}}} \left(\frac{1}{C} - \frac{1}{C_{\mathcal{O}}} \right) + \frac{A'_{\mathcal{O}}}{2 A_{\mathcal{O}}} E^2 L_z \left(\frac{I_{AB}}{C} - I_{ABC} \right) \\
&\quad + \frac{C'_{\mathcal{O}}}{2 C_{\mathcal{O}}} L_z \left(E^2 I_{ABC} - \frac{L_z^2}{C} I_{BC} + \epsilon I_{BC} \right) \\
{W_{XL}}^{t}{}_{t} &= E^2 \left(I_{AB} - \frac{I_B}{A} \right) + A_{\mathcal{O}} \left(E^2 \frac{I_{AB}}{A} - L_z^2 I_{ABC} + \epsilon I_{AB} \right) \\
{W_{XL}}^{t}{}_{r} &= E B_{\mathcal{O}} \ell^r_{\mathcal{O}} \left(I_{AB} - \frac{I_B}{A} \right) \\
{W_{XL}}^{t}{}_{\phi} &= E L_z \left(I_{AB} - \frac{I_B}{A} + C_{\mathcal{O}} \left(\frac{I_{BC}}{A} - I_{ABC} \right) \right) \quad {W_{XL}}^{\theta}{}_{\theta} = \frac{C_{\mathcal{O}}}{L_z} \sin\phi \\
{W_{XL}}^{r}{}_{t} &= E \ell^r \left(I_B - A_{\mathcal{O}} I_{AB} \right) \quad {W_{XL}}^{r}{}_{r} = B_{\mathcal{O}} \ell^r_{\mathcal{O}} \ell^r I_B \\
{W_{XL}}^{r}{}_{\phi} &= L_z \ell^r \left(I_B - C_{\mathcal{O}} I_{BC} \right) \\
{W_{XL}}^{\phi}{}_{t} &= E L_z \left(A_{\mathcal{O}} I_{ABC} - I_{BC} + \frac{I_B - A_{\mathcal{O}} I_{AB}}{C} \right) \\
{W_{XL}}^{\phi}{}_{r} &= B_{\mathcal{O}} \ell^r_{\mathcal{O}} L_z \left(\frac{I_B}{C} - I_{BC} \right) \\
{W_{XL}}^{\phi}{}_{\phi} &= L_z^2 \left(\frac{I_B}{C} - I_{BC} \right) + C_{\mathcal{O}} \left(E^2 I_{ABC} - \frac{L_z^2}{C} I_{BC} + \epsilon I_{BC} \right)
\end{aligned}
$$

$$
\begin{aligned}
{W_{LX}}^{t}{}_{t} &= \frac{1}{2A}\left(\ell^r A' - \ell^r_{\mathcal{O}} A'_{\mathcal{O}} + \frac{A'_{\mathcal{O}}\ell^r_{\mathcal{O}} A' \ell^r}{2}\left(L_z^2 I_{ABC} - \epsilon I_{AB}\right)\right) \\
{W_{LX}}^{t}{}_{\phi} &= \frac{L_z E}{4A} A' \ell^r C'_{\mathcal{O}} \ell^r_{\mathcal{O}} I_{ABC} \\
{W_{LX}}^{t}{}_{r} &= \frac{E}{2AA_{\mathcal{O}}\ell^r_{\mathcal{O}}}\left(A'\ell^r - A'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right) + \frac{EL_z^2}{4A} A'\ell^r\left(\frac{A'_{\mathcal{O}}}{A_{\mathcal{O}}} - \frac{C'_{\mathcal{O}}}{C_{\mathcal{O}}}\right) I_{ABC} \\
&\quad - \frac{A'\ell^r I_{AB}\epsilon A'_{\mathcal{O}} E}{4AA_{\mathcal{O}}} \\
{W_{LX}}^{r}{}_{t} &= \frac{E}{2AB\ell^r}\left(A'_{\mathcal{O}}\ell^r_{\mathcal{O}} - A'\ell^r\right) + \frac{EL_z^2}{4B} A'_{\mathcal{O}}\ell^r_{\mathcal{O}}\left(\frac{C'}{C} - \frac{A'}{A}\right) I_{ABC} \\
&\quad + \frac{A'_{\mathcal{O}}\ell^r_{\mathcal{O}} A' E \epsilon}{4AB} I_{AB} \\
{W_{LX}}^{r}{}_{r} &= \frac{E^2\left(A'_{\mathcal{O}}\ell^r_{\mathcal{O}} - A'\ell^r\right)}{2ABA_{\mathcal{O}}\ell^r\ell^r_{\mathcal{O}}} + \frac{L_z^2\left(C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right)}{2BCC_{\mathcal{O}}\ell^r\ell^r_{\mathcal{O}}} \\
&\quad + \frac{E^2 L_z^2}{4B}\left(\frac{A'}{A} - \frac{C'}{C}\right)\left(\frac{C'_{\mathcal{O}}}{C_{\mathcal{O}}} - \frac{A'_{\mathcal{O}}}{A_{\mathcal{O}}}\right) I_{ABC} \\
&\quad + \frac{\epsilon}{4B}\left(E^2 \frac{A'_{\mathcal{O}} A'}{A_{\mathcal{O}} A} I_{AB} - L_z^2 I_{BC} \frac{C'_{\mathcal{O}} C'}{C_{\mathcal{O}} C}\right) \\
{W_{LX}}^{r}{}_{\phi} &= \frac{L_z}{2BC\ell^r}\left(C'_{\mathcal{O}}\ell^r_{\mathcal{O}} - C'\ell^r\right) + \frac{E^2 L_z}{4B} C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\left(\frac{C'}{C} - \frac{A'}{A}\right) I_{ABC} \\
&\quad + \epsilon \frac{L_z C'}{4BC} C'_{\mathcal{O}}\ell^r_{\mathcal{O}} I_{BC} \\
{W_{LX}}^{\theta}{}_{\theta} &= \frac{\cos\phi}{2C}\left(C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right) - \left(\frac{L_z}{C} + \frac{C'\ell^r C'_{\mathcal{O}}\ell^r_{\mathcal{O}}}{4CL_z}\right)\sin\phi \\
{W_{LX}}^{\phi}{}_{t} &= -\frac{L_z E}{4C} C'\ell^r A'_{\mathcal{O}}\ell^r_{\mathcal{O}} I_{ABC} \quad {W_{LX}}^{\phi}{}_{r} = \frac{L_z\left(C'_{\mathcal{O}}\ell^r_{\mathcal{O}} - C'\ell^r\right)}{2CC_{\mathcal{O}}\ell^r_{\mathcal{O}}} \\
&\quad + \frac{L_z E^2}{4C} C'\ell^r\left(\frac{C'_{\mathcal{O}}}{C_{\mathcal{O}}} - \frac{A'_{\mathcal{O}}}{A_{\mathcal{O}}}\right) I_{ABC} + \frac{C'_{\mathcal{O}} L_z C' \ell^r}{4CC_{\mathcal{O}}} \epsilon I_{BC} \\
{W_{LX}}^{\phi}{}_{\phi} &= \frac{1}{2C}\left(C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right) - \frac{E^2}{4C} C'\ell^r C'_{\mathcal{O}}\ell^r_{\mathcal{O}} I_{ABC} \\
&\quad - \frac{\epsilon I_{BC}}{4C} C'\ell^r C'_{\mathcal{O}}\ell^r_{\mathcal{O}}
\end{aligned}
$$

$$
\begin{aligned}
{W_{LL}}^{t}{}_{r} &= \frac{E}{2A} A' \ell^r B_{\mathcal{O}} \ell^r_{\mathcal{O}} I_{AB} \\
{W_{LL}}^{t}{}_{t} &= \frac{A_{\mathcal{O}}}{A} + \frac{A' \ell^r}{2A} \left(E^2 I_{AB} - L_z^2 A_{\mathcal{O}} I_{ABC} + \epsilon I_{AB} A_{\mathcal{O}} \right) \\
{W_{LL}}^{t}{}_{\phi} &= \frac{E L_z}{2A} A' \ell^r \left(I_{AB} - C_{\mathcal{O}} I_{ABC} \right) \\
{W_{LL}}^{r}{}_{t} &= \frac{E \left(A - A_{\mathcal{O}} \right)}{AB\ell^r} + \frac{E}{2B} \left(L_z^2 \frac{C'}{C} I_{BC} - E^2 \frac{A'}{A} I_{AB} \right) + \frac{L_z^2 E A_{\mathcal{O}}}{2B} \left(\frac{A'}{A} - \frac{C'}{C} \right) I_{ABC} \\
&\quad - \epsilon \frac{A' A_{\mathcal{O}} E}{2AB} I_{AB} \\
{W_{LL}}^{r}{}_{r} &= \frac{B_{\mathcal{O}} \ell^r_{\mathcal{O}}}{B \ell^r} + \frac{B_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2B} \left(L_z^2 \frac{C'}{C} I_{BC} - E^2 \frac{A'}{A} I_{AB} \right) \\
{W_{LL}}^{r}{}_{\phi} &= \frac{L_z \left(C - C_{\mathcal{O}} \right)}{BC\ell^r} + \frac{L_z}{2B} \left(L_z^2 \frac{C'}{C} I_{BC} - E^2 \frac{A'}{A} I_{AB} \right) \\
&\quad + \frac{E^2 L_z C_{\mathcal{O}}}{2B} \left(\frac{A'}{A} - \frac{C'}{C} \right) I_{ABC} - \epsilon \frac{L_z C' C_{\mathcal{O}}}{2BC} I_{BC} \\
{W_{LL}}^{\theta}{}_{\theta} &= \frac{C_{\mathcal{O}}}{C} \cos \phi + \frac{C_{\mathcal{O}}}{2CL_z} C' \ell^r \sin \phi \\
{W_{LL}}^{\phi}{}_{t} &= \frac{E L_z}{2C} C' \ell^r \left(A_{\mathcal{O}} I_{ABC} - I_{BC} \right) \\
{W_{LL}}^{\phi}{}_{r} &= -\frac{L_z B_{\mathcal{O}} \ell^r_{\mathcal{O}}}{2C} C' \ell^r I_{BC} \\
{W_{LL}}^{\phi}{}_{\phi} &= \frac{C_{\mathcal{O}}}{C} + \frac{C' \ell^r}{2C} \left(E^2 C_{\mathcal{O}} I_{ABC} - L_z^2 I_{BC} + \epsilon I_{BC} C_{\mathcal{O}} \right).
\end{aligned}
$$

A.4. Optical tidal matrix in aligned coordinate tetrad

$$
\begin{aligned}
\mathcal{R}^{t}{}_{t} &= \frac{E^2 + \epsilon A}{EB\ell^r} \mathcal{R}^t_r + \left(\frac{L_z}{2ABC} \right)^2 \left(A'C(AB)' + AB \left(A'C' - 2A''C \right) \right) \\
\mathcal{R}^{t}{}_{r} &= \frac{\left(2ABA'' - A'(AB)' \right) E\ell^r}{4A^3 B} \\
\mathcal{R}^{t}{}_{\phi} &= \frac{E L_z A' C'}{4A^2 BC} \quad \mathcal{R}^{r}{}_{t} = -\frac{A}{B} \mathcal{R}^{t}{}_{r} \quad \mathcal{R}^{r}{}_{r} = -\frac{E}{B\ell^r} \mathcal{R}^{t}{}_{r} - \frac{L_z}{B\ell^r} \mathcal{R}^{\phi}{}_{r} \\
\mathcal{R}^{r}{}_{\phi} &= \frac{C}{B} \mathcal{R}^{\phi}{}_{r} \quad \mathcal{R}^{\phi}{}_{t} = -\frac{A}{C} \mathcal{R}^{t}{}_{\phi} \\
\mathcal{R}^{\phi}{}_{r} &= \frac{L_z \ell^r}{4BC^2} \left(B'C' + \frac{B}{C} \left(C'^2 - 2CC'' \right) \right) \\
\mathcal{R}^{\theta}{}_{\theta} &= \mathcal{R}^{\phi}{}_{\phi} + \frac{L_z^2}{4BC^3} \left(C'^2 - 4BC \right) \\
\mathcal{R}^{\phi}{}_{\phi} &= -\left(\frac{E}{2ABC} \right)^2 \left(C'C(AB)' + AB \left(C'^2 - 2CC'' \right) \right) + \frac{L_z^2 - \epsilon C}{L_z B \ell^r} \mathcal{R}^{\phi}{}_{r}.
\end{aligned}
\tag{113}
$$

A.5. Optical tidal matrix in the semi-null tetrad ($\epsilon = 0$)

$$\mathcal{R}^{\mathbf{1}}{}_{\mathbf{1}} = \left(\frac{L_z}{2BC}\right)^2 \left(B\left(C'' - 4B\right) + \left(BC'\right)' + \frac{2B}{C}\left(C'^2 - 2CC''\right)\right) - \left(\frac{E}{2ABC}\right)^2 \left(AB\left(C'^2 - 2CC''\right) + CC'(AB)'\right)$$

$$\mathcal{R}^{\mathbf{2}}{}_{\mathbf{2}} = \left(\frac{L_z}{2ABC}\right)^2 \left(A'C(AB)' + AB\left(A'C' - 2CA''\right)\right) - \left(\frac{E}{2ABC}\right)^2 \left(AB\left(C'^2 - 2CC''\right) + CC'(AB)'\right)$$

$$\mathcal{R}^{\mathbf{2}}{}_{\mathbf{0}} = \Psi\mathcal{R}^{\mathbf{2}}{}_{\mathbf{2}} - \frac{E\ell^r Q}{4(ABC)^{\frac{3}{2}} L_z}\left(AB\left(C'^2 - 2CC''\right) + CC'(AB)'\right) \qquad \mathcal{R}^{\mathbf{3}}{}_{\mathbf{2}} = \frac{\mathcal{R}^{\mathbf{2}}{}_{\mathbf{0}}}{Q}$$

$$\mathcal{R}^{\mathbf{3}}{}_{\mathbf{0}} = \frac{Q}{(2BC)^2}\left(B\left(C'^2 - 2CC''\right) + CC'B'\right) - \left(\frac{E}{\ell^r L_z}\sqrt{\frac{C}{AB}} + 2\frac{\Psi}{Q}\right)\left(\Psi\mathcal{R}^{\mathbf{2}}{}_{\mathbf{2}} - \mathcal{R}^{\mathbf{2}}{}_{\mathbf{0}}\right) + \frac{\Psi^2}{Q}\mathcal{R}^{\mathbf{2}}{}_{\mathbf{2}}. \tag{114}$$

A.6. BGO's in the semi-null tetrad

$$W_{XL}{}^{\mathbf{0}}{}_{\mathbf{0}} = W_{XL}{}^{\mathbf{3}}{}_{\mathbf{3}} = \lambda \quad W_{XL}{}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{\sqrt{CC_{\mathcal{O}}}}{L_z}\sin\phi$$

$$W_{XL}{}^{\mathbf{2}}{}_{\mathbf{2}} = \sqrt{AA_{\mathcal{O}}BB_{\mathcal{O}}CC_{\mathcal{O}}}\,\ell^r\ell^r_{\mathcal{O}} I_{ABC}$$

$$W_{XL}{}^{\mathbf{2}}{}_{\mathbf{0}} = \frac{EQ}{L_z}\ell^r\sqrt{ABC}\left(C_{\mathcal{O}}I_{ABC} - I_{AB}\right) - \lambda\Psi$$

$$W_{XL}{}^{\mathbf{3}}{}_{\mathbf{2}} = \frac{E}{L_z}\ell^r_{\mathcal{O}}\sqrt{A_{\mathcal{O}}B_{\mathcal{O}}C_{\mathcal{O}}}\left(CI_{ABC} - I_{AB}\right) + W_{XL}{}^{\mathbf{2}}{}_{\mathbf{2}}\frac{\Psi}{Q} \tag{115}$$

$$W_{XL}{}^{\mathbf{3}}{}_{\mathbf{0}} = \frac{Q}{L_z^2}\left(E^2\left(CC_{\mathcal{O}}I_{ABC} - \frac{C + C_{\mathcal{O}}}{2}I_{AB}\right) + L_z^2\left(I_B - \frac{C + C_{\mathcal{O}}}{2}I_{BC}\right)\right) + \frac{\Psi}{Q}W_{XL}{}^{\mathbf{2}}{}_{\mathbf{0}} + \frac{\lambda\Psi^2}{2Q}$$

$$
\begin{aligned}
&{W_{XX}}^{\mathbf{0}}{}_{\mathbf{0}} = {W_{XX}}^{\mathbf{3}}{}_{\mathbf{3}} = 1 \quad {W_{XX}}^{\mathbf{1}}{}_{\mathbf{1}} = \sqrt{\frac{C}{C_{\mathcal{O}}}}\left(\cos\phi - \frac{C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\sin\phi}{2L_z}\right)\\
&{W_{XX}}^{\mathbf{2}}{}_{\mathbf{0}} = -\sqrt{ABC}\frac{C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\ell^r EQ}{2L_z}I_{ABC} - \Psi\\
&{W_{XX}}^{\mathbf{2}}{}_{\mathbf{2}} = \sqrt{\frac{ABC}{A_{\mathcal{O}}B_{\mathcal{O}}C_{\mathcal{O}}}}\left(\frac{\ell^r}{\ell^r_{\mathcal{O}}} + \frac{\ell^r}{2}\left(L_z^2A'_{\mathcal{O}} - E^2C'_{\mathcal{O}}\right)I_{ABC}\right)\\
&{W_{XX}}^{\mathbf{3}}{}_{\mathbf{2}} = \frac{E}{2L_z\sqrt{A_{\mathcal{O}}B_{\mathcal{O}}C_{\mathcal{O}}}}\Bigg(CI_{ABC}\left(L_z^2A'_{\mathcal{O}} - E^2C'_{\mathcal{O}}\right)\\
&\qquad + L_z^2\left(C'_{\mathcal{O}}I_{BC} - A'_{\mathcal{O}}I_{AB}\right) + \frac{2\left(C - C_{\mathcal{O}}\right)}{\ell^r_{\mathcal{O}}}\Bigg) + \frac{\Psi}{Q}{W_{XX}}^{\mathbf{2}}{}_{\mathbf{2}}\\
&{W_{XX}}^{\mathbf{3}}{}_{\mathbf{0}} = \frac{Q}{2L_z^2}\left(C - C_{\mathcal{O}} + C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\left(I_{BC}L_z^2 - E^2I_{ABC}C\right)\right) + {W_{XX}}^{\mathbf{2}}{}_{\mathbf{0}}\frac{\Psi}{Q} + \frac{\Psi^2}{2Q}
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
&{W_{LL}}^{\mathbf{0}}{}_{\mathbf{0}} = {W_{LL}}^{\mathbf{3}}{}_{\mathbf{3}} = 1 \qquad {W_{LL}}^{\mathbf{1}}{}_{\mathbf{1}} = \sqrt{\frac{C_{\mathcal{O}}}{C}}\left(\cos\phi + \frac{C'\ell^r\sin\phi}{2L_z}\right)\\
&{W_{LL}}^{\mathbf{2}}{}_{\mathbf{2}} = \sqrt{\frac{A_{\mathcal{O}}B_{\mathcal{O}}C_{\mathcal{O}}}{ABC}}\left(\frac{\ell^r_{\mathcal{O}}}{\ell^r} + \frac{\ell^r_{\mathcal{O}}}{2}\left(E^2C' - L_z^2A'\right)I_{ABC}\right)\\
&{W_{LL}}^{\mathbf{3}}{}_{\mathbf{2}} = \sqrt{A_{\mathcal{O}}B_{\mathcal{O}}C_{\mathcal{O}}}\frac{C'\ell^r\ell^r_{\mathcal{O}}E}{2L_z}I_{ABC} + \frac{\Psi}{Q}{W_{LL}}^{\mathbf{2}}{}_{\mathbf{2}}\\
&{W_{LL}}^{\mathbf{2}}{}_{\mathbf{0}} = \frac{QE}{2L_z\sqrt{ABC}}\Bigg(C_{\mathcal{O}}I_{ABC}\left(E^2C' - L_z^2A'\right)\\
&\qquad + L_z^2\left(A'I_{AB} - C'I_{BC}\right) + \frac{2\left(C_{\mathcal{O}} - C\right)}{\ell^r}\Bigg) - \Psi\\
&{W_{LL}}^{\mathbf{3}}{}_{\mathbf{0}} = \frac{Q}{2L_z^2}\left(C_{\mathcal{O}} - C + C'\ell^r(C_{\mathcal{O}}E^2I_{ABC} - L_z^2I_{BC})\right) + {W_{LL}}^{\mathbf{2}}{}_{\mathbf{0}}\frac{\Psi}{Q} + \frac{\Psi^2}{2Q}
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
&{W_{LX}}^{\mathbf{1}}{}_{\mathbf{1}} = \frac{C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}}{2\sqrt{CC_{\mathcal{O}}}}\cos\phi - \sqrt{\frac{C}{C_{\mathcal{O}}}}\left(\frac{L_z}{C} + \frac{C'\ell^rC'_{\mathcal{O}}\ell^r_{\mathcal{O}}}{4CL_z}\right)\sin\phi\\
&{W_{LX}}^{\mathbf{2}}{}_{\mathbf{2}} = \frac{1}{2\sqrt{AA_{\mathcal{O}}BB_{\mathcal{O}}CC_{\mathcal{O}}}}\Bigg(\frac{E^2C' - L_z^2A'}{\ell^r_{\mathcal{O}}} - \frac{E^2C'_{\mathcal{O}} - L_z^2A'_{\mathcal{O}}}{\ell^r}\\
&\qquad - \frac{I_{ABC}}{2}\left(E^2C' - L_z^2A'\right)\left(E^2C'_{\mathcal{O}} - L_z^2A'_{\mathcal{O}}\right)\Bigg)\\
&{W_{LX}}^{\mathbf{2}}{}_{\mathbf{0}} = \frac{EQ}{4L_z}\frac{C'_{\mathcal{O}}\ell^r_{\mathcal{O}}}{\sqrt{ABC}}\left(L_z^2A' - E^2C'\right)I_{ABC} + \frac{EQ}{2L_z\ell^r}\frac{\left(C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right)}{\sqrt{ABC}}\\
&{W_{LX}}^{\mathbf{3}}{}_{\mathbf{0}} = \frac{Q}{2L_z^2}\left(C'\ell^r - C'_{\mathcal{O}}\ell^r_{\mathcal{O}}\right) - I_{ABC}C'\ell^rC'_{\mathcal{O}}\ell^r_{\mathcal{O}}\frac{E^2Q}{4L_z^2} + \frac{\Psi}{Q}{W_{LX}}^{\mathbf{2}}{}_{\mathbf{0}}.
\end{aligned}
\tag{118}
$$

Note that these relations take the simplest form in the intermediate SNT, as defined in section 3.5, in which we simply have $\Psi = 0$.

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Chapter 4

Testing the null energy condition with precise distance measurements

In this chapter, we present a very general and unexpected result about the distance measurements based on light propagation in curved spacetimes. It states that if general relativity holds, the matter satisfies the NEC and the light travels along null geodesics, then for a source of light placed anywhere between the observer and the first focal point the parallax distance measured by the observer cannot get smaller than the angular diameter distance. The result is completely independent of any spacetime symmetries or motions of the source or the observer. The result rests upon the projected GDE and the light ray bundle formalism, while the techniques used are very similar to those found in standard focusing theorems.

In this paper we study the inequality with three different parallax distance definitions: the determinant-averaged, the trace-averaged, and the directional parallax distances. The result holds equally for the first two definitions, although the proof of the second one is more technically involved than of the first one. On the other hand, the directional parallax distance is shown not to satisfy the inequality if the ray encounters strong tidal forces. The paper is concluded with a list of rough estimates of the positions of focal points and the distance slip throughout and beyond the Milky Way. For this we assumed a certain continuous matter distribution model. We conclude that direction averaging is crucial unless Weyl contribution is on average negligible and uncorrelated with the direction of the baseline of observation.

Author's contribution

The results were derived in collaboration with my supervisor prof. M. Korzyński. My contribution includes:

- making the connection between the BGO and the lightray bundle formalisms, especially the fact that μ is related to the lightray bundle with initially parallel rays;
- estimation of μ in Sec. IV;
- development of Taylor series expansion for BGOs and distance measures used in Sec. III;

Meanwhile, the contribution of prof. M. Korzyński includes:

- the idea of the paper;
- the complete proof for the inequality of trace-averaged parallax distance and the counterexample in the case of strong Weyl curvature in Sec. III.

The draft was written and published jointly by me and M. Korzyński.

Testing the null energy condition with precise distance measurements

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We present an inequality between two types of distance measures from an observer to a single light source in general relativity. It states that for a given source and observer the distance measured by the trigonometric parallax is never shorter than the angular diameter distance provided that the null energy condition holds and that there are no focal points in between. This result is independent of the details of the spacetime geometry or the motions of the observer and the source. The proof is based on the geodesic bilocal operator formalism and on well-known properties of infinitesimal light ray bundles. Observation of the violation of the distance inequality would mean that on large scales either the null energy condition does not hold or that light does not travel along null geodesics.

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I. INTRODUCTION

Measuring the distance to a given light source is one of the fundamental problems of astronomy. Many methods have been developed for that purpose depending on the nature of the source and the distance. The three probably most straightforward ones are the trigonometric parallax measurement, the determination of the angular size of an extended source of known size (i.e., a standard ruler) and the measurement of the energy flux from an isotropic source with known absolute luminosity (i.e., a standard candle). By definition, all three methods must give the same result in a flat spacetime and in the absence of relative motions between the source and the observer. However, in the presence of spacetime curvature and relative motions the three methods are inequivalent. Therefore, in general relativity (GR) we distinguish the angular diameter distance $D_{\rm ang}$, also known as the area distance, defined via the solid angle occupied by the image of a standard ruler, the luminosity distance $D_{\rm lum}$, defined by the measured energy flux from a standard candle, and the parallax distance $D_{\rm par}$, defined via the apparent displacement of the image given the displacement of the observer along a baseline [1,2]. Note that in the presence of curvature the trigonometric parallax may depend on the baseline orientation. However, it is possible to define a baseline-averaged quantity, combining the parallax effects from two orthogonal directions [2].

$D_{\rm ang}$, $D_{\rm lum}$ and the redshift z measured between a fixed source and observer are related by the well-known Etherington's reciprocity relation $D_{\rm lum} = (1+z)^2 D_{\rm ang}$ independently of the spacetime geometry [1,3–10]. In case of the baseline-averaged parallax distance $D_{\rm par}$ the relation to other distance measures is more complicated and depends on the curvature tensor along the line of sight. In fact, for short distances the relative difference between $D_{\rm ang}$ and $D_{\rm par}$, called the *distance slip* μ, depends only on the matter content along the line of sight: the leading order correction is given by an integral of the component $T_{\mu\nu}l^\mu l^\nu$ of the stress-energy tensor, where l^μ is the tangent vector to the null geodesic connecting the observer and the source. Namely, for short distances we have

$$\mu = 1 - \frac{D_{\rm ang}^2}{D_{\rm par}^2} = 8\pi G \int_{\lambda_\mathcal{O}}^{\lambda_\mathcal{E}} T_{\mu\nu} l^\mu l^\mu (\lambda_\mathcal{E} - \lambda) \mathrm{d}\lambda + O(\mathrm{Riem}^2), \tag{1}$$

where λ is the affine parameter of the connecting null geodesic, $\lambda_\mathcal{O}$ corresponds to the observation point and $\lambda_\mathcal{E}$ to the source, while $O(\mathrm{Riem}^2)$ denotes terms involving higher powers of the curvature [2,11].

In this paper we show that the sign of the difference between $D_{\rm par}$ and $D_{\rm ang}$ for a given source and a given observer is directly related to the *null energy condition* (NEC). On the perturbative level this already follows from (1): the leading, linear term in curvature is obviously nonnegative if the NEC condition holds, because in this case we have $0 \le R_{\mu\nu} l^\mu l^\nu = 8\pi G T_{\mu\nu} l^\mu l^\nu$ for any null l^μ. The main theorems of this paper extend this inequality to the non-perturbative level: we show that if the NEC is satisfied then $D_{\rm par} \ge D_{\rm ang}$ at least up to a well-defined, finite distance between the observer and the light source. More precisely, for a fixed observation point and variable source position along a null geodesic the inequality is guaranteed to hold from the observation point up to the so-called focal point, where the parallax distance reaches

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infinite value. Note that this blow-up may happen for a finite value of the affine parameter. Past that point, however, $D_{\rm par}$ becomes finite again and it is possible that $D_{\rm par} < D_{\rm ang}$ there even if the NEC holds globally [12]. This restriction means that the distance inequality is not global. Stated in a more physical language, the inequality means that both the matter along line of sight, resulting in Ricci focusing of the null geodesics, and the tidal forces, producing their shear, can only increase $D_{\rm par}$ in comparison with $D_{\rm ang}$, at least up to the first focal point.

Recall that the *null energy condition* (NEC) for all null vectors l^μ reads

$$T_{\mu\nu} l^\mu l^\nu \geq 0. \tag{2}$$

It is one of the weakest classical pointlike energy conditions with relatively simple and important applications. In general relativity, it is equivalent to the null convergence condition (NCC)

$$R_{\mu\nu} l^\mu l^\nu \geq 0, \tag{3}$$

which ensures that at every point light rays experience gravity as an attractive force. This property implies that any light ray bundle eventually has to reach a caustic over a finite distance provided that suitable initial condition holds. More precisely, the standard focusing theorem states that the ray bundle must reach a point where its expansion diverges provided that the expansion is negative at a single point, and NCC holds [13]. Surprisingly, this simple result is necessary for proving such basic black hole properties like formation of event horizons and singularities under the gravitational collapse, some of the black hole laws, various no hair theorems, and some versions of the positivity of ADM mass [14].

As we have already noted, if we assume the Einstein equations then the NEC is equivalent to the NCC. It is satisfied for most reasonable types of matter like dust, radiation, fluid, or classical electromagnetic fields [15]. It is also insensitive to the presence of cosmological constant. The validity of the NEC can also be used to put bounds on various properties of the FLRW Universe [16]. On the other hand, it has also been noted that quantum effects tend to violate the NEC as well as its even weaker averaged version [17]. The NEC can also fail in the presence of nonstandard matter fields [14,17,18], including fluids with barotropic index $w < -1$ or for holographic dark energy models with even smaller barotropic index [19].

In modified theories of gravity NEC and NCC are in general not equivalent. The NCC can fail even in presence of standard matter if the field equations contain additional terms, like in the case of $f(R)$ gravity [20,21] or other extended metric theories [22–24]. Moreover, light propagation may also work differently than in GR: outside the Riemannian geometry regime the light may follow null curves which are neither autoparallel nor extremal, and this implies that the optical equations contain additional geometric terms, effectively acting as additional, nonclassical matter fields [25–27]. These terms affect the focusing and defocusing properties of the spacetime and, consequently, the relation between NEC and the properties of light rays.

In this paper we assume that standard general relativity holds. Therefore, we will assume NEC, which we in turn treat as equivalent to NCC. However, the reasoning should still apply to any metric theory in which light travels along null geodesics and NCC holds.

The main result of this paper, i.e., the distance inequality, provides a method to test the NEC using sufficiently precise, simultaneous distance measurements by parallax and angular (or luminosity) to a number of light sources. The observation of the violation of the NEC would require a serious reevaluation of the fundamentals of physics. We point out, however, that the precision required seems beyond what is currently possible.

The proof of the main theorem of the paper, that is, Theorem III.1, makes use of the bi-local approach to light propagation in curved spacetimes, developed in [2,12], and of the standard infinitesimal ray bundle formalism [1], closely related to the null congruence formalism. We first show that the distance slip μ between the observation point $\mathcal{O}$ and any source located along a past-directed null geodesic from $\mathcal{O}$ is related to the ratio of the cross-sectional areas of a particular infinitesimal bundle of null rays, with cross sections taken at $\mathcal{O}$ and at the source. We then show that this cross-sectional area cannot increase along the null geodesic as we move away from the observation point in the past direction, at least up to the first focal point, which completes the proof. The main argument is similar to the reasoning used in the proof of the standard focusing theorem.

A. Notation and conventions

Greek letters $(\alpha, \beta, \ldots)$ run from 0 to 3, lowercase Latin indices run from 1 to 3 and uppercase Latin indices run from 1 to 2. They all enumerate tensor components in the coordinate tetrad. Boldface versions of indices cover the same range but denote components in a parallel transported tetrad. The dot stands for the derivative with respect to the affine parameter along the null geodesic. Subscript $\mathcal{O}$ and $\mathcal{E}$ denote evaluation of the quantity at respectively the point of observation and emission, i.e., $f_{\mathcal{O}} \equiv f(\lambda_{\mathcal{O}})$. We assume the speed of light $c = 1$ throughout the paper.

B. Structure of the paper

In Sec. II, we briefly review the bilocal geodesic operator (BGO) formalism and relate these operators to the magnification and parallax matrices as well as angular diameter and parallax distances. Then we introduce the notion of the infinitesimal ray bundle and present two types of bundles which we will use later. Finally, we recall Sachs optical equations and their connection to the BGOs. In Sec. III we present our main results. In the first part we prove the inequality for the parallax distance $D_{\rm par}$ with baseline averaging performed using the determinant of the parallax matrix. In the second part we show that a similar conclusion follows for the parallax distance with the baseline averaging via trace, as proposed earlier by a number of authors. Lastly, we explain why our result cannot be extended to single baseline parallax measurements. We gather our final remarks in Sec. IV, including a short discussion of the prospects for measurement.

II. PRELIMINARIES

Let (M, g) be a Lorentzian spacetime, with signature $(-,+,+,+)$. Let $\mathcal{O}, \mathcal{E} \in M$ be points contained in a geodesically convex set. $\mathcal{O}$ will denote the observation point, lying in the causal future of the emission point $\mathcal{E}$. Let $\gamma_0 : [\lambda_\mathcal{O}, \lambda_\mathcal{E}] \to M$ be the unique geodesic connecting $\mathcal{O}$ and $\mathcal{E}$, and we assume here that γ_0 is null. By convention we assume that the affine parameter λ runs backwards in time, i.e., $\lambda_\mathcal{O} < \lambda_\mathcal{E}$. Let $N_\mathcal{O}, N_\mathcal{E} \subset M$ be locally flat neighborhoods of $\mathcal{O}$ and $\mathcal{E}$ respectively extending in all four dimensions. Let l^μ be the vector tangent to γ_0. Let ξ^μ denote the deviation vector (Jacobi field) along γ_0, satisfying the first order *geodesic deviation equation* (GDE):

$$\nabla_l \nabla_l \xi^\mu = R^\mu{}_{ll\nu} \xi^\nu, \tag{4}$$

where the *optical tidal tensor* $R^\mu{}_{ll\nu}$ is defined as $R^\mu{}_{ll\nu} \equiv R^\mu{}_{\alpha\beta\nu} l^\alpha l^\beta$. This is a linear, second order ordinary differential equation for ξ^μ. It is possible to rewrite it as an equation for four bitensors, forming together the formal resolvent of the GDE [2]. In this language the general solution to (4) can be expressed as

$$\begin{aligned} \xi^\mu(\lambda) &= W_{XX}{}^\mu{}_\nu(\lambda)\xi^\nu(\lambda_\mathcal{O}) + W_{XL}{}^\mu{}_\nu(\lambda)\nabla_l \xi^\nu(\lambda_\mathcal{O}) \\ \nabla_l \xi^\mu(\lambda) &= W_{LX}{}^\mu{}_\nu(\lambda)\xi^\nu(\lambda_\mathcal{O}) + W_{LL}{}^\mu{}_\nu(\lambda)\nabla_l \xi^\nu(\lambda_\mathcal{O}), \end{aligned} \tag{5}$$

where W_{XX}, W_{XL}, W_{LX} and W_{LL} are 4 bitensors, or two-point tensors, acting from the observation point $\mathcal{O}$ to the point λ on the geodesic, called the *bilocal geodesic operators* (BGO's). The BGO's are functionals of the curvature along γ_0 defined via appropriate ordinary differential equations involving the components of the optical tidal tensor as coefficients [2,28].

We introduce the *seminull tetrad* (SNT) of the form $(u^\mu, e^\mu_\mathbf{A}, l^\mu)$, parallel propagated along γ_0. It comprises the null tangent vector l^μ, two orthonormal spacelike vectors $e^\mu_\mathbf{A}$, $\mathbf{A} = 1, 2$, orthogonal to l^μ, spanning a spacelike two-dimensional subspace called the screen space or transverse space, and a normalized timelike vector u^μ orthogonal to $e^\mu_\mathbf{A}$. u^μ is commonly identified with the 4-velocity of an observer measuring the position and the direction of the propagation of photons. It satisfies $u^\mu l_\mu = Q$, where Q is a positive constant. Given the fixed observation point $\mathcal{O}$, the equations for the transverse components of W_{XX} and W_{XL} expressed in the SNF read [2]:

$$\begin{aligned} \ddot{W}_{XX}{}^\mathbf{A}{}_\mathbf{B} - R^\mathbf{A}{}_{ll\mathbf{C}} W_{XX}{}^\mathbf{C}{}_\mathbf{B} &= 0 \\ W_{XX}{}^\mathbf{A}{}_\mathbf{B}(\lambda_\mathcal{O}) &= \delta^\mathbf{A}{}_\mathbf{B} \\ \dot{W}_{XX}{}^\mathbf{A}{}_\mathbf{B}(\lambda_\mathcal{O}) &= 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \ddot{W}_{XL}{}^\mathbf{A}{}_\mathbf{B} - R^\mathbf{A}{}_{ll\mathbf{C}} W_{XL}{}^\mathbf{C}{}_\mathbf{B} &= 0 \\ W_{XL}{}^\mathbf{A}{}_\mathbf{B}(\lambda_\mathcal{O}) &= 0 \\ \dot{W}_{XL}{}^\mathbf{A}{}_\mathbf{B}(\lambda_\mathcal{O}) &= \delta^\mathbf{A}{}_\mathbf{B}. \end{aligned} \tag{7}$$

With this machinery we may introduce the parallax matrix, the magnification matrix and the optical distance measures. Suppose we project the BGOs onto the SNT, with the timelike vector corresponding to the observer's 4-velocity $u^\mu_\mathcal{O}$. Their projection onto the screen space spanned by $e^\mu_\mathbf{A}$ is related to the matrices.

The magnification matrix $M^\mathbf{A}{}_\mathbf{B}$ relates the transverse vectors representing the spatial extent of a luminous body to the vectors on the screen space representing the angular size on an observer's sky [2] (in the gravitational lensing theory this quantity is usually defined using angular variables, and therefore rescaled with respect to $M^\mathbf{A}{}_\mathbf{B}$ as defined here). We have

$$M^\mathbf{A}{}_\mathbf{B} = \frac{1}{(u^\mu_\mathcal{O} l_{\mathcal{O}\mu})} (W_{XL}{}^{-1})^\mathbf{A}{}_\mathbf{B}, \tag{8}$$

see [2]. The submatrix $W_{XL}{}^\mathbf{A}{}_\mathbf{B}$, whose inverse appears in the formula above, is also known as the Jacobi matrix.

The parallax matrix on the other hand relates the transverse displacement of an observer with the apparent change of the position of a body on the observer's sky [2] and is given by

$$\Pi^\mathbf{A}{}_\mathbf{B} = \frac{1}{(u^\mu_\mathcal{O} l_{\mathcal{O}\mu})} (W_{XL}{}^{-1})^\mathbf{A}{}_\mathbf{C} W_{XX}{}^\mathbf{C}{}_\mathbf{B}. \tag{9}$$

In a flat spacetime both linear operators are proportional to a unit matrix. Therefore both the trigonometric parallax and the magnification do not depend on the direction of the baseline or the orientation of the source's shape. However, in the general case the direction in the transverse space is important. Both linear operators can be used to define direction-averaged distances to the observed source of light. In the BGO formalism it is natural to do this as follows: the angular diameter distance, or the area distance, is defined via the determinant of $M^\mathbf{A}{}_\mathbf{B}$:

$$D_{\rm ang} = |\det M^\mathbf{A}{}_\mathbf{B}|^{-1/2} = (u^\mu_\mathcal{O} l_{\mathcal{O}\mu}) |\det W_{XL}{}^\mathbf{A}{}_\mathbf{B}|^{1/2}. \tag{10}$$

In a more operational language, it is given by the ratio of the cross-sectional area of a luminous object and the solid angle taken by its image. It is a well-known quantity in relativistic geometrical optics. The parallax distance averaged by determinant has been introduced in [2]:

$$D_{\mathrm{par}} = |\det \Pi^{\mathbf{A}}{}_{\mathbf{B}}|^{-1/2} = (u^{\mu}_{\mathcal{O}} l_{\mathcal{O}\mu})|\det W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}|^{1/2}|\det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}|^{-1/2}. \quad (11)$$

Both distance measures depend on the state of motion of the observer, given by $u^{\mu}_{\mathcal{O}}$, and the spacetime geometry along the line of sight. However, they do not depend on the state of motion of the light emitter.

Finally, we recall the definition of the main quantity of this work, i.e., the *distance slip* μ [2]:

$$\mu = 1 - \sigma \frac{D^2_{\mathrm{ang}}}{D^2_{\mathrm{par}}} = 1 - \frac{\det \Pi^{\mathbf{A}}{}_{\mathbf{B}}}{\det M^{\mathbf{A}}{}_{\mathbf{B}}}, \quad (12)$$

where $\sigma = \pm 1$ is a sign correction given by $\sigma = \mathrm{sgn}\det \Pi^{\mathbf{A}}{}_{\mathbf{B}}/\mathrm{sgn}\det M^{\mathbf{A}}{}_{\mathbf{B}}$. It is independent of the states of motion of both the observer and the source. It vanishes in a flat spacetime, but not necessary in a curved one.

From (8)–(11) we can rewrite the equations above as

$$\mu = 1 - \det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} \quad (13)$$

and

$$\sigma = \mathrm{sgn}\det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}. \quad (14)$$

A. Infinitesimally thin bundles

The infinitesimally thin ray bundle formalism is a complementary method to describe light propagation between two points. We make use of it in this paper, because the propagation equations describing the bundles can be easily used to prove inequalities involving observable quantities. We will briefly review the infinitesimally thin ray bundle formalism, as described in [1].

By an infinitesimal ray bundle with an elliptical cross section we mean the set

$$S = \{c^I \xi^{\mathbf{A}}_I | c^1, c^2 \in \mathbb{R}, c^I c^J \delta_{IJ} \le 1\} \quad (15)$$

where $\xi^{\mathbf{A}}$ satisfies the GDE in the SNT:

$$\ddot{\xi}^{\mathbf{A}}_I = R^{\mathbf{A}}{}_{ll\mathbf{B}} \xi^{\mathbf{B}}_I, \quad (16)$$

and the index I enumerates linearly independent solutions not proportional to l^{μ}. Note that we take into account only the two transverse components of the vectors. This is possible because we may impose the condition $\xi^{\mathbf{0}} = 0$, or equivalently

$$\xi^{\mu} l_{\mu} = 0, \quad (17)$$

along the whole γ_0, and because the component $\xi^{\mathbf{3}}$ does not couple with the other three. $\xi^{\mathbf{3}}$ is also irrelevant from the point of view of geometric optics [2,29]. The cross section of S by the screen spanned by $e^{\mu}_{\mathbf{A}}$ is spacelike and Lorentz-invariant everywhere along the geodesic. Especially important for the main result is its cross-sectional area:

$$\mathcal{A} = \pi \epsilon_{\mathbf{AB}} \xi^{\mathbf{A}}_1 \xi^{\mathbf{B}}_2, \quad (18)$$

where $\epsilon_{\mathbf{AB}}$ is the area two-form [30] and $\xi^{\mathbf{A}}_I$ are the two linearly independent solutions of (16) projected on the screen space. The area defined by (18) is a signed quantity which may change sign when the bundle degenerates to a point or to a line. We assume throughout the work that the initial value of $\mathcal{A}$ is chosen to be positive, i.e.,

$$\mathcal{A}(\lambda_{\mathcal{O}}) \equiv \mathcal{A}_{\mathcal{O}} > 0. \quad (19)$$

It is customary to rewrite the GDE (16) for the two generators of an infinitesimal bundle in terms of the so-called kinematical bundle variables. We first need to decompose the transverse part of the optical tidal tensor into the trace and the traceless part as follows:

$$R^{\mathbf{A}}{}_{\mu\nu\mathbf{B}} l^{\mu} l^{\nu} = -\frac{1}{2} R_{ll} \delta^{\mathbf{A}}{}_{\mathbf{B}} + C^{\mathbf{A}}{}_{\mu\nu\mathbf{B}} l^{\mu} l^{\nu}, \quad (20)$$

where $R_{ll} = R_{\mu\nu} l^{\mu} l^{\nu}$ denotes the $l-l$ component of the Ricci tensor and $C^{\alpha}{}_{\mu\nu\beta}$ is the Weyl tensor. This decomposition holds even though we are taking the trace only with respect to the two-dimensional subspace of the tangent space, see for example [2,31].

The bundle evolution along the null geodesic is most conveniently described in terms of the deformation tensor $B^{\mathbf{A}}{}_{\mathbf{B}}$ [32–34], defined via

$$\dot{\xi}^{\mathbf{A}}_I(\lambda) = B^{\mathbf{A}}{}_{\mathbf{B}}(\lambda) \xi^{\mathbf{B}}_I(\lambda) \quad (21)$$

for both $I = 1, 2$. The infinitesimal bundle can always be extended to a full congruence of null geodesics. In that case we have the relation

$$B^{\mathbf{A}}{}_{\mathbf{B}} = \nabla_{\mathbf{B}} l^{\mathbf{A}}, \quad (22)$$

where l^{μ} denotes the vector field generating the congruence.

$B^{\mathbf{A}}{}_{\mathbf{B}}$ decomposes into the kinematical variables (also known as the optical scalars), i.e., the scalar expansion θ, traceless symmetric shear $\sigma_{\mathbf{AB}}$, and antisymmetric twist $\omega_{\mathbf{AB}}$, according to

$$B_{\mathbf{AB}} = \frac{1}{2}\theta \delta_{\mathbf{AB}} + \sigma_{\mathbf{AB}} + \omega_{\mathbf{AB}}. \quad (23)$$

Each of them satisfies an appropriate ODE along the null geodesic, known as the null Raychaudhuri equations, Sachs optical equations or transport equations. Here we only consider twist-free (or surface-forming) bundles, for which $\omega_{\mathbf{AB}} = 0$ along the whole γ_0. In this case the equations for θ and shear $\sigma_{\mathbf{AB}}$ read [33]

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{\theta^2}{2} - \sigma_{\mathbf{AB}}\sigma^{\mathbf{AB}} - R_{ll} \tag{24}$$

$$\frac{\mathrm{d}\sigma_{\mathbf{AB}}}{\mathrm{d}\lambda} = -\theta\sigma_{\mathbf{AB}} + C_{\mathbf{A}ll\mathbf{B}}, \tag{25}$$

where $C_{\mathbf{A}ll\mathbf{B}} = C_{\mathbf{A}\mu\nu\mathbf{B}} l^\mu l^\nu$. These equations are sometimes written in a different form, using a set of complex scalars, but we prefer the tensorial representation. The area of the cross section satisfies its own evolution equation, given in terms of the expansion:

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\lambda} = \mathcal{A}\theta. \tag{26}$$

In this paper we mainly consider two examples of infinitesimal ray bundles. The first one is the *infinitesimal ray bundle parallel at* $\mathcal{O}$, or *the parallel bundle at* $\mathcal{O}$ for short, see Fig. 1. As the name suggests, it satisfies the condition of being strictly parallel at $\mathcal{O}$, i.e.,

$$\theta(\lambda_{\mathcal{O}}) = 0 \tag{27}$$

$$\sigma_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0 \tag{28}$$

$$\omega_{\mathbf{AB}}(\lambda_{\mathcal{O}}) = 0. \tag{29}$$

It is related to the transverse components of W_{XX}: namely, we have

$$\xi_I^{\mathbf{A}}(\lambda) = W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)\xi_I^{\mathbf{B}}(\lambda_{\mathcal{O}}) \tag{30}$$

for this bundle. Due to the orthogonality condition (17) ξ_I^μ has only transverse components plus a component proportional to l^μ. The latter is irrelevant from the point of view of the geometry of cross sections, see the Sachs shadow theorem [30], so it is the two transverse components of ξ_I^μ given by (30) that define the distance measures. Finally, Eqs. (30) and (18) give

$$\mathcal{A}(\lambda) = (\det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}})\mathcal{A}_{\mathcal{O}}. \tag{31}$$

From this and (13) we see that

$$\mu = 1 - \frac{\mathcal{A}(\lambda)}{\mathcal{A}_{\mathcal{O}}}, \tag{32}$$

where μ is calculated for the emission point at λ and the observation point at $\mathcal{O}$.

The other bundle we consider in this paper is the *infinitesimal ray bundle with a vertex at point* $\mathcal{E}$, or simply the *vertex bundle from* $\mathcal{E}$. It satisfies the condition of vanishing at $\mathcal{E}$, i.e., $\tilde{\xi}_I^{\mathbf{A}}(\lambda_{\mathcal{E}}) = 0$ for $I = 1, 2$, see Fig. 2(a). Let $\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)$ denote the transverse components of the bitensor defined just like $W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)$, but with the initial point taken at $\mathcal{E}$ instead of $\mathcal{O}$. The vertex bundle is then related to this bitensor via:

$$\tilde{\xi}_I^{\mathbf{A}}(\lambda) = \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)\dot{\tilde{\xi}}_I^{\mathbf{B}}(\lambda_{\mathcal{E}}). \tag{33}$$

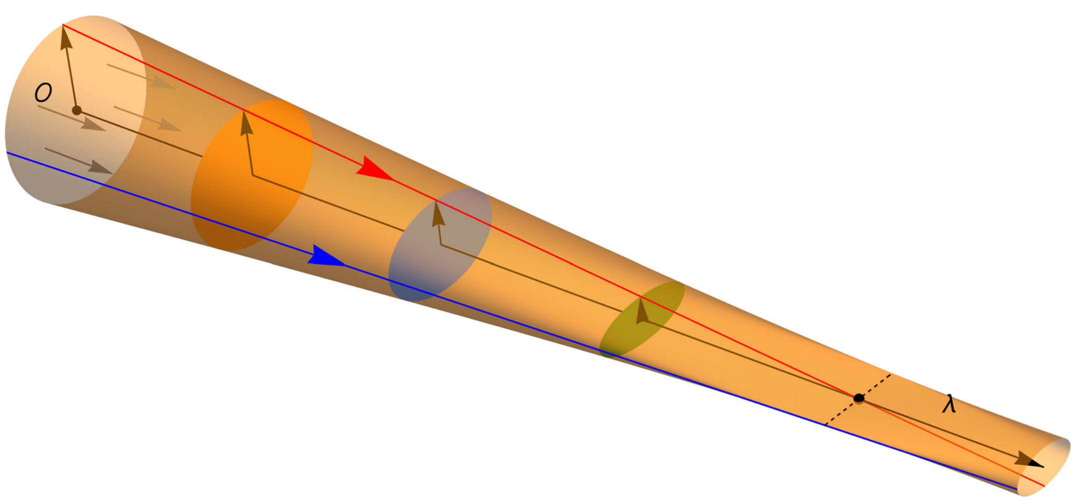
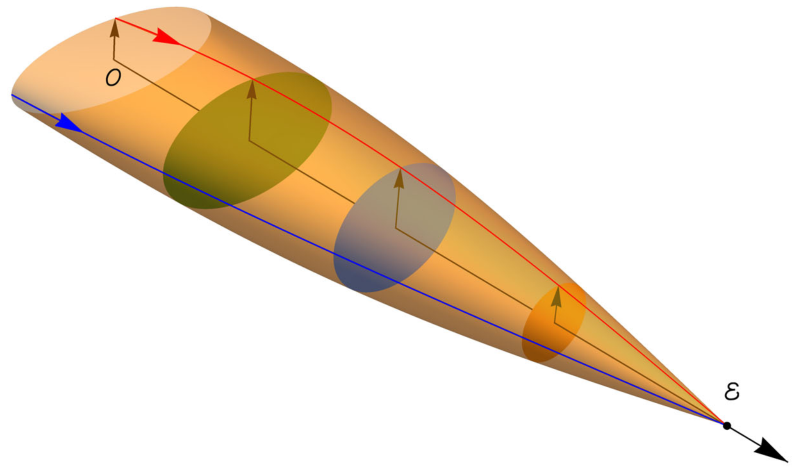
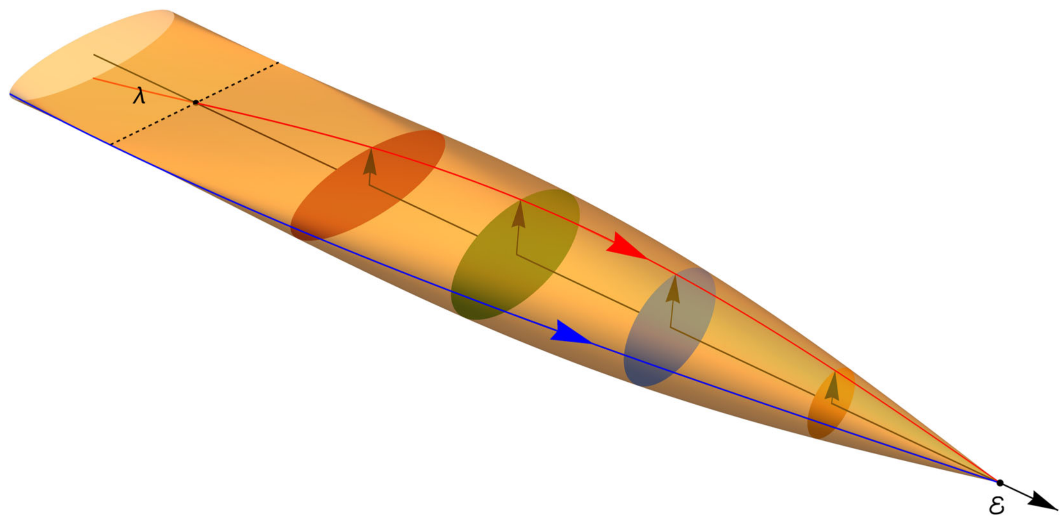


FIG. 1. A bundle of rays that runs parallel at $\mathcal{O}$. Its shape undergoes a deformation under the spacetime curvature. Along the fiducial geodesic a shape of cross section that is circular at $\mathcal{O}$ changes its size and becomes increasingly elliptical. In the generic case it eventually degenerates either to a line or to a point. The point on the geodesic where this degeneracy happens is called the focal point.



(a)



(b)

FIG. 2. (a) A bundle of rays that forms a vertex at $\mathcal{E}$. Its shape undergoes a deformation under the spacetime curvature. Along the fiducial geodesic a shape of cross section that is circular at $\mathcal{E}$ changes its size and becomes increasingly elliptical. (b) The same bundle of rays that forms a vertex at $\mathcal{E}$. At a larger distance it may eventually degenerate either to a line or to a point. The point on the geodesic where this degeneracy happens is called the conjugate point.

The asymptotic behavior of the infinitesimal bundle near $\mathcal{E}$ is described by the Taylor expansion

$$\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda) = (\lambda - \lambda_{\mathcal{E}})\delta^{\mathbf{A}}{}_{\mathbf{B}} + O((\lambda - \lambda_{\mathcal{E}})^2), \quad (34)$$

see equations (7) with $\mathcal{E}$ as the base point. It follows that the expansion $\tilde{\theta}$ has a simple pole at $\lambda_{\mathcal{E}}$ [1]:

$$\tilde{\theta}(\lambda) = 2(\lambda - \lambda_{\mathcal{E}})^{-1} + O(1). \quad (35)$$

This is an example of a singular point of an infinitesimal ray bundle. In the next section we will define this notion more precisely.

By analogy we may also consider an infinitesimal ray bundle with a vertex at $\mathcal{O}$, related to the bitensor W_{XL} instead of $\tilde{W}_{XL}$.

B. Singular points of a bundle

The description of a ray bundle using the shear and expansion can break down at isolated points even though the perturbed geodesics constituting the bundle are perfectly regular there. This typically happens when the bundle collapses along one or two directions, forming a self-intersection. In the language of the Sachs frame this happens if the determinant of the two transverse solutions of the GDE vanishes, i.e., $\det(\xi_1^{\boldsymbol{A}}, \xi_2^{\boldsymbol{B}}) = 0$. The expansion diverges in this case to $\pm\infty$ and changes sign.

We introduce the following definition:

Definition II.1. Point $\mathcal{P}$ along a null geodesic γ_0 is a *regular point* of a ray bundle iff $\theta(\lambda)$, $\omega_{AB}(\lambda)$ and $\sigma_{AB}(\lambda)$ are smooth at $\mathcal{P}$. A point that is not regular will be called *singular*.

We fix the null geodesic γ_0 and the observer's position $\mathcal{O}$ along γ_0. We can now introduce two types of singular points along a null geodesic, defined by the properties of the vertex and initially parallel ray bundles.

We call $\mathcal{P}$ a *conjugate point* with respect to $\mathcal{O}$ iff the vertex bundle from $\mathcal{O}$ refocuses back at $\mathcal{P}$ at least along one transverse direction. This property is equivalent to the existence of a Jacobi field along γ_0, satisfying the GDE and vanishing at $\mathcal{O}$ and $\mathcal{P}$, but not identically zero [1]. It is easy to check that this happens iff $\det W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = 0$ between $\mathcal{O}$ and $\mathcal{P}$. Conjugate points correspond to the intersection of the fiducial geodesic with a caustic and are points of infinite magnification of images of objects located at $\mathcal{P}$ as seen in $\mathcal{O}$.

We call $\mathcal{P}$ a *focal point* iff an infinitesimal bundle of rays running parallel at $\mathcal{O}$ refocuses at $\mathcal{P}$ along at least one direction. This happens when $\det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} = 0$, see Fig 1.

The reader may check that at both the focal point and the conjugate point the expansion θ of the corresponding bundle has a singularity.

C. Integral formula for the cross-sectional area

The ODE (26) can be integrated exactly assuming that $\theta(\lambda)$ is a regular function. Namely, if there are no singular points between $\mathcal{O}$ and λ, the solution simply reads

$$\mathcal{A}(\lambda) = \mathcal{A}_{\mathcal{O}} \exp\left(\int_{\lambda_{\mathcal{O}}}^{\lambda} \theta(\lambda')\mathrm{d}\lambda'\right). \tag{36}$$

This formula will play an important role in the proof of the main result. Let us stress that the condition of regularity of $\theta(\lambda)$ is crucial here, because Eq. (36) may break down after a singular point such as a focal point. This is evident if we note that $\mathcal{A}(\lambda)$ may switch sign past a focal point, which is obviously inconsistent with Eq. (36), in which the signs of $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{A}(\lambda)$ must be the same.

III. THE MAIN THEOREMS

Definition We say that the *null energy condition* (NEC) holds at a set $O \subset M$ iff at every point in O we have $R_{\mu\nu} l^\mu l^\nu \geq 0$ for all null vectors l^μ.

Theorem III.1 Let $\mathcal{O}$ and $\mathcal{E}$ be two points along a null geodesic γ_0 such that $\mathcal{O}$ lies in the causal future of $\mathcal{E}$ and let the NEC hold along γ_0 between $\mathcal{O}$ and $\mathcal{E}$. Assume also that between $\mathcal{O}$ and $\mathcal{E}$ there are no singular points of the infinitesimal bundle of rays parallel at $\mathcal{O}$. Then we have

$$\mu \geq 0 \tag{37}$$

for an observer at $\mathcal{O}$ and a source at $\mathcal{E}$. Moreover, $\mu = 0$ iff the transverse components of the optical tidal tensor $R^{\mathbf{A}}{}_{\mu\nu\mathbf{B}} l^\mu l^\nu$ vanish along γ_0 between $\mathcal{O}$ and $\mathcal{E}$.

Proof We begin with the inequality. The right-hand side of the Eq. (24) for the derivative of θ is obviously non-positive. Since initially $\theta(\lambda_{\mathcal{O}}) = 0$ we see that $\theta(\lambda) \leq 0$ everywhere between $\mathcal{O}$ and $\mathcal{E}$:

$$\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \theta(\lambda)\mathrm{d}\lambda \leq 0. \tag{38}$$

From the integral formula (36) we see that $\mathcal{A}(\lambda) \leq \mathcal{A}_{\mathcal{O}}$, so from (32) we have $\mu \geq 0$. This completes the proof of the first part of the theorem.

Assume now $\mu = 0$ between $\mathcal{O}$ and $\mathcal{E}$. It follows from (32) that $\mathcal{A}(\lambda) = \mathcal{A}_{\mathcal{O}}$. Substituting this to the integral formula (36) we obtain

$$\exp\left(\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \theta(\lambda)\mathrm{d}\lambda\right) = 1, \tag{39}$$

or equivalently

$$\int_{\lambda_{\mathcal{O}}}^{\lambda_{\mathcal{E}}} \theta(\lambda)\mathrm{d}\lambda = 0. \tag{40}$$

By the regularity assumption $\theta(\lambda)$ is continuous on the closed interval between $\lambda_{\mathcal{O}}$ and $\lambda_{\mathcal{E}}$ and we have also shown that $\theta(\lambda) \leq 0$ on this interval. Therefore Eq. (40) is only possible if $\theta(\lambda) = 0$ everywhere on this interval. Substituting this condition to (24) we obtain $-\sigma_{\mathbf{AB}}\sigma^{\mathbf{AB}} - R_{ll} = 0$, which implies that both $\sigma_{\mathbf{AB}} = 0$ and $R_{ll} = 0$ everywhere between $\mathcal{O}$ and $\mathcal{E}$. Finally, we substitute the former relation to (25) to obtain $C_{\mathbf{A}\mu\nu\mathbf{B}} l^\mu l^\nu = 0$.

We have thus proved that both the contracted Ricci tensor and the transverse components of the contracted Weyl tensor vanish. It follows that all transverse components of the optical tidal tensor must vanish between $\mathcal{O}$ and $\mathcal{E}$, see (20). This completes the proof of the second part of the theorem.

This theorem does not automatically imply an inequality between the distance measures because of the sign ambiguity in the definition of μ, see the rhs of Eq. (12). We therefore need one more result regarding the sign-defining factor σ in (12):

Proposition III.2. Under the assumptions of Theorem III.1 we have $\sigma = 1$.

Proof. From (36) we see easily that $\mathcal{A}(\lambda) > 0$. Therefore from (31) we have $\det W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} > 0$. Then $\sigma = 1$ follows from (14).

The following result follows now Theorem III.1, Eq. (12) and Proposition III.2

Corollary III.3. Under the assumptions of Theorem III.1 we have

$$D_{\mathrm{par}} \geq D_{\mathrm{ang}} \tag{41}$$

for any observer at $\mathcal{O}$ and any light source at $\mathcal{E}$. Moreover, $D_{\mathrm{par}} = D_{\mathrm{ang}}$ iff the transverse components of the optical tidal tensor $R^{\boldsymbol{A}}{}_{\mu\nu\boldsymbol{B}} l^\mu l^\nu$ vanish along γ_0 between $\mathcal{O}$ and $\mathcal{E}$.

Proof. From Proposition III.2 and (12) we have $\mu = 1 - \frac{D_{\mathrm{ang}}^2}{D_{\mathrm{par}}^2}$. Together with the positivity of D_{ang} and D_{par}, this implies that the inequality $\mu \geq 0$ is equivalent to $D_{\mathrm{par}} \geq D_{\mathrm{ang}}$ and the equality $\mu = 0$ is equivalent to $D_{\mathrm{par}} = D_{\mathrm{ang}}$. The Corollary follows then trivially from Theorem III.1.

A. Trace-based baseline-averaged parallax distance

In Räsänen [35], Rosquist [36] as well as Ellis *et al.* [7] (republished as [8]), Jordan *et al.* [37] (republished as [38]) a different method of baseline averaging for the parallax distance has been proposed. Effectively it differs from D_{par} by using the trace of the parallax matrix instead of the determinant for baseline direction averaging. We will now prove a result analogous to Theorem III.1 and Corollary III.3 for the parallax distance averaged by trace. Let $\tilde{D}_{\mathrm{par}}$ denote the trace-based parallax distance:

$$\tilde{D}_{\mathrm{par}} = \frac{2}{|\Pi^{\mathbf{A}}{}_{\mathbf{A}}|}. \tag{42}$$

This can be expressed also as

$$\tilde{D}_{\mathrm{par}} = 2(l_{\mathcal{O}\mu} u^\mu_{\mathcal{O}})|\tilde{\theta}|^{-1}, \tag{43}$$

where $\tilde{\theta}$ is the expansion of the vertex bundle emanating from $\mathcal{E}$, evaluated at $\mathcal{O}$, see Fig. 2(a). If we rescale the parametrization λ to fit the observer's frame, ensuring $l_{\mathcal{O}\mu} u^\mu_{\mathcal{O}} = 1$, this expression simplifies to

$$\tilde{D}_{\mathrm{par}} = 2|\tilde{\theta}|^{-1}. \tag{44}$$

We prove now the following theorem:

Theorem III.4. Let $\mathcal{O}$ and $\mathcal{E}$ be two points along a null geodesic γ_0 such that $\mathcal{O}$ lies in the causal future of $\mathcal{E}$ and let the NEC hold along γ_0 between $\mathcal{O}$ and $\mathcal{E}$. Assume also that between $\mathcal{O}$ and $\mathcal{E}$ there are no singular points of the infinitesimal bundle of rays with the vertex at $\mathcal{E}$, except the point $\mathcal{E}$ itself, and that its expansion $\tilde{\theta}$ does not vanish between $\mathcal{O}$ and $\mathcal{E}$. Then we have

$$\tilde{D}_{\mathrm{par}} \geq D_{\mathrm{ang}} \tag{45}$$

for an observer at $\mathcal{O}$ and a source at $\mathcal{E}$. Moreover, $\tilde{D}_{\mathrm{par}} = D_{\mathrm{ang}}$ iff the transverse components of the optical tidal tensor $R^{\boldsymbol{A}}{}_{\mu\nu\boldsymbol{B}} l^\mu l^\nu$ vanish along γ_0 between $\mathcal{O}$ and $\mathcal{E}$.

Proof. The proof uses the properties of an infinitesimal bundle with a vertex at $\mathcal{E}$ in a similar way as the proof of Theorem III.1 uses the infinitesimal bundle parallel at $\mathcal{O}$. We begin by relating the angular diameter distance to the properties of this bundle.

The angular diameter distance is related to the determinant of the Jacobi matrix at $\mathcal{E}$, with the vertex positioned at $\mathcal{O}$, see (10). However, it can be easily related to the Jacobi map with the roles of $\mathcal{O}$ and $\mathcal{E}$ reversed.

Let $\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)$ denote the Jacobi map with the vertex at $\mathcal{E}$, i.e., satisfying

$$\frac{d^2}{d\lambda^2}\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} - R^{\mathbf{A}}{}_{ll\mathbf{C}}\tilde{W}_{XL}{}^{\mathbf{C}}{}_{\mathbf{B}} = 0 \tag{46}$$

$$\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\mathcal{E}) = 0 \tag{47}$$

$$\frac{d}{d\lambda}\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\mathcal{E}) = \delta^{\mathbf{A}}{}_{\mathbf{B}}. \tag{48}$$

From the symplectic property of the GDE we have a simple relation between the Jacobi matrix from $\mathcal{E}$ up to $\mathcal{O}$ and the one calculated the other way round:

$$W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda_{\mathcal{E}}) = -\tilde{W}_{XL\mathbf{B}}{}^{\mathbf{A}}(\lambda_{\mathcal{O}}), \tag{49}$$

i.e., the two matrices are the transpose of each other, with a sign flip [1,10]. Therefore we can replace $\det W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}$ by $\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}$ in (10):

$$D_{\mathrm{ang}} = (l_{\mathcal{O}\mu} u^\mu_{\mathcal{O}})|\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}|^{-1/2}, \tag{50}$$

where $\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}$ is the Jacobi map from $\mathcal{E}$ to $\mathcal{O}$ [39].

$\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda)$, on the other hand, can be related by another ODE to the deformation tensor $\tilde{B}^{\mathbf{A}}{}_{\mathbf{B}}$ of the bundle with vertex at $\mathcal{E}$:

$$\frac{d}{d\lambda}\tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = \tilde{B}^{\mathbf{A}}{}_{\mathbf{C}}\tilde{W}_{XL}{}^{\mathbf{C}}{}_{\mathbf{B}}, \tag{51}$$

The bundle is twist-free, so it decomposes into expansion $\tilde{\theta}$ and shear $\tilde{\sigma}_{\mathbf{AB}}$ according to $\tilde{B}^{\mathbf{A}}{}_{\mathbf{B}}(\lambda) = \frac{1}{2}\tilde{\theta}\delta^{\mathbf{A}}{}_{\mathbf{B}} + \tilde{\sigma}^{\mathbf{A}}{}_{\mathbf{B}}$. It follows from (51) that

$$\frac{d}{d\lambda}\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = \tilde{\theta}(\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}). \tag{52}$$

We now define an auxiliary function $f(\lambda)$ in the following way: we fix the emission point $\mathcal{E}$ and vary the observation point, corresponding to the affine parameter value λ. We then take the ratio of the distances squared, measured between $\mathcal{E}$ and the point λ:

$$f(\lambda) = \frac{D_{\mathrm{ang}}^2}{\tilde{D}_{\mathrm{par}}^2}. \tag{53}$$

From (50) and (43) we get an expression for $f(\lambda)$ in terms of quantities related to the vertex bundle at $\mathcal{E}$:

$$f(\lambda) = \frac{\tilde{\theta}^2 |\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}|}{4}. \tag{54}$$

We will now derive an integral formula for $f(\lambda)$. We begin by differentiating Eq. (54) and applying (52):

$$\frac{df}{d\lambda} = \frac{1}{4}\left(2\tilde{\theta}\frac{d\tilde{\theta}}{d\lambda}|\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}| + \tilde{\theta}^3|\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}|\right) \tag{55}$$

In the first term use the propagation equation (24) for $\tilde{\theta}$ to obtain

$$\frac{df}{d\lambda} = \frac{1}{4}\tilde{\theta}|\det \tilde{W}_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}|(-\tilde{\sigma}_{\mathbf{AB}}\tilde{\sigma}^{\mathbf{AB}} - R_{ll}), \tag{56}$$

or, equivalently,

$$\frac{df}{d\lambda} = \frac{f}{\tilde{\theta}}(-\tilde{\sigma}_{\mathbf{AB}}\tilde{\sigma}^{\mathbf{AB}} - R_{ll}). \tag{57}$$

This equation can be solved by separation of variables, but we need the initial data. Recall that in the bundle with the vertex at $\mathcal{E}$ we have asymptotic expansions (34) and (35). The reader may check using (54) that this implies that $f \to 1$ as $\lambda \to \lambda_{\mathcal{E}}$. With this knowledge we may integrate the ODE (57) to

$$f(\lambda) = \exp\left(-\int_{\lambda_{\mathcal{E}}}^{\lambda} \tilde{\theta}^{-1}(\tilde{\sigma}_{\mathbf{AB}}\tilde{\sigma}^{\mathbf{AB}} + R_{ll})\mathrm{d}\lambda'\right). \tag{58}$$

The integrand contains the expansion $\tilde{\theta}$ in the denominator, but the integral is regular at $\mathcal{E}$ and everywhere along the interval considered because of our assumptions regarding the behavior of $\tilde{\theta}$ (i.e., no zeros and a pole at $\mathcal{E}$). Moreover, we note that $\tilde{\theta} < 0$ inside the interval we consider, because it is negative near $\lambda_{\mathcal{E}}$ due to (35) and from the assumptions we know that it cannot vanish or change sign in the interval we consider. It also follows that we can write $\tilde{\theta} = -|\tilde{\theta}|$. The integration in the formula above proceeds from larger $\lambda_{\mathcal{E}}$ down to smaller λ, so we swap the integration limits and absorb this way the minus sign from $\tilde{\theta}$ to obtain:

$$f(\lambda) = \exp\left(-\int_{\lambda}^{\lambda_{\mathcal{E}}} |\tilde{\theta}|^{-1}(\tilde{\sigma}_{\mathbf{AB}}\tilde{\sigma}^{\mathbf{AB}} + R_{ll})\mathrm{d}\lambda'\right). \tag{59}$$

Now, the last steps of the proof proceed just like in the proof of Theorem III.1: the integrand is manifestly nonnegative if NEC holds. Therefore we have $f \leq 1$, with the equality happening only if and only if both R_{ll} and $\tilde{\sigma}_{\mathbf{AB}}$ vanish between $\mathcal{E}$ and λ. The vanishing of the shear tensor $\tilde{\sigma}_{\mathbf{AB}}$ implies that the transverse components of the Weyl part of the optical tidal matrix $C_{\mathbf{A}ll\mathbf{B}}$ must also vanish because of (25).

B. Single baseline parallax distance

While it may not be obvious from the proofs we presented above, the baseline averaging of the trigonometric parallax effect is necessary for the distance inequality to work. Obviously it is possible to define a type of parallax distance defined through measurements with a single baseline. Let the transverse unit vector $n^{\mathbf{A}}$ denote the baseline direction, with $n^{\mathbf{A}}n_{\mathbf{A}} = 1$. As the simplest example we consider here

$$D_{\mathrm{par}}^{n} = |\Pi_{\mathbf{AB}}n^{\mathbf{A}}n^{\mathbf{B}}|^{-1}, \tag{60}$$

definition considered for example in [40]. D_{par}^{n} coincides with the baseline-averaged parallax distances in flat spacetimes, in which $\Pi^{\mathbf{A}}{}_{\mathbf{B}} = D^{-1}\delta^{\mathbf{A}}{}_{\mathbf{B}}$, D denoting the distance in the observer's frame. However, it turns out D_{par}^{n} does not have to obey the inequality $D_{\mathrm{ang}} \leq D_{\mathrm{par}}^{n}$ even if the NEC is satisfied. As an example consider the situation when we have a vacuum solution ($R_{ll} = 0$), but nonvanishing Weyl tensor $C_{\mathbf{A}ll\mathbf{B}}$ causing shear of null geodesics along γ_0. Obviously the NEC holds in a vacuum spacetime. The most general Taylor expansions for $W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}}$ and $W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}}$ around $\lambda_{\mathcal{O}}$ read [12]:

$$W_{XX}{}^{\mathbf{A}}{}_{\mathbf{B}} = \delta^{\mathbf{A}}{}_{\mathbf{B}} + \frac{(\lambda-\lambda_{\mathcal{O}})^2}{2}R^{\mathbf{A}}{}_{ll\mathbf{B}} + O((\lambda-\lambda_{\mathcal{O}})^3) \tag{61}$$

$$W_{XL}{}^{\mathbf{A}}{}_{\mathbf{B}} = (\lambda-\lambda_{\mathcal{O}})\delta^{\mathbf{A}}{}_{\mathbf{B}} + \frac{(\lambda-\lambda_{\mathcal{O}})^3}{6}R^{\mathbf{A}}{}_{ll\mathbf{B}} + O((\lambda-\lambda_{\mathcal{O}})^4), \tag{62}$$

with $R^{\mathbf{A}}{}_{ll\mathbf{B}}$ evaluated at $\mathcal{O}$. It follows that the parallax matrix has the Taylor expansion

$$\Pi_{\mathbf{AB}} = (l_{\mathcal{O}\mu}u_{\mathcal{O}}^{\mu})^{-1}(\lambda-\lambda_{\mathcal{O}})^{-1}\left(\delta_{\mathbf{AB}} + \frac{(\lambda-\lambda_{\mathcal{O}})^2}{3}R_{\mathbf{A}ll\mathbf{B}}\right) + O((\lambda-\lambda_{\mathcal{O}})^2), \tag{63}$$

while the expansion for D_{par}^{n} reads

$$D_{\mathrm{par}}^{n} = (l_{\mathcal{O}\mu}u_{\mathcal{O}}^{\mu})(\lambda-\lambda_{\mathcal{O}})\left(1 - \frac{(\lambda-\lambda_{\mathcal{O}})^2}{3}R_{\mathbf{A}ll\mathbf{B}}n^{\mathbf{A}}n^{\mathbf{B}}\right) + O((\lambda-\lambda_{\mathcal{O}})^4). \tag{64}$$

We compare the latter expression with the Taylor series for D_{ang}:

$$D_{\mathrm{ang}} = (l_{\mathcal{O}\mu}u_{\mathcal{O}}^{\mu})(\lambda-\lambda_{\mathcal{O}})\left(1 - \frac{(\lambda-\lambda_{\mathcal{O}})^2}{12}R_{ll}\right) + O((\lambda-\lambda_{\mathcal{O}})^4). \tag{65}$$

Assuming vacuum we have $R_{ll} = 0$ and $R_{\mathbf{A}ll\mathbf{B}} = C_{\mathbf{A}ll\mathbf{B}}$ and hence

$$D^n_{\rm par} = D_{\rm ang}\left(1 - \frac{(\lambda-\lambda_{\mathcal{O}})^2}{3} C_{\mathbf{A}//\mathbf{B}} n^{\mathbf{A}} n^{\mathbf{B}}\right) + O((\lambda-\lambda_{\mathcal{O}})^3). \tag{66}$$

Obviously we have $D^n_{\rm par} < D_{\rm ang}$ near $\mathcal{O}$ provided that $C_{\mathbf{A}//\mathbf{B}} n^{\mathbf{A}} n^{\mathbf{B}} > 0$, i.e., $n^{\mathbf{A}}$ is chosen such that the images undergo stretching by the tidal forces in its direction.

IV. CONCLUSIONS

The two main results of this paper, i.e., Theorems III.3 and III.4, mean that any observation of a systematic difference between the angular diameter distance and the baseline-averaged parallax distance with $D_{\rm ang} > D_{\rm par}$ would be difficult to reconcile with general relativity and the theory of light propagation as we understand them today. In particular, the effects of shear of the ray bundle due to tidal fields along the line of sight could not explain away a result of this kind. One would have to give up either the null energy condition or the assumption that light travels along null geodesics. Therefore, systematic comparison of both distance measurements may be considered an experimental test of the null energy condition, assuming that the general relativity and the geometric optics approximation for light propagation hold. Since $D_{\rm ang}$ is related to $D_{\rm lum}$ and the redshift through the Etherington's reciprocity relation, it is in principle possible to perform the measurement of distance slip μ using standard candles for which we have also precise redshift and trigonometric parallax measurements.

We note, however, that the measurements of the difference between the two distance measures seem impossible today, because the annual parallax effects are too small over the distances in which we can measure the trigonometric parallax. We can provide an order-of-magnitude estimate of the distance slip from the integral formula (1): for the mass density comparable with the mass density in the thin disc of the Milky Way $\rho = 1\ \mathrm{M}_\odot\,\mathrm{pc}^{-3}$ [41], negligible pressure terms in $T_{\mu\nu}$ and the distance of 20 kpc, comparable to the largest distance measured by trigonometric parallax [42], we get $\mu \approx 2 \times 10^{-4}$. A successful measurement of μ for a single source would then require the determination of both distance measures and the redshift with relative error not greater than 10^{-4}, way below current limitations [43–45]. This problem could possibly be overcome with a sufficiently large sample of sources. However, it would still require very precise standardization of the standard candles, good control of all possible sources of systematics as well as very precise redshift measurements.

Another type of parallax measurement has been proposed by Kardashev [46]. The measurement would use the displacement due to the motion of the Solar System with respect to the CMB rest frame as the baseline. The baseline grows in this case linearly in time and the signal is measured as secular variations of angular separations between distant sources. Longer baseline should in principle allow for parallax measurements on cosmological distances, although the signal seems still too low for modern instruments. Moreover, the foreground signal due to the Galactic aberration drift needs to be removed first [47]. The idea of cosmic parallax was also developed by many other authors [11,35,36,40,47–52]. Interestingly, the estimates for the distance slip on cosmological distances in the standard ΛCDM model (satisfying the NEC) yield fairly large values, reaching for example $\mu = 0.2$ near $z = 1$ [11]. Measurements of μ on such distances and determination its sign could test for the NEC violation by dark energy, an obvious sign of physics beyond the ΛCDM model.

We also point out three caveats regarding the distance inequality. The first two are related to the limitations of the mathematical approach. In the proofs we have used the first order geodesic deviation equation around a null geodesic. This means that we assume that the linear, curvature term in the geodesic deviation equation describes very well the behavior of all relevant light rays. This may fail, for example, if light passes through a region of very quickly varying gravitational potential across the null ray bundle considered (physical width of around 1 AU). In particular, it may fail in case of a microlensing event, i.e., a small massive body passing through the line of sight. It can also fail if light rays undergo significant nongravitational bending, for example due to the presence of ionized medium of variable density along the line of sight, or if the geometric optics approximation is not applicable.

Second, the inequality works only up to the first focal point, whose position in a given direction is not known beforehand. However, focal points between the Earth and galactic sources should be very rare, confined to rather special, fairly strongly lensed rays. Assuming that the line of sight is filled uniformly with mass density of $100\ \mathrm{M}_\odot\,\mathrm{pc}^{-3}$, scale of the density inside the bulge of the Milky Way, and ignoring the Weyl tensor contribution, we may predict the first focal point to appear at around 140 kpc. For uniform mass density comparable with the thin disc ($1\ \mathrm{M}_\odot\,\mathrm{pc}^{-3}$) the distance to the focal point grows to over 1 Mpc, while the density of the dark matter halo of $10^{-2}\ \mathrm{M}_\odot\,\mathrm{pc}^{-3}$ yields 14 Mpc, the mass density estimates taken again from [41]. The assumption of uniform mass density along the light of sight makes these estimates very conservative. We conclude that we should not expect the formation of such points anywhere around galactic distances. Moreover, since ${W_{XX}}^{\mathbf{A}}{}_{\mathbf{B}}$ usually changes sign of the determinant at the focal point, we may expect the parallax matrix ${\Pi^{\mathbf{A}}}_{\mathbf{B}}$ past the focal point to deviate far from proportionality to the unit matrix. This in turn may lead to unusual dependence of the two-dimensional parallax angles on the Earth's position, an effect that is in principle observable. Therefore sources past focal point could in principle be detected and removed from the data.

Third, as we have seen in Sec. III B, if the parallax is measured only along one baseline then the test is prone to errors in the presence of strong shear induced by the Weyl tensor. Therefore, if shear is not negligible then the baseline averaging step is crucial: we must be able to measure the parallax in two orthogonal directions for the measurement to yield a reliable NEC test. In case of the annual parallax due to Earth's motion this requirement excludes sources too close to the ecliptic. The problem is obviously even more serious for the cosmic parallax, in which only one baseline is available for all sources. Tests on cosmological distances require therefore prior estimation of the Weyl tensor contribution to the parallax matrix over large distances. Note, however, that even in the presence of moderate shear this problem might in principle be overcome if the single-baseline parallax and angular diameter distance have been measured for a sufficiently large sample of sources at different positions on the sky. Since the Weyl tensor along the line of sight depends on the position of the source, it varies relatively quickly across the sky and it is uncorrelated with the fixed baseline direction. We may expect that its impact will average to 0, at least in the linear order given a large sample of sources. What is left from the averaging is thus the bare, baseline-independent effect of the Ricci tensor. Quantification of the impact of shear on the NEC test, its error budget and the question of feasibility are beyond the scope of this paper. We stress here that unlike the full baseline-averaged measurement, this type of single-baseline measurement relies on additional assumptions about the metric tensor, i.e., either the vanishing of the Weyl contribution to the parallax matrix or the random, uncorrelated nature of the Weyl tensor over the whole sky.

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Chapter 5

Conclusion

In this thesis I present the theory and applications of general relativistic transfer matrices for null geodesics known as the bilocal geodesic operators. The transfer matrix method is well known in studies of linear systems. In General Relativity the Jacobi propagators play the role of transfer matrices for deviations of null and timelike geodesics. However, the transfer matrix theory for the propagation of light as presented in the literature seems incomplete, as it cannot accommodate either redshift or position drift effects. The formalism presented here allows to consider all possible lowest-order nontrivial optical effects.

The bilocal geodesic operator formalism was originally presented as a resolvent operator for the geodesic deviation equation. In this thesis I present a different representation of the problem. Instead of working with the null geodesics on the base manifold, I consider their lifts to the tangent bundle. Then the dimensionality of the resolvent operator becomes twice as big, but the governing differential equations reduce from the 2nd order to the 1st order. Furthermore, the metric structure of the base manifold permits a covariant splitting of vector fields on the tangent bundle into two invariant parts – the horizontal and the vertical subspace. It can be shown that each subspace is isomorphic to a copy of the tangent space to the base manifold. This implies that the resolvent operator can be decomposed into four distinct bilocal operators on the base manifold, known as the BGOs.

In the first paper we presented expressions for an analytic form of the BGOs for 4D static spherically symmetric spacetimes and applied them to the Schwarzschild spacetime. The solution was obtained in two ways. The first method requires differentiation of the solution to the geodesic equation with respect to its initial conditions. The result then needs to be updated for the final result to be fully covariant. The fix is based on the standard decomposition into the horizontal and vertical subspace of the tangent bundle. This decomposition is coordinate-invariant by construction.

The second approach involves a direct integration of the GDE. Symmetries of the spacetime together with the symplecticity of the GDE give rise to the first integrals, or quadratures. The next step involves expressing the BGOs in a parallel-transported semi-null tetrad. It turns out that after the projections the expressions simplify dramatically, especially in the screen space. The results are then applied to the Schwarzschild spacetime to study the behaviour of the angular diameter and parallax distance measures as well as their associated distance slip. Depending on the impact parameter of the fiducial null geodesic we expect a formation of special points along the fiducial light ray, where distance measures either vanish or diverge.

In the second paper we prove an inequality between the parallax distance and angular diameter distance under rather general conditions. Namely, if we assume the validity of General Relativity, the null energy condition, and the propagation of light in vacuum, then it can be shown that for any given source of light the parallax distance defined by arithmetic or geometric mean cannot grow larger than the angular diameter distance. Averaging of the parallax effect over both linearly independent baselines is crucial in order to avoid contributions of the Weyl tensor at linear order, which break the inequality for certain directions of the baseline. The result is blind to the symmetries of spacetime, while the proof shares some of its methods with the well-known focusing theorems. However, experimental confirmation of this inequality seems to be too difficult. The measurement in the Milky Way requires precise measurements of distances to 10^5 sources on the opposite side of the disk together with good modeling of the matter distribution in the galaxy. It is also possible to consider the cosmological parallax due to the motion of the solar system with respect to the CMB. Again, the expected signal with current technologies is too low, but the secular growth of the effect makes it in principle measurable.

Future directions

The topic can be extended in several different ways. Firstly, in comparison with the number of exact solutions of Einstein equations, there is only a handful of metrics with general exact solutions to the geodesic deviation equation (see appendix B). Therefore, it would be interesting to find the BGOs for other spacetimes with symmetries and classify them by various optical effects.

Secondly, position [63, 65, 64, 66] and redshift [133, 134, 95, 125, 38] drift effects have been considered only for a few models of the Universe. At the moment there are several observatories under construction [11, 47, 102], which will enable the measurement of redshift drift effects for the first time. A better understanding of drift effects and conditions under which they vanish is welcome.

Thirdly, some of the BGOs so far have been considered only formally, i.e. as a part of the total resolvent operator. It is still not clear whether they or their combinations correspond to any observable effects. Moreover, it is known that the Jacobi map satisfies Etherington's duality relation which holds for all Lorentzian spacetimes with Levi-Civita connection [126]. One may wonder if there are more duality relations expressible in terms of all the BGOs.

In practice, the spacetime geometry of the Universe cannot be expressed as an exact solution to Einstein's equations or their modifications. Instead, we have to assume a parametrized model and find the best fit with the least possible errors. Any statement about a well-fitted model has to be corroborated with an appropriate statistical analysis of the observational data [89, 24, 116, 78, 127, 33]. Nonetheless, the lowest-order perturbation theory and its statistical analysis both in the context of cosmological perturbations and weak gravitational lensing provide an extremely good agreement between the theory and observations. An equivalent perturbative and statistical study of new optical effects, such as the drifts, and their connections to the standard observables could make it easier to bring the formalism to practice. The robustness of data interpretation can be further increased by reducing its dependence on the model assumption and enforcing covariant formulation of the formalism. Recent studies suggest that redshift drift cosmography admits a metric-independent cosmological data analysis [50, 103, 3, 2, 1]. Extension of these ideas to other observables, such as the position drift, could be interesting.

The applicability of the formalism relies heavily on the measurability of new optical observables. The spatial resolution of individual light sources is possible only in the immediate vicinity of the Solar system and during strong lensing scenarios. On the other hand, the time of arrival of emitted particles can be measured with much higher precision. This serves as the motivation for the local surface of communication introduced in [61]. In this article, the BGOs are replaced by a different set of operators mapping position perturbations at both endpoints to their vector perturbations, but both sets of operators are equivalent in the sense that they involve the same functionals and variables. Even so, not all properties of the BGOs have been translated to the properties of the local surface communication. Particularly intriguing is the connection between the tangent bundle structure in the BGO formalism and the product manifold structure in [61].

Finally, the BGO formalism only describes the linear geodesic deviation equation. The full geodesic deviation is much more involved due its nonlinearity in terms of curvature tensors [9, 10][1] and path deviations as well as its non-locality. Still, the geometrical picture presented in Ch.2 could work as a starting point to generalize the geodesic deviation beyond the normal convex neighbourhood in a covariant manner [128, 114, 49, 21, 20]

[1]Motivation for this can be given by taking a difference of integral curves of $\mathbf{G}_{(p,\mathbf{X}_p)}$ evaluated at positions differing by a vector $\mathbf{Y}_p$ and expanding the difference around the reference integral curve, which yields $\dot{Y}^i = \sum_{k=1}^{\infty} \frac{1}{k!} G^i{}_{,j_1 \cdots j_k} Y^{j_1} \dots Y^{j_k}$.

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Appendix A

Derivation of GDE from geodesic flow

In this section we give a more precise derivation of Eq. (2.36) from Eq. (2.35). Recall that in the basis $\{f_i\} = \{\mathbf{f}_{x,\alpha}, \mathbf{f}_{v,\beta}\}$ (Eq. (2.8)) the geodesic spray reads

$$\left(G^i\right) = \begin{pmatrix} v^\alpha \\ -\Gamma^\beta{}_{\mu\nu}\left(x^\rho\right) v^\mu v^\nu \end{pmatrix}. \tag{A.1}$$

On the tangent bundle we have the induced coordinate system $\left(\chi^i\right) = \left(x^\kappa, v^\delta\right)$. We choose the ordering of the partial derivatives to match the ordering of the coordinates. Then the partial derivatives of G^i can be listed as

$$\left(G^i_{,j}\right) = \begin{pmatrix} 0 & \delta^\alpha{}_\delta \\ -\Gamma^\beta{}_{\mu\nu,\kappa}\left(x^\rho\right) v^\mu v^\nu & -2\Gamma^\beta{}_{\mu\delta}\left(x^\rho\right) v^\mu \end{pmatrix}. \tag{A.2}$$

Now return to Eq.(2.35) and write out the ODEs for Y^i expressed in the associated basis $\{f_i\}$:

$$\begin{pmatrix} \dfrac{d}{d\lambda} Y^\alpha_x \\ \dfrac{d}{d\lambda} Y^\beta_v \end{pmatrix} = \begin{pmatrix} Y^\alpha_v \\ -\Gamma^\beta{}_{\mu\nu,\kappa} v^\mu v^\nu Y^\kappa_x - 2\Gamma^\beta{}_{\mu\delta} v^\mu Y^\delta_v \end{pmatrix}.$$

As it was explained in Sec. 2.4, the splitting can be made invariant with respect to coordinate transformations (2.13) by the change of variables (2.21). After a bit of calculation and term rearranging the equations can be rewritten as

$$\begin{pmatrix} \dfrac{d}{d\lambda} Y^\alpha_H + \Gamma^\alpha{}_{\mu\nu} Y^\mu_H v^\nu \\ \dfrac{d}{d\lambda} Y^\beta_V + \Gamma^\beta{}_{\mu\nu} Y^\mu_V v^\nu \end{pmatrix} = \begin{pmatrix} Y^\alpha_V \\ R^\beta{}_{\mu\nu\rho} v^\mu v^\nu Y^\rho_H \end{pmatrix},$$

where $R^\beta{}_{\mu\nu\rho}$ is a collection of Christoffel symbols and their derivatives constituting the components Riemann tensor on the base manifold. Note that the terms on the left-hand side cannot be yet reduced to the covariant derivatives, because these expressions are components of a vector in $T_{(p,\mathbf{x}_p)}TM$, and TM has not been supplied either with a metric or a connection.

The final step of reduction is the utilization of the isomorphisms iso^H and iso^V (2.24). This way we can make the following identifications:

$$\begin{aligned} T_{(p,\mathbf{x}_p)}TM \ni Y^\alpha_H\, \mathbf{e}_{H\alpha} &\longrightarrow \tilde{Y}^\alpha_H \frac{\partial}{\partial x^\alpha} \in T_pM \\ \frac{dY^\alpha_H}{d\lambda} + \Gamma^\alpha{}_{\mu\nu} v^\mu Y^\nu_H &\longrightarrow \frac{d\tilde{Y}^\alpha_H}{d\lambda} + \Gamma^\alpha{}_{\mu\nu} l^\mu \tilde{Y}^\nu_H \equiv \nabla_l \tilde{Y}^\alpha_H, \end{aligned} \tag{A.3}$$

where $\tilde{Y}_H$ is a vector field along the geodesic γ, whose components are equal to those of Y_H, i.e. $\tilde{Y}_H^\alpha = Y_H^\alpha$. A similar construction can be used for the vertical part. Therefore the isomorphisms allow us to regard the equations for horizontal and vertical parts as proper expressions on the spacetime M. M, on the other hand, is equipped with a metric tensor and a covariant derivative. As a result, we end up with a system of equations on the base manifold

$$\begin{cases} \nabla_l \tilde{Y}_H^\alpha & = \tilde{Y}_V^\alpha \\ \nabla_l \tilde{Y}_V^\beta & = R^\beta{}_{\mu\nu\rho} l^\mu l^\nu \tilde{Y}_H^\rho. \end{cases} \tag{A.4}$$

(recall that $v^\mu = \dot{\gamma}^\mu$, because we are dealing with a lift of a geodesic, and we have introduced the notation $l^\mu = \dot{\gamma}^\mu$). This system is equivalent to the GDE for $\tilde{Y}_H^\alpha$. We can check that by substituting $\tilde{Y}_V$ in the second equation with the expression from the first one, obtaining this way $\nabla_l \nabla_l \tilde{Y}_H^\alpha = R^\alpha{}_{\mu\nu\rho} l^\mu l^\nu \tilde{Y}_H^\rho$.

Appendix B

List of exact solutions to GDE

The geodesic deviation equation constitutes a set of coupled linear second order ordinary differential equations. Although the analysis of ODEs is simpler than that of partial differential equations, finding a compact solution to a system of ODEs is not an easy task [12, 23]. However, most contemporary articles contain a very limited list of examples of exact solutions. In this appendix we provide a list of spacetimes for which the corresponding GDE has been integrated analytically.

- Constant curvature spacetimes: [122, 123, 84]
- Spherically symmetric spacetimes (without deriving the full BGOs): [82]
- Static spherically symmetric spacetimes: [51, 26, 34, 35, 46, 117, 118, 119, 120]
- Schwarzschild spacetime: [83, 77, 25, 73, 86, 15, 101, 100, 76, 58, 92]
- Newtonian gravity with oblate Earth: [44]
- Reissner–Nordström spacetime: [4, 96, 119]
- Kerr spacetime: [87, 81, 12]
- FLRW spacetime: [83, 17, 74, 29, 75]
- conformally flat spacetime [85]
- plane symmetric cosmological models: [56, 57]
- Bianchi cosmological models: [104, 98, 106]
- pp-wave: [4]
- rotating disk spacetime: [67]
- miscellaneous: [80, 97, 36, 87, 105, 23, 124]

This compilation consists of solutions for null or timelike trajectories for either general or special trajectories. The solution methods vary greatly, ranging from direct integration to application of Newman-Penrose scalars. Nevertheless, derivation of BGOs for these spacetimes is still an open problem, which is alleviated by the existence of these solutions.