Ph.D. Thesis Evaluation

Thesis Title: **"Generalized Calogero-Moser-Sutherland systems, quantization, topological methods and relationships with quantum chaos"**

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The doctoral dissertation "Generalized Calogero-Moser-Sutherland systems, quantization, topological methods and relationships with quantum chaos" by Katarzyna Kowalczyk-Murynka deals with various classical and quantum aspects of the generalized Calogero-Moser systems.

The original Calogero-Moser model, formulated nearly half a century ago, describes one-dimensional particles with inverse-square pairwise interactions and provides the prime example of an integrable and solvable, both on the classical and quantum level, many-body system.

In the generalized Calogero-Moser systems (also known as Euler C-M systems) the common coupling constant for all interacting pairs is replaced by additional dynamical variables providing a sort of internal (spin) degrees of freedom. The internal degrees of freedom in the so-called matrix formulation of the generalized C-M models are described by two-particle elements L_{ij} (measuring the coupling strengths between particles) of an anti-Hermitian matrix L. In the vectorial formulation of the models, the additional variables are introduced by assigning a complex vector to each particle. The couplings arise as a scalar product of the relevant vectors.

The complete integrability and interesting features of the interaction potential of the (generalized) C-M model made that over the decades it has played a "Jack-in-the-box" role in mathematical and theoretical physics popping up as a recurrent and unifying motif in various apparently different areas as gauge and string theories, condensed matter physics (including Quantum Hall Effect and functional statistics), quantum chaotic systems. Therefore various aspects of the Calogero-Moser model (and its generalizations) have been exhaustively studied resulting in a huge number of published articles.

Nevertheless, there are still open issues awaiting investigations. In particular, the two main research problems investigated by Ms. Katarzyna Kowalczyk-Murynka in her thesis, that is the dynamics of the internal degrees of freedom and quantization of generalized Calogero-Moser models belong to these open questions.

The dissertation of Ms. Katarzyna Kowalczyk-Murynka begins with an abstract and introduction which are followed by six chapters and two appendices.

The introduction provides basic facts and references concerning the (generalized) C-M systems as well as examples of their applicability in various areas of mathematics and physics, in particular, in the field of quantum chaos. In fact, the application of the generalized C-M models to the description of the dynamics and statistical properties of quantum spectra of chaotic systems was one of the main motivations for the author to study these models.

Advanced, although standard, methods, definitions and notations of mathematical and theoretical physics used in the thesis are briefly presented in chapter two. It includes the elements of group theory and differential geometry, the Hamiltonian formalism within the framework of symplectic geometry, the method of Hamiltonian reduction, and finally the procedure of canonical quantization.

In chapter three the author collects the known key facts concerning the classical generalized C-M models, in particular, the unitary reduction of linear matrix model resulting in generalized C-M dynamics. In addition, the spectrum and wave functions of the ordinary quantum C-M model and its relationship with the Quantum Hall Effect as well as some properties of the system with spin state exchange are briefly outlined.

The next two main chapters of the thesis present the contribution of Ms. Katarzyna Kowalczyk-Murynka to the field. The results concerning the classical generalized C-M system contained in chapter four include:

a) The proof that the matrix C-M model given by the Hamiltonian (3.17) and derived as a unitary reduction of free motion in the Hermitian (or orthogonal) matrices configuration space is equivalent to the vectorial formulation defined by the set of N complex vectors of equal length. These vectors are obtained from suitably adjusted anti-Hermitian matrix L by a Cholesky decomposition. The equivalence of the two formulations allows considering the matrix two-body degrees of freedom as functions of one-particle observable with internal degrees of freedom.
b) The classification of the L matrix orbits of generalized C-M system by the rank $d-$ the dimension of space of internal vectorial degrees of freedom. (it is possible due to the proven equivalence of both formulations) and showing that the rank $d = 1$ corresponding to the ordinary C-M model is the only rank for which one deals with the stationary and coinciding coupling constant $|L_{ij}(t)|^2 = g_{ij} = g$.
c) The analysis of the interplay between the internal L_{ij} and the spatial x_i, p_j degrees of freedom. This involves:
   - comparison of the trajectories generated by the ordinary C-M Hamiltonian with non-dynamical coupling constants with the trajectories corresponding to the generalized C-M model with dynamical $|L_{ij}(t)|^2$ couplings and showing that matrix model with purely imaginary matrix $L_{ij}(0) = ig_{ij}$ can be used to approximate the dynamics of ordinary C-M model with high accuracy;
   - the examination of the influence of particle collisions on the L_{ij} variables;
   - the study of reachable sets of L matrices (i.e. sets of the matrices $L(t)$ that can be obtained from a chosen L assuming arbitrary initial positions and momenta) and the explicit construction of these sets for the three-body generalized C-M model.
d) A construction of a new integrable many-body system with internal degrees of freedom by combining the matrix and vectorial degrees of freedom in an extended phase space. The proposed $SU(N)$- symmetric Hamilton function being a kind of minimal coupling between vectorial and matrix degrees of freedom generates a spiral solutions in the matrix space. The unitary reduction of this model provides the reduced Hamiltonian which contains the standard inverse-square pairwise interaction as well as a long-distance Kepler pairwise one with the dynamical non-stationary points.

The analysis of the two quantization schemes of classical C-M models with internal degrees of freedom is presented in chapter five.

The first scheme consists in a direct canonical quantization of reduced matrix C-M model. The basic spatial phase space variables are replaced by the self-adjoint operators acting in the position representation in the Hilbert space of square integrable functions $\Psi(\boldsymbol{R}^n)$. The operators $\hat{L}$ corresponding to the internal degrees of freedom span the $su(N)$ (or $so(N)$) algebra. Its finite-dimensional irreducible representations act on $\boldsymbol{C^n}$. This approach leads to the non-trivial many-body quantum Hamiltonian with internal degrees of freedom given by equation (5.4) and acting on a product $\mathbf{L^2(R^n)} \otimes \mathbf{C^n}$. Ms. Katarzyna Kowalczyk-Murynka has analyzed the eigenproblem given by this three-particle Hamiltonian with spin operators playing the role of the quantum $\hat{L}$ operators. She solved it exactly in the case of spin one. She has partially generalized the result to N particle case and the operators of internal degrees of freedom given by defining representation of $so(N)$ algebra.

Referring to the classical equivalence of the matrix and vectorial formulations of C-M model Ms. Kowalczyk-Murynka has studied the quantum two-particle operators $\hat{L}$ in a form of tensor product of single-particle operators acting on subspace of internal states. Assuming the bosonic creation and annihilation rules for the single-particle operators she constructed the Hamiltonian with the interactions depending on the excitation number. She demonstrated that in classical limit one gets the C-M model with non-dynamical coupling constants.

The second considered quantization scheme starts with the canonical quantization of a free (or harmonic) system in the phase space of matrices that is followed by the quantum reduction yielding a quantum C-M Hamiltonian. It contains an additional attractive term in the real symmetric case. This suggests that reduction of a canonically quantized matrix model may provide a richer set of solutions than the canonical quantization of reduced system.

In my opinion all research results obtained by Ms. Kowalczyk-Murynka are interesting and make a substantial and important contribution to the knowledge and understanding of the C-M models with internal degrees of freedom (this, in turn, may be useful in the field of quantum chaotic systems), even without completely exploring and solving all addressed questions. I particularly appreciate the construction of new many-body integrable systems with the internal degrees of freedom and the analysis of the relationship between the matrix and vectorial formulation of generalized C -M model. These results have been published in Physica D. In fact, the other results concerning the reachable sets of the internal dynamics, quantization of the considered system are likely to be published in the upper range journals.

The dissertation shows clearly and critically what Ms. Katarzyna Kowalczyk-Murynka has done and why. All weaknesses in the research are indicated and explained how they affect the main conclusions. The thesis is very well structured with each chapter and section having a clear function and presented in a logical order. Writing is clear and concise, with attractive functional figures and layout.

In summary my conclusion is that PhD thesis of Ms. Katarzyna Kowalczyk-Murynka presents original research results of great importance and high scientific quality.

The thesis certainly meets all requirements laid down for the degree of Ph.D. in physics by the statutes in the Journal of Laws of the Republic of Poland (Dz.U. 2020, poz.85) and is ready to be defended.

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