Generalized Calogero-Moser-Sutherland systems,
quantization, topological methods and relationships with quantum chaos

Katarzyna Kowalczyk-Murynka

Abstract
Calogero-Moser-Sutherland Hamiltonians describe one-dimensional many-body systems with inverse-square interactions. They can be generalized by substituting the coupling constants in the interaction terms with additional dynamical variables, often referred to as internal or spin degrees of freedom. Classical, completely integrable N-body systems of this type, with the internal degrees of freedom spanning the $\mathfrak{so}(N)$, $\mathfrak{su}(N)$ or $\mathfrak{sp}(N)$ Lie algebra, arise in the surprisingly opposite context, namely energy level repulsion in quantum chaotic systems. On the other hand the quantum Calogero-Moser-Sutherland models appear in the area of topological quantum matter, Quantum Hall Effect in particular, as their eigenfunctions are obtained as 1D representations of Laughlin states projected on a single Landau level.

The classical dynamics of the internal degrees of freedom were studied to some extent, but the details relevant to the problem of level repulsion, such as the existence of a finer classification of models within the orthogonal, unitary and symplectic classes, still need to be resolved. As for the quantum case, specific models with internal degrees of freedom have been investigated, but they typically loose this internal structure in the classical limit. Models whose spin degrees of freedom survive the quantum-classical transition, together with the potential presence of topological effects, are not fully understood.

The perhaps most remarkable characteristic of the classical Calogero-Moser-Sutherland systems, taking the interactions and internal degrees of freedom into account, is their complete integrability. This trait is a direct consequence of the fact that these systems can be obtained *via* Hamiltonian reduction of very simple (such as free or harmonic), integrable dynamics in spaces of matrices. The reduction procedure, though defined rigorously in the language of symplectic geometry, in this case simplifies to diagonalizing an $N \times N$ time-dependent matrix $X(t)$, assigning its eigenvalues to instantaneous positions of the N particles and eliminating the eigenvectors from the equations of motion. So obtained functions on a reduced phase space can be quantized canonically by the

Dirac's prescription. On the other hand, simple dynamics in a matrix phase space can be quantized and then reparametrized by eigenvalues and eigenvectors.

Within the framework of Hamiltonian mechanics I have shown that the Lie-algebraic, matrix generalization of the Calogero-Moser model is equivalent to an alternative, vectorial formulation in which the internal state of each particle in encoded in a complex vector. The dimension of the subspace spanned by these vectors (equivalent to the rank of the matrix from the corresponding Lie algebra) is a constant of motion which I have used to classify the models. I have proven that the models with purely imaginary matrices encoding the initial values of internal variables have special properties: they approximate the phase space trajectories of the non-generalized Calogero-Moser system with distinct coupling constants and have the smallest reachable sets. Additionally, by combining the matrix and vector degrees of freedom and reducing it as described above, I have obtained a new integrable model with $1/x$ interactions. I have obtained two different Hamiltonians from the procedures of canonical quantization of the matrix model and reducing a free quantum model. The reduction of the quantum model introduces an additional term $-\frac{\hbar^2}{4m}\sum_{i<j}\frac{\alpha(2-\alpha)}{(x_i-x_j)^2}$ where $\alpha = 1, 2$ for the $\mathfrak{so}(N)$ and $\mathfrak{su}(N)$ cases respectively, which is an attractive term in the orthogonal setting. I have identified some of the irreducible representations of the Lie algebra spanned by the internal observables, for which the internal degrees of freedom introduce diagonal matrices into the Hamiltonian, and I have found the spectrum and eigenfunctions for $N = 3$.

The classical results can be used in the further study of energy level repulsion. The expected level spacing distribution for the purely imaginary matrices is different from the one which is known for general $\mathfrak{su}(N)$ matrices. Similarily, one should expect differences in probability distributions depending on the rank of the matrices. The study of the quantum systems can be continued in two main directions: the quantum reduction can be done in the symplectic case, as well as the spectra and eigenfunctions can be found (if not exactly, then at least approximately) for higher numbers of particles and dimensions of the space of internal states. The finite-dimensional space of internal states, once well understood, can serve as an additional dimension, thus yield the system two-dimensional and prone to topological effects.

The original results presented in the thesis were published in the following papers:

- K. Kowalczyk-Murynka, M. Kuś. Matrix and vectorial generalized Calogero–Moser models. *Physica D: Nonlinear Phenomena*, 440: 133491, 2022
- K. Kowalczyk-Murynka, M. Kuś. Calogero-Moser models with internal degrees of freedom revisited, arXiv:2010.10215.