Centrum Fizyki Teoretycznej

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Nowe zasady wariacyjne w Ogólnej Teorii Względności i ich związki z zasadą Hilberta

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Warszawa, 2022

Spis treści

Rozdział 1

Wprowadzenie

W celu znalezienia poprawnego opisu tzw. ciemnej materii i/lub ciemnej energii wielu autorów rozważa ostatnio uogólnienia Ogólnej Teorii Względności bazujące na nieliniowych lagranżjanach, tj. funkcjach Lagrange'a w nieliniowy sposób zależnych od tensora krzywizny. Szczególny przypadek lagranżjanu L, zależącego nieliniowo od skalara krzywizny R lub nawet od całego tensora Ricciego $R_{\mu\nu}$, ale nie zależącego od tensora Weyla, był rozważany przez wielu (przykładowo [1] i [2]). Prawdopodobnie pierwszą, dobrze umotywowaną fizycznie, propozycją takiej teorii, był nieliniowy "R^2"-lagranżjan A. Sacharova [3]. Należy on do rodziny teorii "$f(R)$", gdzie $L = \sqrt{|g|}f(R)$ (patrz również [4]).

W matematycznym ujęciu te teorie są równoważne standardowej wersji Ogólnej Teorii Względności z oddziaływaniem z dodatkowymi polami materii. Dla teorii typu $f(R)$ ta równoważność została udowodniona już w roku 1987 (patrz [5] i [6]). Później ten rezultat został rozszerzony na lagranżjany zależące od tensora Ricciego ([6] i [7]) oraz na takie, które zależą od całego tensora Riemanna $R^{\lambda}_{\kappa\sigma\mu}$ [8].

Okazuje się, że ta spektakularna "uniwersalność" teorii Einsteina rozciąga się również na teorie wyższego rzędu, gdzie lagranżjan zależy od wyższych pochodnych kowariantnych $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\mu}$ krzywizny. Ten rezultat został opublikowany w pracy [9]. Natomiast w pracy [10] została przedstawiona struktura geometryczna leżąca u podstawy tego wyniku.

Rozważania zawarte w pracy [10] wychodzą od pytania: ile *niezależnych* stopni swobody jest zawartych w pochodnych kowariantnych rzędu k krzywizny, $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\mu}$, jeśli weźmiemy pod uwagę wszystkie tożsamości Bianchi'ego I i II rodzaju razem z ich pochodnymi? W odpowiedzi na to pytanie w naturalny sposób pojawia się pojęcie tensora krzywizny rzędu k. Zostało pokazane, że wszystkie niezależne tożsamości, spełniane przez te pochodne kowariantne, mogą być przedstawione jako tożsamości Bianchi'ego kolejnych rzędów k, dla $k = 1, 2, \ldots$ (patrz równania (4.2) oraz (4.3)). Do analizy tensorów wyższych rzędów zostało wykorzystane alternatywne ujęcie teorii koneksji na rozmaitości. W pełni równoważne standardowej podręcznikowej wersji, znacząco upraszcza zarówno wariacyjne, jak i kanoniczne sformułowanie Ogólnej Teorii Względności (patrz [8]). Tutaj tworzy naturalne ramy dla teorii krzywizny wyższych rzędów.

Główny wynik tej pracy jest następujący: tensory krzywizny do rzędu k zawierają tę samą informację, co tensor Riemanna oraz jego pochodne kowariantne do rzędu k-1. Ta sama informacja jest opisana jako zbiór *niezależnych* geometrycznych obiektów, co jest kluczowe w poprawnym sformułowaniu problemów wariacyjnych wyższych rzędów ([14] i [15]) oraz w dowodzie ich równoważności ze standardową Ogólną Teorią Względności.

W pracy [9] została pokazana uniwersalność standardowej Ogólnej Teorii Względności w szerszym kontekście, mianowicie dla langranżjanu zależącego również od wyższych pochodnych kowariantnych krzywizny $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}$:

$$L = L\left(R^{\lambda}_{\kappa\sigma\nu}, \nabla_{\mu} R^{\lambda}_{\kappa\sigma\nu}, \nabla_{\mu_2}\nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}, \cdots, \nabla_{\mu_n} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}\right) . \tag{1.1}$$

Idea dowodu opiera się na wariacji metodą Palatiniego, gdzie metryka g oraz koneksja Γ są traktowane jako *a priori* niezależne geometryczne obiekty. Jeśli nie mamy do czynienia z klasycznym lagranżjanem Hilberta, gdzie wariacja w odniesieniu do Γ daje równanie metryczne

$$\nabla_{\lambda} g_{\mu\nu} = 0 , \tag{1.2}$$

jako jedno z równań Eulera-Lagrange'a oraz w konsekwencji implikuje prostą zależność pomiędzy metryką a koneksją:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_{\nu} g_{\kappa\mu} + \partial_{\mu} g_{\kappa\nu} - \partial_{\kappa} g_{\mu\nu}\right) , \tag{1.3}$$

to w przypadku dowolnego lagranżajnu L, będącego funkcją metryki, tensora krzywizny oraz (być może) jego pochodnych kowariantnych, zero po prawej stronie równania (1.2) zostaje zastąpione skomplikowaną kombinacją tych wszystkich wielkości, jak również koneksja Γ przestaje być metryczną koneksją Levi-Civity (1.3). Naiwnie można by sądzić, że wynikająca z tego teoria nie jest równoważna oryginalnej, gdzie metryczność koneksji jest założona *a priori*, tj. kiedy Γ jest zaledwie skrótowym zapisem kombinacji (1.3) metryki g i jej pochodnych. Taki wniosek jest jednak błędny. W rzeczy samej, rozkładając nie-metryczną koneksję Γ na metryczną część $\overset{\circ}{\Gamma}$ oraz pozostałe pole tensorowe:

$$\Gamma^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + N^{\lambda}_{\mu\nu} ,$$

możemy przepisać całą teorię w kategoriach tensora metrycznego oddziałującego z nowymi "polami materii", opisanymi za pomocą nie-metrycznego pola tensorowego N. Co więcej, zostało pokazane, że oddziaływanie pomiędzy metryką a ową materią, wynikające z tej nowej teorii, jest tego rodzaju, który występuje w konwencjonalnej teorii Einsteina. Okazuje się, iż w przypadku lagranżjanu Sacharova (tj. zawierającego wyraz $c \cdot R^2$), jak również w przypadku jakiejkolwiek innej teorii typu "$f(R)$", całość informacji o N jest zawarta w pojedynczym polu skalarnym (patrz [5] i [6]) i stąd taka teoria jest równoważna konwencjonalnej teorii Einsteina, gdzie metryka oddziałuje ze skalarnym polem materii. Jedynym unikatowym aspektem takiej teorii jest konkretna postać lagranżjanu materii nowego pola skalarnego, jednoznacznie zdefiniowana poprzez oryginalny lagranżjan $f(R)$.

Powyższa konstrukcja dla dowolnego lagranżjanu zależnego od metryki, całego tensora krzywizny $R^{\lambda}_{\kappa\sigma\mu}$ oraz pewnych pól materii, została dobrze opisana w pracy [8]. To, co pozostaje, to udowodnić, że teoria wyższego rzędu (czyli taka, w której lagranżjan zależy od pochodnych krzywizny) może być w równoważny sposób zapisana jako teoria pierwszego rzędu (tj. taka, w której lagranżjan zależy od krzywizny oraz od pól materii, ale już nie od pochodnych krzywizny). Procedura "obniżania rzędu różniczkowgo teorii", gdzie wyższe pochodne są traktowane jako nowe zmienne, jest standardową procedurą w analizie równań różniczkowych cząstkowych. Jej nowym, opisanym tutaj aspektem jest to, że taka procedura "obniżania rzędu" może być przeprowadzona w sposób, który zachowuje: 1) wariacyjny charakter oraz: 2) dyfeomorficzną niezmienniczość teorii.

Rozdział 2

Zasady wariacyjne wyższych rzędów w teorii grawitacji

2.1 Równania Eulera-Lagrange'a

Rozważmy gęstość lagranżjanu wyższego rzędu

$$L = L(\varphi^K, \varphi^K_\mu, \varphi^K_{\mu_1\mu_2} \cdots, \varphi^K_{\mu_1\cdots\mu_n})\,, \tag{2.1}$$

gdzie

$$\varphi^K_{\mu_1\cdots\mu_k} := \partial_{\mu_1}\cdots\partial_{\mu_k}\varphi^K\,, \tag{2.2}$$

dla $k = 0, 1, 2, \ldots, n$. Może być ona źródłem zasady wariacyjnej n-tego rzędu, która, naiwnie, mogłaby zostać sformułowana jako zasada wariacyjna pierwszego rzędu z $(n-1)$-szym dżetem pola φ jako zmienną konfiguracyjną. Stąd niezależne zmienne opisujące pola to $\varphi^K_{\mu_1\cdots\mu_k}$ dla $k = 0, 1, 2, \ldots, n-1$. Równania pola można otrzymać z następującej relacji symplektycznej:

$$\begin{aligned}
\delta L\left(\varphi^K, \cdots, \varphi^K_{\mu_1\cdots\mu_n}\right) &= \partial_\lambda\left(p^\lambda_K\delta\varphi^K + p^{\mu\lambda}_K\delta\varphi^K_\mu + \cdots + p^{\mu_1\cdots\mu_{n-1}\lambda}_K\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\right)\\
&= (\partial_\lambda p^\lambda_K)\delta\varphi^K + \left(p^\mu_K + \partial_\lambda p^{\mu\lambda}_K\right)\delta\varphi^K_\mu + \cdots\\
&\quad + \left(p^{\mu_1\cdots\mu_{n-1}}_K + \partial_\lambda p^{\mu_1\cdots\mu_{n-1}\lambda}_K\right)\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\\
&\quad + p^{\mu_1\cdots\mu_n}_K\delta\varphi^K_{\mu_1\cdots\mu_n}\,,
\end{aligned} \tag{2.3}$$

lub równoważnie

$$p^{\mu_1\cdots\mu_n}_K = \frac{\partial L}{\partial\varphi^K_{\mu_1\cdots\mu_n}} \tag{2.4}$$

$$p^{\mu_1\cdots\mu_{n-1}}_K + \partial_\lambda p^{\mu_1\cdots\mu_{n-1}\lambda}_K = \frac{\partial L}{\partial\varphi^K_{\mu_1\cdots\mu_{n-1}}} \tag{2.5}$$

$$\cdots = \cdots \tag{2.6}$$

$$p^\mu_K + \partial_\lambda p^{\mu\lambda}_K = \frac{\partial L}{\partial\varphi^K_\mu} \tag{2.7}$$

$$\partial_\lambda p^\lambda_K = \frac{\partial L}{\partial\varphi^K} \tag{2.8}$$

Podstawiając po kolei pierwsze równanie do drugiego, drugie do trzeciego, itd. otrzymujemy równanie Eulera-Lagrange'a:

$$0 = \frac{\partial L}{\partial \varphi^K} - \partial_\mu \frac{\partial L}{\partial \varphi^K_\mu} + \partial_{\mu_1}\partial_{\mu_2}\frac{\partial L}{\partial \varphi^K_{\mu_1\mu_2}} + \cdots + (-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\frac{\partial L}{\partial \varphi^K_{\mu_1\cdots\mu_n}}, \tag{2.9}$$

razem z definicjami wszystkich pędów kanonicznie sprzężonych $\left(p^\mu_K, p^{\mu_1\mu_2}_K, \cdots, p^{\mu_1\cdots\mu_n}_K\right)$.

Jest jednakże pewnien problem z takim sformułowaniem: odpowiednia "struktura symplektyczna" nie jest taką w rzeczywistości, ponieważ jest zdegenerowana. Ta degeneracja może być jednak dość łatwo usunięta poprzez odpowiednią redukcję symplektyczną. Ta procedura została matematycznie ściśle przeanalizowana w pracy [15]. Zostało tam pokazane, iż najprostszym na to sposobem jest zażądanie całkowitej symetrii dla wszystkich składowych pędu:

$$p^{\mu_1\cdots\mu_k}_K = p^{(\mu_1\cdots\mu_k)}_K. \tag{2.10}$$

A priori tylko pęd najwyższego rzędu (czyli dla $k = n$) spełnia warunek całkowitej symetryczności, wynikający z równania (2.4), podczas gdy pędy niższych rzędów mogą również zawierać niesymteryczną część, która nie wpływa na ewolucję pola (równanie Eulera-Lagrange'a (2.9) zostało wyprowadzone bez narzuconych warunków na jakiekolwiek symetrie). Stąd warunek symetrii (2.10) odgrywa rolę warunku cechowania, co pozwala przedstawiać *unikalny* fizyczny stan pola za pomocą pojęć położeń i pędów. Funkcja tworząca (2.3) razem z warunkiem cechowania (2.10) określają *w sposób jednoznaczny* kanoniczne (Hamiltonowskie) sformułowanie teorii pola, zdefiniowanej przez zasadę wariacyjną wyższego rzędu (cf. [15] for details).

W teorii grawitacji rolę zmiennej konfiguracyjnej pełni koneksja: $\varphi^K = \Gamma^\lambda_{\kappa\sigma}$. Odpowiadające temu pędy $P^{\kappa\sigma\mu_1\cdots\mu_k}_\lambda$ będą symetryczne w indeksach $(\mu_1\cdots\mu_k)$.

2.2 Redukcja rzędu teorii

Każdy układ równań różniczkowych cząstkowych można traktować jako układ równań pierwszego rzędu (różniczkowego), jeśli wyższe pochodne potraktuje się jak nowe zmiennne pola. Ta prosta obserwacja znajduje zastosowanie również do zasad wariacyjnych – wystarczy dokonać transformacji Legendre'a pomiędzy pochodną najwyższego rzędu oraz pędem najwyższego rzędu. W tym celu funkcję tworzącą (2.3) można przepisać następująco:

$$\begin{aligned}\delta L &= (\partial_\lambda p^\lambda_K)\delta\varphi^K + \left(p^\mu_K + \partial_\lambda p^{\mu\lambda}_K\right)\delta\varphi^K_\mu + \cdots + p^{\mu_1\cdots\mu_{n-1}}_K \delta\varphi^K_{\mu_1\cdots\mu_{n-1}} \\ &\quad + \partial_\lambda\left(p^{\mu_1\cdots\mu_{n-1}\lambda}_K \delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\right),\end{aligned} \tag{2.11}$$

a ostatni wyraz przekształcić jako:

$$\partial_\lambda\left(p^{\mu_1\cdots\mu_{n-1}\lambda}_K \delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\right) = \delta\left[\partial_\lambda\left(p^{\mu_1\cdots\mu_{n-1}\lambda}_K \varphi^K_{\mu_1\cdots\mu_{n-1}}\right)\right] - \partial_\lambda\left(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p^{\mu_1\cdots\mu_{n-1}\lambda}_K\right).$$

Różniczka zupełna:

$$\delta\left[\partial_\lambda\left(p^{\mu_1\cdots\mu_{n-1}\lambda}_K \varphi^K_{\mu_1\cdots\mu_{n-1}}\right)\right] = \delta\left(p^{\mu_1\cdots\mu_n}_K \varphi^K_{\mu_1\cdots\mu_n}\right) + \delta\left[\left(\partial_\lambda p^{\mu_1\cdots\mu_{n-1}\lambda}_K\right)\varphi^K_{\mu_1\cdots\mu_{n-1}}\right] \tag{2.12}$$

może być przeniesiona na lewą stronę równania i w ten sposób (po kilku krokach, szczegółowo opisanych w pracy [9]) otrzymujemy nową funkcję tworzącą:

$$\begin{aligned}\delta\mathcal{L} &= (\partial_\lambda p^\lambda_K)\delta\varphi^K + \left(p^\mu_K + \partial_\lambda p^{\mu\lambda}_K\right)\delta\varphi^K_\mu + \cdots + p^{\mu_1\cdots\mu_{n-1}}_K \delta\varphi^K_{\mu_1\cdots\mu_{n-1}} \\ &\quad - \partial_\lambda\left(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p^{\mu_1\cdots\mu_{n-1}\lambda}_K\right).\end{aligned} \tag{2.13}$$

Ostatni wyraz opisuje zasadę wariacyją dla zmiennej $p_K^{\mu_1\cdots\mu_n}$. Ta wariacja jest (różniczkowo) pierwszego rzędu, a rolę pędu kanonicznie sprzężonego przyjmuje następująca wielkość:

$$\Pi^{K\lambda}_{\mu_1\cdots\mu_n} := -\varphi^K_{(\mu_1\cdots\mu_{n-1}}\delta^\lambda_{\mu_n)} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda p_K^{\mu_1\cdots\mu_n}\right)}\,, \tag{2.14}$$

zgodnie z tożsamością:

$$-\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda} = \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta p_K^{\mu_1\cdots\mu_n}\,. \tag{2.15}$$

I w konsekwencji:

$$-\partial_\lambda\left(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda}\right) = \left(\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n}\right)\delta p_K^{\mu_1\cdots\mu_n} + \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta\left(\partial_\lambda p_K^{\mu_1\cdots\mu_n}\right)\,. \tag{2.16}$$

Równania pola otrzymane z nowej funkcji tworzącej wyglądają następująco:

$$\begin{aligned}\delta\mathcal{L} &= (\partial_\lambda p_K^\lambda)\delta\varphi^K + \left(p_K^\mu + \partial_\lambda p_K^{\mu\lambda}\right)\delta\varphi^K_\mu + \cdots + p_K^{\mu_1\cdots\mu_{n-1}}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\\ &+ \left(\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n}\right)\delta p_K^{\mu_1\cdots\mu_n} + \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta\left(\partial_\lambda p_K^{\mu_1\cdots\mu_n}\right)\,,\end{aligned} \tag{2.17}$$

i razem z definicją (2.14) pędu Π są równoważne równaniom (2.4) – (2.8). Przyjmują one następująca postać:

$$\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n} = -\varphi^K_{\mu_1\cdots\mu_n} = \frac{\partial\mathcal{L}}{\partial p_K^{\mu_1\cdots\mu_n}} \tag{2.18}$$

$$p_K^{\mu_1\cdots\mu_{n-1}} = \frac{\partial\mathcal{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-1}}} \tag{2.19}$$

$$p_K^{\mu_1\cdots\mu_{n-2}} + \partial_\lambda p_K^{\mu_1\cdots\mu_{n-2}\lambda} = \frac{\partial\mathcal{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-2}}} \tag{2.20}$$

$$\cdots = \cdots \tag{2.21}$$

$$p_K^\mu + \partial_\lambda p_K^{\mu\lambda} = \frac{\partial\mathcal{L}}{\partial\varphi^K_\mu} \tag{2.22}$$

$$\partial_\lambda p_K^\lambda = \frac{\partial\mathcal{L}}{\partial\varphi^K} \tag{2.23}$$

Równanie (2.18) (razem z definicją (2.14) pędu Π) jest równaniem Eulera-Lagrange'a zasady wariacyjnej pierwszego rzędu, zdefiniowanej przez lagranżjan:

$$\mathcal{L} = \mathcal{L}\left(j^{n-1}(\varphi), p_K^{\mu_1\cdots\mu_n}, \partial_\lambda p_K^{\mu_1\cdots\mu_n}\right)\,,$$

podczas gdy równania (2.19) – (2.23) opisują zasadę wariacyjną rzędu $(n-1)$, określoną przez zależność $\mathcal{L}$ od $(n-1)$-go dżetu zmiennej konfiguracyjnej $x \to \varphi^K(x)$.

Zatem zasada wariacyjna rzędu n dla pola φ^K może być równoważnie zastąpiona taką, która jest rzędu $(n-1)$ dla pola φ^K oraz rzędu 1 dla pomocniczego pola (pola "materii") $p_K^{\mu_1\cdots\mu_n}$. Konsekwentne stosowanie tego twierdzenia umożliwia zredukowanie rzędu dowolnej zasady wariacyjnej do 1. Koszt takiej operacji to wprowadzenie dodatkowych "pól materii".

Ponieważ tensor krzywizny $R^\lambda_{\kappa\sigma\nu}$ zawiera pierwsze pochodne współczynników koneksji $\Gamma^\lambda_{\kappa\sigma}$, lagranżjan (1.1) jest (różniczkowego) rzędu $(n+1)$ w odniesieniu do zmiennych konfiguracyjnych $\Gamma^\lambda_{\kappa\sigma}$ (odgrywających rolę pól φ^K). Zatem aby obniżyć rząd (1.1), można wykorzystać metodę opisaną wyżej. Jednakże ani same współczynniki koneksji, ani ich pochodne nie są obiektami tensorowymi i w konsekwencji takie bezpośrednie zastosowanie wspomnianej metody nie byłoby niezmiennicze względem transformacji współrzędnych. W celu zachowania tej niezmienniczości teorii pochodne Γ oraz same Γ należy zapisywać w postaci tylko takich kombinacji, które odpowiadają tensorowi krzywizny oraz jego pochodnym. Obiekty te spełniają jednak wiele tożsamości (np. tożsamości Bianchi'ego I i II rodzaju oraz ich poc000hodne kowariantne), co sprawia, że *a priori* trudno byłoby wybrać spośród nich wielkości prawdziwie niezależne, które pozwoliłyby wyrazić gęstość funkcji Lagrange'a (1.1) za pomocą niezależnych obiektów.

Powyższy problem był inspiracją do rozwinięcia przeze mnie koncepcji *tensorów krzywizny wyższych rzędów*, oryginalnie zaproponowanej przez prof. Kijowskiego. W uzyskaniu opisanych dalej wynków istotną rolę odegrało zastosowanie odmiennego niż standardowe ujęcia teorii koneksji.

Rozdział 3

Teoria krzywizny

3.1 Koneksja jako pole układów inercjalnych

Pod pojęciem koneksji Γ na rozmaitości M, $\dim M = n$, zazwyczaj rozumie się koneksję w $\mathrm{GL}(n, \mathbb{R})$ wiązce głównej układów odniesienia nad M. Taka wiązka układów nie jest dowolną wiązką główną, lecz jest wyposażona w dodatkową strukturę: *formę klejącą (?)*. Za sprawą tej struktury tak zdefiniowana koneksja nie jest obiektem nieredukowalnym, lecz rozpada się kanonicznie na dwie nieredukowalne części: *koneksję symetryczną* oraz tensorowy obiekt zwany *torsją*. W pracy [10] została zaproponowana inna definicja koneksji symetrycznej - jako fundamentalnego (nieredukowalnego) obiektu geometrycznego. Weźmy kanoniczną *wiązkę układów odniesienia* $\mathcal{R}(M)$ nad M. Koneksja jest wówczas zdefiniowana jako cięcie tej wiązki:

$$\Gamma : M \to \mathcal{R}(M) . \tag{3.1}$$

Innymi słowy: koneksja jest polem układów odniesienia. Raz wybrana na M, definiuje w każdym punkcie $\mathbf{m} \in M$ "uprzywilejowany" układ odniesienia $\Gamma(\mathbf{m}) \in \mathcal{R}(M)$, który będzie nazywany *inercjalnym układem odniesienia* w $\mathbf{m} \in M$.

Pojęcie inercjalnego układu odniesienia zostało wprowadzone przez Newtona w jego pierwszym prawie dynamiki: "ruch ciała swobodnie spadającego jest jednostajny i prostoliniowy", tj. spełnia równanie

$$\ddot{x}^k := \frac{\mathrm{d}^2}{\mathrm{d}t^2} \, x^k(t) = 0 , \tag{3.2}$$

w uprzywilejowanym układzie współrzędnych (x^k), który Newton nazwał "inercjalnym".

Istnienie inercjalnych układów odniesienia było oryginalnie uważane za *globalną* strukturę Wszechświata. W świetle Szczególnej Teorii Względności unowocześniona wersja pierwszego prawa dynamiki jest czysto *lokalna*. Stad powyższe Newtonowskie sformułowanie musi zostać zastąpione sformułowaniem Einsteinowskim: dla każdego punktu $\mathbf{m} \in M$ istnieje układ współrzędnych (y^α) w otoczeniu $\mathbf{m}$ taki, że równanie różniczkowe opisujące trajektorie ciała spadającego swobodnie redukuje się *w tym konkretnym punkcie* do:

$$\ddot{y}^\alpha = 0 . \tag{3.3}$$

Tu pochodna oznacza pochodną po czasie własnym, podczas gdy Newtonowska wersja odnosiła się do (nieistniejącego) "absolutnego czasu".

Równanie (3.3) można łatwo przepisać w innym układzie współrzędnych (x^λ) na M. W ten sposób uzyskuje się następujące *równanie ruchu swobodnie spadających ciał*, spełnione w dowolnym układzie współrzędnych (zarówno tu, jak i w dalszej części pracy używana jest konwencja sumacyjna Einsteina):

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0\,, \tag{3.4}$$

gdzie $\Gamma^\lambda_{\mu\nu}$ oznacza następującą kombinację współczynników transformacji pomiędzy oboma układami współrzędnych

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial y^\alpha} \cdot \frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}\,. \tag{3.5}$$

Widać, że nowy układ współrzędnych jest również inercjalny w $\mathbf{m}$ wtedy i tylko wtedy, gdy powyższe drugie pochodne znikają w $\mathbf{m}$. Na podstawie tego można sformułować następującą definicję:

Układ odniesienia w punkcie $\mathbf{m} \in M$ jest klasą równoważności lokalnych układów współrzędnych względem relacji "$\sim_\mathbf{m}$", gdzie dwa układy współrzędnych w otoczeniu punktu $\mathbf{m}$ określa się jako równoważne wtedy i tylko wtedy, gdy drugie pochodne jednych współrzędnych po drugich znikają w $\mathbf{m}$:

$$((x^\mu) \sim_\mathbf{m} (y^\alpha)) \Longleftrightarrow \left(\frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(\mathbf{m}) = 0\right)\,. \tag{3.6}$$

Łatwo można sprawdzić, że powyższe jest w rzeczy samej relacją równoważności. Klasę równoważności układu (y^α) oznacza się jako $[(y^\alpha)]_\mathbf{m}$ i nazywa "układem odniesienia w $\mathbf{m} \in M$".

Mając układ współrzędnych (x^λ) w otoczeniu punktu $\mathbf{m}$, można sparametryzować dowolny układ odniesienia $[(y^\alpha)]_\mathbf{m}$ w $\mathbf{m}$ poprzez następującą, symetryczną z definicji, tablicę wielkości:

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial y^\alpha} \cdot \frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(\mathbf{m})\,, \tag{3.7}$$

gdzie (y^α) jest reprezentantem klasy $[(y^\alpha)]_\mathbf{m}$. Łatwo można pokazać, że wielkości $\Gamma^\lambda_{\mu\nu}$ nie zależą od wyboru reprezentanta. W szczególności układ odniesienia $\left[(x^\lambda)\right]_\mathbf{m}$ odpowiada trywialnej tablicy $\Gamma^\lambda_{\mu\nu} = 0$. Co więcej, ta parametryzacja jest surjekcją, tzn. dla danej tablicy $\Gamma^\lambda_{\mu\nu}$ istnieje unikalny układ współrzędnych w $\mathbf{m}$, mianowicie: $[(y^\alpha)]_\mathbf{m}$ taki, że wielkości (3.7) zgadzają się z tą tablicą. Jej reprezentant może być zdefiniowany przykładowo jako:

$$y^\lambda := x^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu}x^\mu x^\nu\,. \tag{3.8}$$

Powyższy wzór obowiązuje we współrzędnych wyśrodkowanych w $\mathbf{m}$, tj. takich, że $\mathbf{m} = (0,\ldots,0)$. W innym przypadku x^μ musi zostać zastąpione przez $(x^\mu - m^\mu)$, gdzie $\mathbf{m} = (m^\mu)$. W ten sposób zbiór wszystkich układów odniesienia $\mathcal{R}(M)$ nabiera struktury afinicznej wiązki włóknistej nad M oraz $(x^\lambda, \Gamma^\lambda_{\mu\nu})$ są lokalnym współrzędnymi na $\mathcal{R}(M)$, zgodnymi z tą strukturą.

W danym układzie współrzędnych (x^λ) koneksja (3.1) (tj. cięcie $\mathcal{R}(M)$) może być w takim razie jednoznacznie zdefiniowana przez zbiór $\frac{n^2(n+1)}{2}$ funkcji: $M \ni \mathbf{m} \to \Gamma(\mathbf{m}) = \left(\Gamma^\lambda_{\mu\nu}(\mathbf{m})\right)$, zwanych *współczynnikami koneksji*. Współrzędne (y^α) nazywa się *inercjalnymi* w $\mathbf{m}$ dla koneksji Γ, jeśli $[(y^\alpha)]_\mathbf{m} = \Gamma(\mathbf{m})$. Współrzędne (x^λ) są inercjalne w $\mathbf{m}$ dla koneksji Γ, jeśli wszystkie funkcje $\Gamma^\lambda_{\mu\nu}$ znikają w $\mathbf{m}$.

Za sprawą (3.8) współczynniki koneksji $(\Gamma^\lambda_{\mu\nu})$ zyskują prostą interpretację geometryczną: opisują kwadratową poprawkę, która jest konieczna, aby przekształcić dowolny układ współrzędnych (x^λ) we współrzędne inercjalne.

3.2 Tensor krzywizny jako miara odstępstwa od afiniczności. Równoważność z tensorem Riemanna

Koneksja jest *płaska*, jeśli istnieje *globalny* inercjalny układ odniesienia, tj. układ współrzędnych, który jest inercjalny nie tylko w jednym punkcie (i jego otoczeniu), lecz wszędzie. Mając dany układ współrzędnych (x^μ) inercjalnych w $\mathbf{m}$, w którym znika $\Gamma^\lambda_{\mu\nu}$, możemy za pomocą szeregu transofrmacji współrzędnych stworzyć konstrukcję, która pozwoli sprawdzić, czy dana koneksja jest płaska. Okazuje się, że wielkość otrzymana na koniec tej operacji:

$$K^\lambda_{\mu\nu\kappa} := \Gamma^\lambda_{\mu\nu\kappa} - \Gamma^\lambda_{(\mu\nu\kappa)} \ , \tag{3.9}$$

gdzie $\Gamma^\lambda_{(\mu\nu\kappa)}$ oznacza część całkowicie symetryczną pochodnych Γ, zapisaywanych jako $\Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m}) := \partial_\kappa \Gamma^\lambda_{\mu\nu}(\mathbf{m})$ stanowi przeszkodę w wyzerowaniu pochodnych Γ, tj. przeciwko jej płaskości. Mierzy zatem, jak *nie-płaska*, jak *zakrzywiona*, jest ta koneksja. Nazywamy ją *tensorem krzywizny*.

Definicja: Tensorem krzywizny koneksji Γ nazywamy tensor, który redukuje się do postaci (3.9) w inercjalnym układzie odniesienia.

Powyższa definicja implikuje następujący wzór, prawdziwy w dowolnym układzie współrzędnych:

$$\begin{aligned} K^\lambda_{\mu\nu\kappa} &= \Gamma^\lambda_{\mu\nu\kappa} - \Gamma^\lambda_{(\mu\nu\kappa)} + \left(\Gamma^\lambda_{\sigma\kappa}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma(\kappa}\Gamma^\sigma_{\mu\nu)}\right) \\ &= \Gamma^\lambda_{\mu\nu\kappa} + \Gamma^\lambda_{\sigma\kappa}\Gamma^\sigma_{\mu\nu} - \left(\Gamma^\lambda_{(\mu\nu\kappa)} + \Gamma^\lambda_{\sigma(\kappa}\Gamma^\sigma_{\mu\nu)}\right) \ . \end{aligned} \tag{3.10}$$

Tensor krzywizny K jest symetryczny w pierwszej parze indeksów oraz jego część całkowicie symteryczna znika tożsamościowo:

$$K^\lambda_{\mu\nu\kappa} = K^\lambda_{\nu\mu\kappa} \quad ; \quad K^\lambda_{(\mu\nu\kappa)} = 0 \ . \tag{3.11}$$

Druga tożsamość jest odpowiednikiem tożsamości Bianchi'ego I-szego rodzaju (patrz (3.13)).

Tak zdefiniowany tensor krzywizny jest równoważny standardowemu tensorowi Riemanna $R^\lambda_{\mu\nu\kappa}$: antysymetryzacja w ostanich dwóch indeksach K daje R, a symetryzacja R w pierwszych dwóch indeksach daje K. Zachodzą między nimi następujące relacje:

$$R^\lambda_{\mu\nu\kappa} = -2K^\lambda_{\mu[\nu\kappa]} = K^\lambda_{\mu\kappa\nu} - K^\lambda_{\mu\nu\kappa} \quad ; \quad K^\lambda_{\mu\nu\kappa} = -\frac{2}{3}R^\lambda_{(\mu\nu)\kappa} = -\frac{1}{3}\left(R^\lambda_{\mu\nu\kappa} + R^\lambda_{\nu\mu\kappa}\right) \ , \tag{3.12}$$

a tożsamości (3.11) dla K są równoważne analogicznym tożsamościom dla R:

$$R^\lambda_{\mu\nu\kappa} = -R^\lambda_{\mu\kappa\nu} \quad ; \quad R^\lambda_{[\mu\nu\kappa]} = 0 \ . \tag{3.13}$$

na podstawie powyższego można wnioskować, iż tensor Riemanna $R^\lambda_{\mu\nu\kappa}$ i tensor krzywizny $K^\lambda_{\mu\nu\kappa}$ to dwie różne, ale całkowicie równoważne reprezentacje jednego obiektu geometrycznego. Jak się okazuje, ta druga jest technicznie o wiele wygodniejsza dla zastosowań Ogólnej Teorii Względności.

Definicja (3.9) może być sformułowana w jeszcze prostszy sposób, jeśli wprowadzimy pojęcie inercjalności wyższego rzędu:

Definicja: Mając daną koneksję Γ na rozmaitości M, układ współrzędnych (x^λ) nazywamy *inercjalnym rzędu 1* w punkcie $\mathbf{m} \in M$, jeśli (x^λ) jest inercjalny oraz jeśli całkowicie symetryczna część pochodnych cząstkowych Γ znika w $\mathbf{m} \in M$:

$$\Gamma^\lambda_{\kappa\sigma}(\mathbf{m}) = \Gamma^\lambda_{(\kappa\sigma\mu)}(\mathbf{m}) = 0 \,. \tag{3.14}$$

Tensor krzywizny jest w tym ujęciu tensorem, którego składowe pokrywają się z cząstkowymi pochodnymi Γ w dowolnym układzie współrzędnych, inercjalnym rzędu 1:

$$K^{\lambda}_{\kappa\sigma\mu} \stackrel{*}{=} \Gamma^{\lambda}_{\kappa\sigma\mu} \, . \tag{3.15}$$

Powyższa konstrukcja może być w naturalny sposób rozsszerzona na obiekty opisujące wyższe pochodne koneksji. W tym celu definiujemy *układy inercjalne wyższych rzędów.*

$$\partial_{\mu_1} \cdots \partial_{\mu_k} \Gamma^{\lambda}_{\kappa\sigma} =: \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} \, . \tag{3.16}$$

Definicja: Mając daną koneksję Γ na rozmaitości M, układ współrzędnych (x^{λ}) nazywamy *inercjalnym rzędu* k w punkcie $\mathbf{m} \in M$, jeśli współczynniki koneksji oraz części całkowicie symetryczne ich pochodnych cząstkowych do rzędu k znikają w $\mathbf{m}$, tzn.:

$$\Gamma^{\lambda}_{\kappa\sigma}(\mathbf{m}) = \Gamma^{\lambda}_{(\kappa\sigma\mu)}(\mathbf{m}) = \Gamma^{\lambda}_{(\kappa\sigma\mu_1\mu_2)}(\mathbf{m}) = \cdots = \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}(\mathbf{m}) = 0 \, . \tag{3.17}$$

W szczególności: "inercjalny" znaczy "inercjalny rzędu 0".

Przykład: Współrzędne normalne są inercjalne maksymalnego rzędu.

Rozdział 4

Tensory krzywizny wyższych rzędów

4.1 Definicje i własności

Definicja: Tensor krzywizny rzędu k jest tensorem $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ takim, że w układzie współrzędnych, który jest inercjalny rzędu k, jego składowe są równe pochodnym cząstkowym współczynników koneksji:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} \overset{*}{=} \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\,. \tag{4.1}$$

Lemat: Powyższa definicja jest poprawna, tzn. $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ istnieje i jest jednoznacznie określony.

Wniosek: Tensory krzywizny spełniają poniższe tożsamości symetrii, analogiczne do (3.11):

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = K^{\lambda}_{(\kappa\sigma)\mu_1\cdots\mu_k} = K^{\lambda}_{\kappa\sigma(\mu_1\cdots\mu_k)}\,, \tag{4.2}$$

$$K^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)} = 0\,. \tag{4.3}$$

Uwaga: W przypadku, gdy współrzędne (x^{λ}) są inercjalne rzędu $(k-1)$, (4.1) przyjmuje następującą postać:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\,. \tag{4.4}$$

Twierdzenie: Tensor krzywizny rzędu k, wyrażony jako funkcja Γ oraz jej pochodnych, ma następującą formę:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)} + f(\text{pochodnych niższego rzędu } \Gamma)\,, \tag{4.5}$$

w dowolnym (niekoniecznie inercjalnym) układzie współrzędnych (x^{λ}).

Przykład 1: Tensor krzywizny rzędu $k = 1$ to po prostu tensor krzywizny:

$$K^{\lambda}_{\kappa\sigma\mu} = \Gamma^{\lambda}_{\kappa\sigma\mu} - \Gamma^{\lambda}_{(\kappa\sigma\mu)} + \Gamma^{\lambda}_{\gamma\mu}\Gamma^{\gamma}_{\kappa\sigma} - \Gamma^{\lambda}_{\gamma(\mu}\Gamma^{\gamma}_{\kappa\sigma)}\,. \tag{4.6}$$

Tożsamość (4.3) jest tożsamością Bianchi'ego 1-go rodzaju.

Przykład 2: Tensor krzywizny rzędu $k = 2$ jest równy:

$$\begin{aligned} K^{\lambda}_{\kappa\sigma\mu\nu} &= \frac{5}{8}\nabla_{\nu}K^{\lambda}_{\kappa\sigma\mu} + \frac{5}{8}\nabla_{\mu}K^{\lambda}_{\kappa\sigma\nu} - \frac{1}{8}\nabla_{\sigma}K^{\lambda}_{\mu\nu\kappa} - \frac{1}{8}\nabla_{\kappa}K^{\lambda}_{\mu\nu\sigma} \\ &= \Gamma^{\lambda}_{\kappa\sigma\mu\nu} - \Gamma^{\lambda}_{(\kappa\sigma\mu\nu)} + 2\Gamma^{\gamma}_{\kappa\sigma(\nu}\Gamma^{\lambda}_{\mu)\gamma} - 2\Gamma^{\gamma}_{(\kappa\sigma\nu}\Gamma^{\lambda}_{\mu)\gamma} + \Gamma^{\gamma}_{(\kappa\sigma}\Gamma^{\lambda}_{\mu\nu)\gamma} \\ &- \Gamma^{\gamma}_{\mu\nu}\Gamma^{\lambda}_{\kappa\sigma\gamma} + 4\Gamma^{\lambda}_{\gamma(\kappa\sigma}\Gamma^{\gamma}_{\mu\nu)} - 2\Gamma^{\lambda}_{\gamma\kappa(\mu}\Gamma^{\gamma}_{\nu)\sigma} - 2\Gamma^{\lambda}_{\gamma\sigma(\mu}\Gamma^{\gamma}_{\nu)\kappa} \\ &+ \Gamma^{\lambda}_{\gamma\alpha}\left(\Gamma^{\gamma}_{(\kappa\sigma}\Gamma^{\alpha}_{\mu\nu)} - \Gamma^{\gamma}_{\mu\nu}\Gamma^{\alpha}_{\kappa\sigma}\right) + 4\Gamma^{\lambda}_{\gamma(\kappa}\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\gamma}_{\nu)\alpha} - 4\Gamma^{\lambda}_{\gamma(\mu}\Gamma^{\alpha}_{\nu)(\kappa}\Gamma^{\gamma}_{\sigma)\alpha}\,, \end{aligned} \tag{4.7}$$

Przykład 3: Tensor krzywizny rzędu $k = 3$ jest równy:

$$\begin{aligned} K^{\lambda}_{\kappa\sigma\mu\nu\gamma} = &\frac{3}{40}\left[6\left(\nabla_{(\gamma}\nabla_{\nu)}K^{\lambda}_{\kappa\sigma\mu} + \nabla_{(\mu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\nu} + \nabla_{(\nu}\nabla_{\mu)}K^{\lambda}_{\kappa\sigma\gamma}\right)\right. \\ &-\left(\nabla_{(\gamma}\nabla_{\sigma)}K^{\lambda}_{\mu\nu\kappa} + \nabla_{(\kappa}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\sigma} + \nabla_{(\mu}\nabla_{\sigma)}K^{\lambda}_{\nu\gamma\kappa}\right. \\ &\left.\left.+\nabla_{(\kappa}\nabla_{\mu)}K^{\lambda}_{\nu\gamma\sigma} + \nabla_{(\nu}\nabla_{\sigma)}K^{\lambda}_{\gamma\mu\kappa} + \nabla_{(\kappa}\nabla_{\nu)}K^{\lambda}_{\gamma\mu\sigma}\right)\right] \\ &-\frac{3}{80}\left[23\left(K^{\lambda}_{\kappa\alpha(\gamma}K^{\alpha}_{\mu\nu)\sigma} + K^{\lambda}_{\sigma\alpha(\gamma}K^{\alpha}_{\mu\nu)\kappa}\right) - 37K^{\alpha}_{\kappa\sigma(\mu}K^{\lambda}_{\nu\gamma)\alpha}\right. \\ &\left.-2\left(K^{\alpha}_{(\mu\nu)\kappa}K^{\lambda}_{(\gamma)\alpha\sigma} + K^{\alpha}_{(\mu\nu)\sigma}K^{\lambda}_{(\gamma)\alpha\kappa}\right)\right]\,. \end{aligned} \tag{4.8}$$

Dowód wzoru (4.7) zawarty jest w dodatku A pracy [10]. Ze wzoru (A.1) w tym dodatku wynika następujące wyrażenie na pochodne kowariantne pierwszego rzędu tensora krzywizny:

$$K^{\lambda}_{\kappa\sigma\nu|\mu} = \nabla_{\mu}K^{\lambda}_{\kappa\sigma\nu} = K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{(\kappa\sigma\nu)\mu} = \frac{1}{3}\left(2K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{\nu\sigma\kappa\mu} - K^{\lambda}_{\nu\kappa\sigma\mu}\right)\,. \tag{4.9}$$

Używając jej, możemy obliczyć pochodne kowariantne tensora Riemanna $R^{\lambda}_{\kappa\sigma\nu} = -2K^{\lambda}_{\kappa[\sigma\nu]}$:

$$R^{\lambda}_{\kappa\sigma\nu|\mu} = \nabla_{\mu}R^{\lambda}_{\kappa\sigma\nu} = -K^{\lambda}_{\kappa\sigma\nu\mu} + K^{\lambda}_{\kappa\nu\sigma\mu}\,. \tag{4.10}$$

Wniosek: Powyższy wzór jest równoważny tożsamości Bianchi'ego 2-go rodzaju:

$$R^{\lambda}_{\kappa[\sigma\nu|\mu]} = \frac{1}{3}\left(R^{\lambda}_{\kappa\sigma\nu|\mu} + R^{\lambda}_{\kappa\mu\sigma|\nu} + R^{\lambda}_{\kappa\nu\mu|\sigma}\right) = -K^{\lambda}_{\kappa(\sigma\nu\mu)} + K^{\lambda}_{\kappa(\nu\sigma\mu)} \equiv 0\,. \tag{4.11}$$

Definicja: W przypadku dowolnego k, tożsamość (4.3) będzie nazywana *tożsamością Bianchi'ego k-tego rodzaju.*

Okazuje się, iż tak zdefiniowane tożsamości zawierają w sobie wszystkie możliwe tożsamości spełniane przez pochodne kowariantne tensora krzywizny (a więc również tensora Riemanna). Mówi o tym poniższe twierdzenie:

Twierdzenie: Każda pochodna *kowariantna* k-tego rzędu tensora krzywizny $K^{\lambda}_{\kappa\sigma\mu}$ może być wyrażona jako (w ogólnym przypadku – nieliniowa) kombinacja składowych tensorów krzywizny $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$ do rzędu $l = (k+1)$.

4.2 Dżety kowariantne koneksji

Dla danej koneksji symetrycznej $\Gamma^{\lambda}_{\mu\nu}$ w otoczeniu punktu $\mathbf{m} \in M$ oraz liczby naturalnej k, istnieje transformacja układu współrzędnych taka, że współczynniki $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \cdots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$ przyjmują dowolne, wybrane wartości w $\mathbf{m}$. Takie przekształcenia odgrywają rolę *transformacji cechowania.* Oczywiście, nie zmieniają one wartości tensorowych wielkości, takich jak tensory krzywizny $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \cdots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$ w $\mathbf{m} = (0, \ldots, 0)$.

Twierdzenie: W każdym punkcie czasoprzestrzeni $\mathbf{m} \in M$ wielkości

$$\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \cdots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}; \Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \cdots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$$

mogą być użyte jako globalne współrzędne we włóknie $J^k_{\mathbf{m}}\mathcal{R}(M)$ wiązki $J^k\mathcal{R}(M)$ k-tego dżetu cięć wiązki $\mathcal{R}(M)$ lokalnych układów odniesienia w czasoprzestrzeni M. (Cięcia tej wiązki to po prostu koneksje symetryczne w M.)

Inaczej mówiąc, k-te dżety koneksji $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{\kappa\sigma\mu}, \cdots, \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$ w naturalny sposób dzielą się na klasy równoważności względem transformacji cechowania. Każda klasa jest jednoznacznie scharakteryzowana przez zbiór tensorów krzywizny $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \cdots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$. W obrębie każdej klasy wielkości $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \cdots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$ mogą być użyte jako globalne współrzędne. Odgrywają rolę "parametrów cechowania", ponieważ mogą zostać dowolnie zmienione poprzez transformacje układu współrzędnych. Stąd całkowita, "niezmiennicza względem cechowania" informacja o k-tym dżecie koneksji jest zawarta w tensorach krzywizny $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \cdots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$. Zbiór tensorów krzywizny do rzędu k może być w takim razie traktowany jako k-ty "kowariantny dżet" Γ. Tensory nie zmieniają się pod wpływem wyżej opisanych transformacji cechowania. Stąd kombinacja Γ i jej pochodnych, które – jak się okazuje – są wielkościami *tensorowymi*, nie może zależeć od parametrów cechowania. Te obserwacje można podsumować następująco:

Propozycja: Funkcja F o wartościach tensorowych w wiązce dżetów $J^k\mathcal{R}(M)$ nie zależy od parametrów cechowania $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \cdots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$, tzn. jest wyłącznie funkcją tensorów krzywizny $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$, $l = 0, 1, \ldots, k$.

W szczególności rozważmy *afiniczną* zasadę wariacyjną dla pola grawitacyjnego. Jest to zasada wariacyjna pierwszego rzędu dla koneksji Γ. Jej równoważność z metryczną zasadą wariacyjną, bazującą na lagranżjanie Hilberta, została udowodniona dawno temu (patrz [12] lub [20]). Afiniczna funkcja Lagrange'a zależy od pierwszego dżetu koneksji. Jednakże, aby była niezmiennicza względem cechowania, musi być stała na klasach równoważnych względem cechowania dżetów, tj. musi być funkacją krzywizny: $K^{\lambda}_{\kappa\sigma\mu}$ lub, równoważnie, $R^{\lambda}_{\kappa\sigma\mu}$. Widać, że to samo jest prawdziwe dla afinicznej zasady wariacyjnej wyższego rzędu: jej lagranżjan może zależeć wyłącznie od kowariantnych dżetów koneksji.

4.3 Główne twierdzenia algebry tensorów krzywizny wyższych rzędów

Twierdzenie 1: Każdy tensor krzywizny wyższego rzędu $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$, $k \geqslant 1$, może zostać zapisany jako (w ogólnym przypadku – nieliniowa) kombinacja tensorów krzywizny 1-*go* rzędu $K^{\lambda}_{\kappa\sigma\mu}$ oraz ich pochodnych kowariantnych $\nabla_{\mu_1} \ldots \nabla_{\mu_l} K^{\lambda}_{\kappa\sigma\mu}$ do rzędu $l = k - 1$. Co więcej, to wyrażenie jest liniowe w pochodnych najwyższego rzędu (czyli dla $l = k - 1$).

To twierdzenie opiera się na następującym lemacie, udowodnionym w dodaktu B pracy [10]:

Lemat: Dla każdego $k \geqslant 1$ istnieje jednoznacznie określona, liniowa kombinacja $S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ pochodnych kowariantnych $\nabla_{\mu_1} \ldots \nabla_{\mu_{k-1}} K^{\lambda}_{\kappa\sigma\mu_k}$ takich, że

$$S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)} + f(\text{pochodnych niższego rzędu } \Gamma)\,, \tag{4.12}$$

w dowolnym (niekoniecznie inercjalnym) układzie współrzędnych (x^{λ}).

Twierdzenie 2:

Tensor krzywizny $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n}$, $n > 1$, może być zapisany jako suma dwóch elementów: 1) liniowej kombinacji pochodnych kowariantnych pierwszego rzędu tensorów rzędu $k-1$: $\nabla_{\mu_n} K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_{n-1}}$ oraz 2) (nieliniowej) funkcji tensorowej tensorów krzywizny niższych rzędów $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$, $l \leqslant n-1$:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n} = S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n} + f\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \cdots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_{n-1}}\right)\,, \tag{4.13}$$

gdzie

$$S^{\lambda}_{\nu_1\nu_2\cdots\nu_{n+2}} = \sum_{\pi} c(\pi) \cdot \nabla_{\nu_{\pi(1)}} K^{\lambda}_{\nu_{\pi(2)}\nu_{\pi(3)}\cdots\nu_{\pi(n+2)}} \tag{4.14}$$

natomiast π jest permutacją $(n+2)$ elementów oraz $c(\pi)$ są stałymi współczynnikami.

Przykład 4: Ten sam tensor krzywizny rzędu $k = 3$ (4.8), wyrażony tym razem jako funkcja pierwszej pochodnej tensora rzędu $k = 2$ oraz nieliniowej kombinacji tensorów rzędu $k = 1$, jest równy:

$$\begin{aligned} K^{\lambda}_{\kappa\sigma\mu\nu\gamma} &= \frac{11}{10}\nabla_{\nu)} K^{\lambda}_{\kappa\sigma(\mu_1\mu_2} + \frac{1}{10}\nabla_{(\nu} K^{\lambda}_{\mu_1\mu_2)\kappa\sigma} - \frac{2}{5}\nabla_{\sigma)} K^{\lambda}_{(\mu_1\mu_2\nu)(\kappa} \\ &+ \frac{3}{80}\left[-25\left(K^{\lambda}_{\kappa\alpha(\gamma} K^{\alpha}_{\mu\nu)\sigma} + K^{\lambda}_{\sigma\alpha(\gamma} K^{\alpha}_{\mu\nu)\kappa}\right) + 31 K^{\alpha}_{\kappa\sigma(\mu} K^{\lambda}_{\nu\gamma)\alpha}\right. \\ &+ \left. 10\left(K^{\alpha}_{(\mu\nu)\kappa} K^{\lambda}_{(\gamma)\alpha\sigma} + K^{\alpha}_{(\mu\nu)\sigma} K^{\lambda}_{(\gamma)\alpha\kappa}\right)\right]\,. \end{aligned} \tag{4.15}$$

Rozdział 5

Niezmiennicza redukcja rzędu teorii

Jak zostało powiedziane w rozdziale 2.2, obniżenia rzędu (1.1) w sposób niezmienniczy (względem transformacji współrzędnych) można dokonać, wykorzystując pojęcie *tensorów krzywizny wyższych rzędów*, które opisują część współrzędnościowo niezmenniczą *dżetów koneksji wyższych rzędów*.

Niezmienniczy lagranżjan nie będzie zatem zależeć od pola grawitacyjnego Γ i jego pochodnych w dowolny sposób, lecz jedynie poprzez niezmiennicze obiekty geometryczne: tensory krzywizny. Zawierają one kompletną informację o tensorze Riemanna i jego pochodnych kowariantnych, podczas gdy równania (4.2) i (4.3) pokazują, ile niezależnych obiektów potrzeba do zapisania tej informacji. Lagranżjan (1.1) może być więc zapisany jako funkcja tensorów krzywizny:

$$L = L(\varphi, \varphi_\kappa, K^\lambda_{\mu\nu\sigma}, \cdots, K^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}), \tag{5.1}$$

gdzie jako φ zostały zapisane pola materii (możliwe indeksy zostały pominięte dla uproszczenia notacji), oddziałujące z polem grawitacyjnym $\Gamma^\lambda_{\mu\nu}$. Wariacja w odniesieniu do pól materii daje oczywiste równania Eulera-Lagrange'a poprzez standardową funkcję tworzącą:

$$\delta L(\varphi, \varphi_\kappa, \cdots) = \partial_\kappa \left(p^\kappa \delta\varphi\right) + \cdots, \tag{5.2}$$

podczas gdy wariacja w odniesieniu do Γ daje pozostałą część funkcji tworzącej, zgodnie z (2.11), ze zmiennymi φ^K zastąpionymi przez $\Gamma^\lambda_{\mu\nu}$:

$$\begin{aligned} \delta L &= \cdots + \partial_\kappa \left(P^{\mu\nu\kappa}_\lambda \delta\Gamma^\lambda_{\mu\nu} + P^{\mu\nu\sigma\kappa}_\lambda \delta\Gamma^\lambda_{\mu\nu\sigma} + \cdots + P^{\mu\nu\sigma_1\cdots\sigma_n\kappa}_\lambda \delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}\right) \\ &= \cdots + \left(\partial_\kappa P^{\mu\nu\kappa}_\lambda\right)\delta\Gamma^\lambda_{\mu\nu} + \left(P^{\mu\nu\sigma}_\lambda + \partial_\kappa P^{\mu\nu\sigma\kappa}_\lambda\right)\delta\Gamma^\lambda_{\mu\nu\sigma} + \cdots \\ &\quad + \left(P^{\mu\nu\sigma_1\cdots\sigma_{n-1}}_\lambda + \partial_\kappa P^{\mu\nu\sigma_1\cdots\sigma_{n-1}\kappa}_\lambda\right)\delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} + P^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda \delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}. \end{aligned} \tag{5.3}$$

Aby obliczyć wartości pojawiających się tu pędów $P^{\mu\nu\sigma_1\cdots\sigma_k}_\lambda$, należy dokonać wariacji gęstości funkcji Lagrange'a (5.1) (tu dla uproszczenia pominięto część "materialną" wyrażenia):

$$\delta L = Q^{\mu\nu\sigma}_\lambda \delta K^\lambda_{\mu\nu\sigma} + Q^{\mu\nu\sigma_1\sigma_2}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\sigma_2} + \cdots + Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}, \tag{5.4}$$

gdzie wielkości Q są gęstościami tensorowymi (w przeciwieństwie do P). Zapisując każdy tensor krzywizny $K^\lambda_{\mu\nu\sigma_1\cdots\sigma_k}$, $k \leqslant n$ jako funkcję współczynników koneksji $\Gamma^\lambda_{\mu\nu}$ oraz ich pochodnych do rzędu k, można ostatecznie wyrazić wswzystkie pędy $P^{\mu\nu\sigma_1\cdots\sigma_k}_\lambda$ jako funkcje pochodnych $\Gamma^\lambda_{\mu\nu}$ i stąd otrzymać równania Eulera-Lagrange'a. W szczególności wyrażenie na pęd najwyższego rzędu jest wyjątkowo proste:

$$P^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda = Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda, \tag{5.5}$$

ponieważ zależność tensora krzywizny od najwyższych pochodnych koneksji jest bardzo prosta (patrz równanie (4.5)).

Stosując procedurę opisaną w rozdziale 2.2, możemy obniżyć różniczkowy rząd zasady wariacyjnej (5.3) w odniesieniu do Γ poprzez ulepszenie pędu najwyższego rzędu $P_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}$ do poziomu pól materii. Jak zostało wspomniane na początku tego rozdziału, taka procedura jest niesatysfakcjonująca, ponieważ wielkość

$$\mathcal{L} = L + \partial_\kappa \left(P_\lambda^{\mu\nu\sigma_1\cdots\sigma_{n-1}\kappa} \Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} \right)$$

nie jest niezmienniczą gęstością skalarną. Możemy ją jednak nieco zmodyfikować tak, że otrzymana w jej wyniku gęstość lagranżjanu będzie prawdziwą gęstością skalarną, a nowe równania pola nie będą zależały od wyboru układu współrzędnych. Mianowicie, korzystając z Twierdzenia 2 z rozdziału 4.3 (patrz równanie (4.13)), możemy użyć $S^\lambda_{\kappa\sigma\mu_1\cdots\mu_n}$ zamiast $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_n}$, by sparametryzować n-ty kowariantny dżet koneksji Γ. Następnie dokonujemy transformacji Legendre'a pomiędzy S a pędem najwyższego rzędu $Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}$. Na koniec, wykorzystując równanie (4.14), definiujemy pęd kanonicznie sprzężony do $Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}$ jako:

$$\Pi^{\lambda\kappa}_{\nu_1\nu_2\cdots\nu_{n+2}} = -\sum_\pi c(\pi) \cdot \delta^\kappa_{\nu_{\pi(1)}} K^\lambda_{\nu_{\pi(2)}\nu_{\pi(3)}\cdots\nu_{\pi(n+2)}} \,, \tag{5.6}$$

który oczywiście spełnia tożsamość:

$$\nabla_\kappa \Pi^{\lambda\kappa}_{\nu_1\nu_2\cdots\nu_{n+2}} = -S^\lambda_{\nu_1\nu_2\cdots\nu_{n+2}} \,. \tag{5.7}$$

Ostatecznie wariacja nowej funkcji tworzącej $\mathcal{L}$ przyjmuje postać:

$$\begin{aligned} \delta\mathcal{L} = {} & \widetilde{Q}_\lambda^{\mu\nu\sigma} \delta K^\lambda_{\mu\nu\sigma} + \widetilde{Q}_\lambda^{\mu\nu\sigma_1\sigma_2} \delta K^\lambda_{\mu\nu\sigma_1\sigma_2} + \cdots \\ & + \cdots + \approx{Q}_\lambda^{\mu\nu\sigma_1\cdots\sigma_{n-1}} \delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} + \partial_\kappa \left(\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n} Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n} \right) \,. \end{aligned} \tag{5.8}$$

Kolejne kroki tej procedury oraz pojawiające się wielkości są szczegółowo opisane w pracy [9]. Wszystkie wielkości występujące we wzorze (5.8) są tensorami lub gęstościami tensorowymi. Mamy tu do czynienia z zasadą wariacyjną pierwszego rzędu dla nowego pola materii $Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}$ oraz zasadą $(n-1)$-go rzędu dla pola grawitacyjnego $\Gamma^\lambda_{\mu\nu}$, analogiczną do zasady n-tego rzędu (5.4).

Stosując tę procedurę $(n-1)$ razy, możemy zredukować ilość grawitacyjnych stopni swobody zawartych w lagranżjanie do jednego. Powstała w ten sposób teoria jest równoważna teorii Einsteina, jak zostało pokazane w [8].

Rozdział 6

Praca: Covariant jets of a connection and higher order curvature tensors

Journal of Geometry and Physics 163 (2021) 104092

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

Covariant jets of a connection and higher order curvature tensors

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ARTICLE INFO

Article history:
Received 16 October 2020
Received in revised form 22 December 2020
Accepted 27 December 2020
Available online 11 January 2021

Keywords:
Curvature
Riemann tensor
Theory of connection
Jet theory

ABSTRACT

A novel approach to the connection theory on a differential manifold is presented here. In this framework, the notion of a k-th order ($k = 1, 2, \dots$) curvature tensor arises in a natural way. For $k = 1$ our curvature tensor is equivalent to the conventional Riemann tensor. This approach provides natural tools for analysis of higher order variational problems in General Relativity Theory.

1. Introduction

To find an appropriate description of the "dark matter" and/or "dark energy", many authors consider recently generalizations of General Relativity Theory based on "non-linear Lagrangians", i.e. Lagrangian functions depending in a non-linear way upon the curvature tensor. The particular case of a Lagrangian L depending non-linearly upon the scalar curvature R or even the complete Ricci tensor $R_{\mu\nu}$, but not depending upon the Weyl tensor, was considered by many authors (cf. [6] and [19]). Probably the first, physically well motivated, proposal of such a theory was the non-linear "R^2"-Lagrangian proposed by A. Sacharov (see [17]). It belongs to the family of "$f(R)$-theories", where $L = \sqrt{|g|}f(R)$ (see also [18]).

Mathematically, these theories are equivalent to conventional version of General Relativity Theory, interacting with extra matter fields. For $f(R)$-theories this equivalence was proved already in 1987 (see [5] and [7]). Later on, this result was extended to Lagrangians depending upon the Ricci tensor (see [7] and [8]) and, finally, to those which depend upon the complete Riemann curvature $R^{\lambda}_{\kappa\sigma\mu}$ (see [10]).

It turns out that this spectacular "universality" of the Einstein theory extends also to theories of higher differential order, when the Lagrangian function depends upon higher order covariant derivatives $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\mu}$ of the curvature. This result will be published elsewhere. Here, we want to present the fundamental geometric structures underlying this result.

The story begins with the question: How many *independent* degrees of freedom is contained within the k-th order covariant derivatives $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\mu}$ of the curvature tensor, if we take into account all the Bianchi-I and Bianchi-II identities together with their derivatives? To answer this question, the notion of a k-th order curvature tensor arises in a natural way (see Section 3). It is shown there that all the independent identities fulfilled by those covariant derivatives can be identified as the Bianchi identities of subsequent orders k, for $k = 1, 2, \dots$ (see (29) and (30)).

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https://doi.org/10.1016/j.geomphys.2020.104092
0393-0440/

The analysis of higher order curvature tensors is, however, preceded (in Section 2) by an alternative version of the theory of a connection on a manifold. Although equivalent to the standard textbook version, it highly simplifies both the variational and canonical formulations of General Relativity Theory (see [10] and [14]). But here, it provides a natural framework for the theory of the higher order curvature.

In Section 4 we introduce the notion of a "covariant jet" of a connection as an equivalence class of jets *modulo* "gauge transformations", also defined there. Finally, Section 5 contains the main result of this paper: curvature tensors up to order k contain the same information as the Riemann tensor and its covariant derivatives up to order k-1. Although the same, the information is now described as a list of *independent* geometric objects, which is the key step towards correct formulation of higher order variational problems (cf. [13] and [12]) and the proof of their equivalence with the conventional General Relativity Theory.

2. Theory of curvature

A connection Γ on a manifold M, $\dim M = n$, is usually understood as a connection in the principal $\mathrm{GL}(n, \mathbb{R})$-bundle of frames ($n$-beins) over M. The bundle of frames is, however, not an arbitrary principal bundle but is equipped with an extra structure: the *solder form*. Due to this structure, a connection defined this way is not an irreducible object but splits canonically into two irreducible components: the *symmetric connection* and the tensorial object called *torsion*. Below, we propose another definition of the symmetric connection, as a fundamental (irreducible) geometric object. For this purpose we are going to construct the canonical *bundle of reference frames* $\mathcal{R}(M)$ over M. The connection will be then defined as a section of this bundle:

$$\Gamma : M \to \mathcal{R}(M) . \tag{1}$$

In other words: connection is a field of reference frames. Once chosen on M, it defines at each point $\mathbf{m} \in M$ the "privileged" frame $\Gamma(\mathbf{m}) \in \mathcal{R}(M)$ which will be called the *inertial frame* at $\mathbf{m} \in M$.

The notion of an inertial frame was introduced by I. Newton in his first law of dynamics: "the motion of a freely falling body is rectilinear and uniform", i.e. satisfies equation

$$\ddot{x}^k := \frac{\mathrm{d}^2}{\mathrm{d}t^2} x^k(t) = 0 , \tag{2}$$

in a privileged coordinates system (x^k), which was called "inertial" by Newton.

Nowadays almost forgotten, the above point of view was formulated and then strongly supported by Albert Einstein who stressed the *active* role of space and time in physics. He considered existence of inertial frames as a fundamental property of our spacetime (cf. [3]) in contrast to then dominating philosophy, which was treating it as merely a *passive "stage" in Theatrum Mundi*, the place where the whole physics is presented to our eyes.

Originally, existence of inertial frames was meant as a *global* structure of the Universe. In view of the Special Relativity Theory, the modern version of the first law must, therefore, be entirely *local*. Hence, the above "Newtonian" formulation must be replaced by its "Einsteinian" version: for every point $\mathbf{m} \in M$ there is a coordinate system (y^α) in a neighborhood of $\mathbf{m}$ such that the differential equation describing trajectories of a freely falling body (e.g. a space ship with its engine turned off) reduces *at this particular point* to:

$$\ddot{y}^\alpha = 0 . \tag{3}$$

Here, "dot" denotes the derivative with respect to an internal clock (e.g. a biological clock which organizes the life of the ship's crew), whereas the Newton's version referred to the (nonexistent) "absolute time".

Eq. (3) can be easily rewritten to another coordinate system (x^λ) on M. This way we obtain the following *equation of motion of freely falling bodies*, valid in arbitrary coordinates (Einstein summation convention is used systematically here):

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 , \tag{4}$$

where by $\Gamma^\lambda_{\mu\nu}$ we denote the following combination of transition coefficients between the two coordinate systems

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial y^\alpha} \cdot \frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu} . \tag{5}$$

We see that the new coordinate system is also "inertial" at $\mathbf{m}$ if and only if the second derivatives in question vanish at $\mathbf{m}$. This observation justifies the following

Definition. A reference frame at a point $\mathbf{m} \in M$ of a manifold M is an equivalence class of local coordinate charts with respect to the relation "$\sim_{\mathbf{m}}$", where two charts in a neighborhood of $\mathbf{m}$ are declared to be equivalent if and only if the second derivatives of any coordinate from one chart with respect to coordinates of the other chart vanish at $\mathbf{m}$:

$$((x^\mu) \sim_{\mathbf{m}} (y^\alpha)) \Longleftrightarrow \left(\frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(\mathbf{m}) = 0 \right) . \tag{6}$$

At a first glance, the relation does not look to be symmetric. But it is easy to check that, indeed, it is a genuine equivalence relation. The equivalence class of the chart (y^α) will be denoted by $[(y^\alpha)]_{\mathbf{m}}$ and called "a reference frame at $\mathbf{m} \in M$".

Given a coordinate chart (x^λ) in a neighborhood of $\mathbf{m}$, we may parameterize any reference frame $[(y^\alpha)]_{\mathbf{m}}$ at $\mathbf{m}$ by the following (symmetric by definition) table of numbers:

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial y^\alpha} \cdot \frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(\mathbf{m}) \, , \tag{7}$$

where (y^α) is a representative of $[(y^\alpha)]_{\mathbf{m}}$. It can be easily checked that the numbers $\Gamma^\lambda_{\mu\nu}$ do not depend upon the choice of a representative. In particular, our reference frame $\left[(x^\lambda)\right]_{\mathbf{m}}$ corresponds to the trivial table $\Gamma^\lambda_{\mu\nu} = 0$. Moreover, the parameterization is surjective: given a table $\Gamma^\lambda_{\mu\nu}$, there is a unique reference frame at $\mathbf{m}$, namely: $[(y^\alpha)]_{\mathbf{m}}$, such that (7) agrees with this table. Its representative is given e.g. by:

$$y^\lambda := x^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu} x^\mu x^\nu \, . \tag{8}$$

The formula is written for coordinates *centered* at $\mathbf{m}$, i.e. such that $\mathbf{m} = (0, \dots, 0)$. Otherwise, x^μ must be replaced by $(x^\mu - m^\mu)$, where $\mathbf{m} = (m^\mu)$. This way the set of all reference frames $\mathcal{R}(M)$ acquires the structure of an affine fiber bundle over M and $(x^\lambda, \Gamma^\lambda_{\mu\nu})$ are local coordinates on $\mathcal{R}(M)$, compatible with this structure.

Given a coordinate chart (x^λ), a connection (1) (i.e. a section of $\mathcal{R}(M)$) can, therefore, be uniquely described by $\frac{n^2(n+1)}{2}$ functions: $M \ni \mathbf{m} \to \Gamma(\mathbf{m}) = \left(\Gamma^\lambda_{\mu\nu}(\mathbf{m})\right)$ called *connection coefficients*. It is a simple exercise to derive their standard transformation laws with respect to any change of coordinates (x^λ). Coordinates (y^α) are called *inertial* at $\mathbf{m}$ for the connection Γ if $[(y^\alpha)]_{\mathbf{m}} = \Gamma(\mathbf{m})$. Our coordinates (x^λ) are inertial at $\mathbf{m}$ for the connection Γ if all the functions $\Gamma^\lambda_{\mu\nu}$ vanish here.

Due to (8), the connection coefficients $(\Gamma^\lambda_{\mu\nu})$ obtain a simple geometric interpretation: they describe the quadratic correction which is necessary to upgrade our working coordinates (x^λ) to the level of inertial coordinates.

Connection is *flat* if there exists a *global* inertial frame, i.e. a coordinate chart which is inertial not just at a single point, but everywhere. Given a connection Γ, how to check whether or not it is flat? First, we can choose coordinates (x^μ) which are inertial at $\mathbf{m}$, i.e. such that $\Gamma^\lambda_{\mu\nu}$ vanish at $\mathbf{m}$. Without any loss of generality we can assume that $\mathbf{m} = (0, 0, \dots, 0)$. Is it possible to "improve" these coordinates in such a way that $\Gamma^\lambda_{\mu\nu}$ vanish also outside of $\mathbf{m}$? As a first step to answer this question let us try to annihilate also the derivatives $\Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m}) := \partial_\kappa \Gamma^\lambda_{\mu\nu}(\mathbf{m})$. Is it possible?

Consider an improved system of coordinates:

$$y^\lambda := x^\lambda + \frac{1}{6} U^\lambda_{\mu\nu\kappa} x^\mu x^\nu x^\kappa + \text{terms of order higher than 3} \, , \tag{9}$$

where coefficients U are totally symmetric: $U^\lambda_{\mu\nu\kappa} = U^\lambda_{(\mu\nu\kappa)}$. We limit ourselves to such coordinate transformations because:

1. terms of order 0 vanish under differentiation (5), i.e. do not influence the connection coefficients $\Gamma^\lambda_{\mu\nu}$;
2. terms of order 1 produce only a linear (with constant coefficients) transformation of $\Gamma^\lambda_{\mu\nu}$ and, whence, a linear homogeneous (tensorial type) transformation of the coefficients $\Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m})$: if they do not vanish before, they will not vanish after such a transformation;
3. non-vanishing terms of order 2 would change, due to (5), the value of Γ at $\mathbf{m}$. We try to avoid it because we have already put $\Gamma^\lambda_{\mu\nu}(\mathbf{m}) = 0$ and we do not want to spoil this.
4. a possible non-symmetric part of U vanishes when contracted with the totally symmetric expression $x^\mu x^\nu x^\kappa$;
5. 4-th and higher order terms produce 2-nd and higher order terms in $\Gamma^\lambda_{\mu\nu}$ and, whence, do not change the value of derivatives $\Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m})$ at $\mathbf{m}$, i.e. at $x^\mu = 0$.

Using (5) we calculate the new connection coefficients $\tilde{\Gamma}^\lambda_{\mu\nu}(\mathbf{m})$. They contain an extra term $U^\lambda_{\mu\nu\kappa} x^\kappa$. Finally, after differentiation, we obtain:

$$\tilde{\Gamma}^\lambda_{\mu\nu\kappa}(\mathbf{m}) = \Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m}) + U^\lambda_{\mu\nu\kappa} \, . \tag{10}$$

Using an arbitrary (but symmetric) tensor $U^\lambda_{\mu\nu\kappa}$ we are able to "kill" the totally symmetric part $\Gamma^\lambda_{(\mu\nu\kappa)}$ of $\Gamma^\lambda_{\mu\nu\kappa}$. The remaining part, if any:

$$K^\lambda_{\mu\nu\kappa} := \Gamma^\lambda_{\mu\nu\kappa} - \Gamma^\lambda_{(\mu\nu\kappa)} \, , \tag{11}$$

constitutes an obstruction against a possibility of annihilating derivatives of Γ, i.e. against its flatness. It measures, therefore, how *non-flat*, i.e. how *curved*, is the connection. We call it the *curvature tensor*.

Definition. The curvature tensor of a connection Γ is a tensor which reduces to (11) in inertial coordinates.

It is easy the prove the following, simple

Lemma. *Above definition implies the following formula, valid in an arbitrary coordinate system:*

$$K^{\lambda}_{\mu\nu\kappa} = \Gamma^{\lambda}_{\mu\nu\kappa} - \Gamma^{\lambda}_{(\mu\nu\kappa)} + \left(\Gamma^{\lambda}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma(\kappa}\Gamma^{\sigma}_{\mu\nu)}\right)$$
$$= \Gamma^{\lambda}_{\mu\nu\kappa} + \Gamma^{\lambda}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\nu} - \left(\Gamma^{\lambda}_{(\mu\nu\kappa)} + \Gamma^{\lambda}_{\sigma(\kappa}\Gamma^{\sigma}_{\mu\nu)}\right) \ . \tag{12}$$

Due to the definition, the curvature tensor K is symmetric in first indices and its totally symmetric part vanishes identically:

$$K^{\lambda}_{\mu\nu\kappa} = K^{\lambda}_{\nu\mu\kappa} \quad ; \quad K^{\lambda}_{(\mu\nu\kappa)} = 0 \ . \tag{13}$$

The last identity is equivalent to Bianchi I-st type identity (see (15)).

The above curvature tensor is equivalent to the standard Riemann tensor $R^{\lambda}_{\mu\nu\kappa}$: antisymmetrization of K in last two indices produces R and symmetrization of R in first two indices produces K. More precisely, the following relations are obvious:

$$R^{\lambda}_{\mu\nu\kappa} = -2K^{\lambda}_{\mu[\nu\kappa]} = K^{\lambda}_{\mu\kappa\nu} - K^{\lambda}_{\mu\nu\kappa} \quad ; \quad K^{\lambda}_{\mu\nu\kappa} = -\frac{2}{3}R^{\lambda}_{(\mu\nu)\kappa} = -\frac{1}{3}\left(R^{\lambda}_{\mu\nu\kappa} + R^{\lambda}_{\nu\mu\kappa}\right) \ , \tag{14}$$

and the identities (13) for K are equivalent to the analogous identities for R:

$$R^{\lambda}_{\mu\nu\kappa} = -R^{\lambda}_{\mu\kappa\nu} \quad ; \quad R^{\lambda}_{[\mu\nu\kappa]} = 0 \ . \tag{15}$$

We conclude that the Riemann tensor $R^{\lambda}_{\mu\nu\kappa}$ and the curvature $K^{\lambda}_{\mu\nu\kappa}$ provide two different, but entirely equivalent, representations of the same geometric object. It turns out, however, that the latter is technically much more convenient for purposes of the General Relativity Theory. This is easily seen e.g. in the fundamental Misner–Thorn–Wheeler monograph [16], where the fundamental variational formula for Einstein equations (formula (21.20) on p. 500) is written in a simplified form, because "the boundary term is too complicated...". A trivial origin of this complication is due to the fact that the momentum canonically conjugate to Γ:

$$P_{\lambda}^{\mu\nu\kappa} := \frac{\partial L}{\partial \Gamma^{\lambda}_{\mu\nu\kappa}} = \frac{\partial L}{\partial K^{\lambda}_{\mu\nu\kappa}} \tag{16}$$

and the derivative of the Lagrangian with respect to the Riemann tensor:

$$Q_{\lambda}^{\mu\nu\kappa} := \frac{\partial L}{\partial R^{\lambda}_{\mu\nu\kappa}} \ , \tag{17}$$

although related by a one-to-one correspondence, have different symmetries, which obscures significantly the variational formula. Using (16) instead of (17), the "mutilated" Misner–Thorn–Wheeler formula can be nicely written (see formula (4.10) in [9]), which constitutes the key step in the analysis of the rich structure of the surface terms in Canonical Relativity (cf. [2]).

Observe finally, that definition (11) can be formulated in a yet simpler way if we introduce the notion of the higher order inertiality:

Definition. Given a connection Γ on a manifold M, we call a coordinate system (x^{λ}) "inertial of order 1" at the point $\mathbf{m} \in M$ if it is inertial and if the totally symmetric part of its partial derivatives vanishes at $\mathbf{m} \in M$:

$$\Gamma^{\lambda}_{\kappa\sigma}(\mathbf{m}) = \Gamma^{\lambda}_{(\kappa\sigma\mu)}(\mathbf{m}) = 0 \, . \tag{18}$$

Curvature tensor is a tensor whose components coincide with partial derivatives of Γ in any inertial coordinates of order 1:

$$K^{\lambda}_{\kappa\sigma\mu} \stackrel{*}{=} \Gamma^{\lambda}_{\kappa\sigma\mu} \, . \tag{19}$$

3. Curvature tensors of higher order

The construction presented above can be naturally extended to objects describing higher order derivatives of a connection. For this purpose we define *higher order inertial systems*. We use here consequently the notation well adapted to the theory of jets, namely:

$$\partial_{\mu_1} \cdots \partial_{\mu_k} \Gamma^{\lambda}_{\kappa\sigma} =: \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} \, . \tag{20}$$

Definition. Given a connection Γ on a manifold M, we call a coordinate system (x^{λ}) "inertial of order k" at the point $\mathbf{m} \in M$ if the connection coefficients and the totally symmetric parts of their partial derivatives up to order k vanish at $\mathbf{m}$, i.e.:

$$\Gamma^{\lambda}_{\kappa\sigma}(\mathbf{m}) = \Gamma^{\lambda}_{(\kappa\sigma\mu)}(\mathbf{m}) = \Gamma^{\lambda}_{(\kappa\sigma\mu_1\mu_2)}(\mathbf{m}) = \cdots = \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}(\mathbf{m}) = 0 \, . \tag{21}$$

In particular: "inertial" means "inertial of order 0".

Example. Normal coordinates are inertial of maximal order.

Given a differentiable manifold M equipped with a connection Γ, the exponential map $T_{\mathbf{m}}M \to M$, defined in a neighborhood of $0 \in T_{\mathbf{m}}M$, is defined by the following formula:

$$\exp(v) = \gamma_v(1)$$

where $t \to \gamma_v(t)$ is an *orthodrome* (or "self-parallel line", or "geodesic line" in case of a metric connection) originating at $\mathbf{m} = \gamma_v(0)$, whose tangent vector equals: $\dot{\gamma}_v(0) = v = v^k \partial_k$.

This mapping gives us a coordinate system, called *normal coordinates* at $\mathbf{m}$. In these coordinates any "radial line", i.e. given by formula $x^k(t) = tv^k$, is an orthodrome (a geodesic line), i.e. fulfills equation:

$$\ddot{x}(t) + \Gamma^k_{lm}(x^k)\dot{x}^l\dot{x}^m = \Gamma^k_{lm}(tv^k)v^l v^m = 0 \,, \tag{22}$$

for every vector (v^k), where "dot" denotes $\frac{d}{dt}$. The well-known polarization theorem of a symmetric bilinear form implies vanishing of the connection coefficients at $t = 0$: $\Gamma^k_{lm}(0) = 0$.

The same argument can be used for higher order derivatives of (22) at $t = 0$. For example, first derivative of (22) with respect to parameter t implies:

$$\Gamma^k_{lmn}(tv^k)v^l v^m v^n = 0 \,, \tag{23}$$

and, consequently,

$$\Gamma^k_{lmn}(0)v^l v^m v^n = \Gamma^k_{(lmn)}(0)v^l v^m v^n = 0 \,.$$

Due to the general polarization theorem (see Appendix C), we obtain

$$\Gamma^k_{(lmn)}(0) = 0 = \Gamma^k_{(lmn)}(\mathbf{m}) \,.$$

Further differentiation of Eq. (22) leads to $\Gamma^k_{(lmni)}(\mathbf{m}) = 0$. Repeating this procedure k times (if our manifold M is differentiable of order k) we prove that the normal coordinate system is inertial of order k, where k is maximal.

Definition. The curvature tensor of order k is a tensor $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ such that, in a particular coordinate system which is inertial of order k, its components are equal to partial derivatives of the connection coefficients:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} \stackrel{*}{=} \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} \,. \tag{24}$$

Lemma. *The above definition is correct, i.e.* $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ *exists and is unique.*

Proof. If a coordinate system (x^λ) *is not* inertial at $\mathbf{m}$, then to calculate the components $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ one has to: 1) choose an inertial system (y^α) of order k and calculate the corresponding components $K^{\alpha}_{\beta\gamma\delta_1\cdots\delta_k}$ according to (24) and then: 2) transform the components back to coordinates (x^λ) according to tensorial transformation laws. We have to prove that the result does not depend upon the choice of (y^α). Consider, therefore, another (inertial of order k at $\mathbf{m}$) system (z^λ). Without any change of generality, we may assume that both systems are centered at $\mathbf{m}$, i.e. $\mathbf{m} = \{y^\alpha = 0\} = \{z^\lambda = 0\}$ because Γ's do not change under the translation of coordinates: $z^\lambda \longrightarrow z^\lambda + \text{const}$. Also, we can assume that the following:

$$A^{\lambda}{}_{\alpha} := \frac{\partial z^\lambda}{\partial y^\alpha}(\mathbf{m}) \tag{25}$$

is a unit matrix: $A^{\lambda}{}_{\alpha} = \delta^{\lambda}_{\alpha}$. Indeed, if the condition is not satisfied, we can define a new system

$$\tilde{y}^\lambda := A^{\lambda}{}_{\alpha} y^\alpha$$

which is also inertial of order k at $\mathbf{m}$ because such a linear (with constant coefficients) change of coordinates implies a tensorial (homogeneous) transformation of components $\Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$. The above remarks imply that we can limit ourselves to the case when the Taylor expansion of the coordinate transformation between those two coordinate systems assumes the following form:

$$z^\lambda = y^\lambda + \sum_{n\geq 2} \frac{1}{n!} \widetilde{U}^{\lambda}{}_{\beta_1\ldots\beta_n} \cdot y^{\beta_1} \ldots y^{\beta_n} \,, \tag{26}$$

where $\widetilde{U}$'s are constant matrices, totally symmetric in lower indices.

In order to simplify our analysis, we can equivalently assume that our coordinate transformation (26) is a subsequent superposition of transformations of the following shape:

$$z^\lambda = y^\lambda + \frac{1}{n!} U^{\lambda}{}_{\beta_1\ldots\beta_n} \cdot y^{\beta_1} \ldots y^{\beta_n} \,, \tag{27}$$

where n assumes subsequent values: $n = 2, 3, \ldots$. It is easy to see that such a transformation does not change the value of components $\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m})$ for $k < n-2$, whereas for $k = n-2$ it implies the following transformation of derivatives of Γ:

$$\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m}) \longrightarrow \Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m}) + U^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}\,. \tag{28}$$

Hence, for $n = 2$, transformation (27) violates the inertiality condition (21) at $k = 0$ if $U^\lambda_{\beta_1\beta_2} \neq 0$, which the subsequent transformations corresponding to $n \geq 2$ cannot repair. For $n = 3$, transformation (27) violates the inertiality condition (21) at $k = 1$ if $U^\lambda_{\beta_1\beta_n\beta_3} \neq 0$, which the subsequent transformations corresponding to $n \geq 3$ cannot repair etc. We conclude that if both systems (y^α) and (z^λ) are inertial of order k, the matrices U vanish identically for $n < k+2$ and, therefore, the components $\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}$ are identical in both systems. Hence, their tensorial transformation to an arbitrary system (x^λ) gives identical result $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}$. □

Corollary. *The curvature tensors fulfill the symmetry identities, analogous to (13):*

$$K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} = K^\lambda_{(\kappa\sigma)\mu_1\cdots\mu_k} = K^\lambda_{\kappa\sigma(\mu_1\cdots\mu_k)}\,, \tag{29}$$

$$K^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)} = 0\,. \tag{30}$$

Remark. In case when (x^λ) are inertial of order $(k-1)$, formula (28) implies that (24) transforms to the following form:

$$K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)}\,. \tag{31}$$

Theorem. *When expressed in terms of Γ's and their derivatives, the curvature tensor of order k has the following form:*

$$K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)} + f(\text{lower order derivatives of } \Gamma)\,, \tag{32}$$

in an arbitrary (not necessarily inertial!) system of coordinates (x^λ).

Proof. To calculate $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}$ we have to find first an inertial frame (y^α) of order k. For this purpose we can use again a superposition of subsequent transformations (27) for $n = 2, 3, \ldots, k+2$. To annihilate $\Gamma^\lambda_{\kappa\sigma}(\mathbf{m})$ we must use $n = 2$ and put

$$U^\lambda_{\kappa\sigma} := \Gamma^\lambda_{\kappa\sigma}(\mathbf{m})\,.$$

To annihilate $\Gamma^\lambda_{(\kappa\sigma\mu)}(\mathbf{m})$ we must use $n = 3$ and put

$$U^\lambda_{\kappa\sigma\mu} := \Gamma^\lambda_{(\kappa\sigma\mu)}(\mathbf{m})\,,$$

etc. Finally, to annihilate $\Gamma^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)}(\mathbf{m})$ we must use $n = k+2$ and put

$$U^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} := \Gamma^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)}(\mathbf{m})\,.$$

As noticed in the previous proof, the subsequent steps do not spoil the result of annihilation obtained in the previous step. However, the transformation number n changes the value of $\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_l}(\mathbf{m})$ for $l \geq n-2$. It is easy to see that this change produces an extra term depending upon derivatives of Γ of order lower than l. We conclude that after the transformation number $n = k+1$ the value of $\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m})$ differs from its original value by an expression containing derivatives of Γ of order lower that k. Finally, transformation number $n = k+2$ produces the change described by formula (28). Hence, the final value of $\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m})$ is given by the right hand side of (32). This ends the proof because the tensorial transformation between (y^α) and (x^λ) at $\mathbf{m}$ is a unit matrix, see (25). □

Example 1. The curvature tensor of order $k = 1$ is simply the curvature tensor *tout court*:

$$K^\lambda_{\kappa\sigma\mu} = \Gamma^\lambda_{\kappa\sigma\mu} - \Gamma^\lambda_{(\kappa\sigma\mu)} + \Gamma^\lambda_{\gamma\mu}\Gamma^\gamma_{\kappa\sigma} - \Gamma^\lambda_{\gamma(\mu}\Gamma^\gamma_{\kappa\sigma)}\,. \tag{33}$$

Identity (30) is the 1-st type Bianchi identity.

Example 2. The curvature tensor of order $k = 2$ equals:

$$\begin{aligned} K^\lambda_{\kappa\sigma\mu\nu} &= \frac{5}{8}\nabla_\nu K^\lambda_{\kappa\sigma\mu} + \frac{5}{8}\nabla_\mu K^\lambda_{\kappa\sigma\nu} - \frac{1}{8}\nabla_\sigma K^\lambda_{\mu\nu\kappa} - \frac{1}{8}\nabla_\kappa K^\lambda_{\mu\nu\sigma} \\ &= \Gamma^\lambda_{\kappa\sigma\mu\nu} - \Gamma^\lambda_{(\kappa\sigma\mu\nu)} + 2\Gamma^\gamma_{\kappa\sigma(\nu}\Gamma^\lambda_{\mu)\gamma} - 2\Gamma^\gamma_{(\kappa\sigma\nu}\Gamma^\lambda_{\mu)\gamma} + \Gamma^\gamma_{(\kappa\sigma}\Gamma^\lambda_{\mu\nu)\gamma} \\ &\quad - \Gamma^\gamma_{\mu\nu}\Gamma^\lambda_{\kappa\sigma\gamma} + 4\Gamma^\lambda_{\gamma(\kappa\sigma}\Gamma^\gamma_{\mu\nu)} - 2\Gamma^\lambda_{\gamma\kappa(\mu}\Gamma^\gamma_{\nu)\sigma} - 2\Gamma^\lambda_{\gamma\sigma(\mu}\Gamma^\gamma_{\nu)\kappa} \\ &\quad + \Gamma^\lambda_{\gamma\alpha}\left(\Gamma^\gamma_{(\kappa\sigma}\Gamma^\alpha_{\mu\nu)} - \Gamma^\gamma_{\mu\nu}\Gamma^\alpha_{\kappa\sigma}\right) + 4\Gamma^\lambda_{\gamma(\kappa}\Gamma^\alpha_{\sigma\mu}\Gamma^\gamma_{\nu)\alpha} - 4\Gamma^\lambda_{\gamma(\mu}\Gamma^\alpha_{\nu)(\kappa}\Gamma^\gamma_{\sigma)\alpha}\,, \end{aligned} \tag{34}$$

see proof in Appendix A. Formula (A.1) of this proof implies the following expression for first order covariant derivatives of the curvature tensor:

$$K^{\lambda}_{\kappa\sigma\nu|\mu} = \nabla_{\mu} K^{\lambda}_{\kappa\sigma\nu} = K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{(\kappa\sigma\nu)\mu} = \frac{1}{3}\left(2K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{\nu\sigma\kappa\mu} - K^{\lambda}_{\nu\kappa\sigma\mu}\right) . \tag{35}$$

Using it we can calculate covariant derivatives of the Riemann tensor $R^{\lambda}_{\kappa\sigma\nu} = -2K^{\lambda}_{\kappa[\sigma\nu]}$:

$$R^{\lambda}_{\kappa\sigma\nu|\mu} = \nabla_{\mu} R^{\lambda}_{\kappa\sigma\nu} = -K^{\lambda}_{\kappa\sigma\nu\mu} + K^{\lambda}_{\kappa\nu\sigma\mu} . \tag{36}$$

Corollary. *Above formula is equivalent to the 2-nd type Bianchi identity:*

$$R^{\lambda}_{\kappa[\sigma\nu|\mu]} = \frac{1}{3}\left(R^{\lambda}_{\kappa\sigma\nu|\mu} + R^{\lambda}_{\kappa\mu\sigma|\nu} + R^{\lambda}_{\kappa\nu\mu|\sigma}\right) = -K^{\lambda}_{\kappa(\sigma\nu\mu)} + K^{\lambda}_{\kappa(\nu\sigma\mu)} \equiv 0 . \tag{37}$$

Definition. In case of an arbitrary order k, identity (30) will be called the "Bianchi identity of order k".

It turns out that these identities contain all possible identities satisfied by covariant derivatives of the curvature (i.e. also Riemann) tensor. This is due to the following

Theorem. *Every k-th order covariant derivative of the curvature tensor $K^{\lambda}_{\kappa\sigma\mu}$ can be expressed as a (non-linear) combination of the components of the curvature tensors $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$ up to order $l = (k+1)$.*

Proof. Taking into account formula (33), the derivative $\nabla_{\mu_k}\cdots\nabla_{\mu_1}K^{\lambda}_{\kappa\sigma\mu}$ is a (non-linear) combination of the connection coefficients $\Gamma^{\lambda}_{\kappa\sigma}$ and their partial derivatives up to order $l = (k+1)$. When calculated in an inertial coordinate system of order l at the point $\mathbf{m} \in M$, these derivatives become curvature tensors (see (24)). Hence, the thesis has been proved in inertial coordinates. Once established in inertial coordinates, this tensorial identity remains valid universally. □

4. Covariant jets of a connection

In the previous section we have used transformations of coordinates:

$$y^{\lambda} = x^{\lambda} + \frac{1}{n!}U^{\lambda}_{\beta_1\ldots\beta_n} \cdot x^{\beta_1}\ldots x^{\beta_n} , \tag{38}$$

for $n = 2, 3, \ldots$ and noticed that for $k = n-2$ they imply the change (28), whereas for $k < n-2$ the quantities $\Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}(\mathbf{m})$ do not change. Since $U^{\lambda}_{\beta_1\ldots\beta_n}$ is totally symmetric in lower indices (or, at least, only its totally symmetric part survives in formula (38)), this observation implies the following simple

Corollary. *Given a symmetric connection $\Gamma^{\lambda}_{\mu\nu}$ in a neighborhood of $\mathbf{m} \in M$ and a natural number k, there is a coordinate transformation such that the coefficients $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \ldots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$ assume an arbitrarily chosen value at $\mathbf{m}$.*

Such transformations play role of *gauge transformations*. Of course, they do not change tensorial quantities like curvature coefficients $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$ at $\mathbf{m} = (0, \ldots, 0)$.

Theorem. *At each spacetime point $\mathbf{m} \in M$ the quantities*

$$\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}; \Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \ldots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$$

can be used as global coordinates in the fiber $J^k_{\mathbf{m}}\mathcal{R}(M)$ of the bundle $J^k\mathcal{R}(M)$ of k-th jets of sections of the bundle $\mathcal{R}(M)$ of local reference frames in spacetime M. (Sections of this bundle are simply symmetric connections in M.)

In other words: k-th jets of a connection $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{\kappa\sigma\mu}, \ldots, \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$ split in a natural way into classes of gauge-equivalent jets. Each class is uniquely characterized by the collection of curvature tensors $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$. Within each class quantities $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \ldots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$ can be used as global coordinates. They play role of "gauge parameters", as they can be arbitrarily changed by coordinate transformations (38). Hence, the complete "gauge invariant" information about the k-th jet of a connection is carried by the curvature tensors $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)$. Collection of curvature tensors up to degree k can be, therefore, treated as the k-th "covariant jet" of Γ. But tensors do not change under above gauge transformation. Hence, a combination of Γ's and their derivatives which turns out to be a *tensor* cannot depend upon gauge parameters. This observation can be summarized as follows:

Proposition. *A tensor-valued function F on the jet bundle $J^k\mathcal{R}(M)$ does not depend upon gauge parameters $\left(\Gamma^{\lambda}_{\kappa\sigma}, \Gamma^{\lambda}_{(\kappa\sigma\mu)}, \ldots, \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)}\right)$, i.e. is a function of curvature tensors $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$, $l = 0, 1, \ldots, k$, exclusively.*

In particular, consider the *affine* variational principle for gravitational field. It is a first order variational principle for the connection Γ. Its equivalence with the metric variational principle, based on the Hilbert Lagrangian function, was proved long ago (see [9] or [11] for an easy, heuristic introduction). The affine Lagrangian function depends upon the first jet of the connection. However, to be gauge invariant, it must be constant on classes of gauge-equivalent jets, i.e. must be a function of the curvature: $K^{\lambda}_{\kappa\sigma\mu}$ or, equivalently, $R^{\lambda}_{\kappa\sigma\mu}$. We see that the same is true for a higher order affine variational formula: its Lagrangian function may depend upon the covariant jet of a connection, exclusively.

5. Main result

Above considerations lead us to the main result of this paper:

Theorem. *Every higher order curvature tensor $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$, $k \geq 1$, can be expressed as a (non-linear) combination of the (1-st order) curvature tensor $K^{\lambda}_{\kappa\sigma\mu}$ and its covariant derivatives $\nabla_{\mu_1}\ldots\nabla_{\mu_l}K^{\lambda}_{\kappa\sigma\mu}$ up to order $l = k-1$. Moreover, this expression is linear with respect to the highest order derivatives (i.e. those of order $l = k-1$).*

The thesis is based on the following lemma, whose proof is highly technical, although it does not go beyond the linear algebra and is presented in Appendix B:

Lemma. *For every $k \geq 1$ there is a unique, linear combination $S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ of covariant derivatives $\nabla_{\mu_1}\ldots\nabla_{\mu_{k-1}}K^{\lambda}_{\kappa\sigma\mu_k}$, such that*

$$S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \Gamma^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)} + f(\text{lower order derivatives of } \Gamma)\,, \tag{39}$$

in an arbitrary (not necessarily inertial!) system of coordinates (x^{λ}).

Proof of the Theorem. Suppose that the thesis is true for $k-1$, like it is for $k = 1$. Due to formula (32), the tensor

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\,, \tag{40}$$

contains Γ's and their derivatives of order $l \leq k-1$. Hence, due to proposition, it is a function of parameters $\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_{k-1}}\right)$. Hence, the thesis is true also for k. □

Example 3. The curvature tensor of order $k = 3$ is equal to:

$$\begin{aligned} K^{\lambda}_{\kappa\sigma\mu\nu\gamma} = &\frac{3}{40}\left[6\left(\nabla_{(\gamma}\nabla_{\nu)}K^{\lambda}_{\kappa\sigma\mu} + \nabla_{(\mu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\nu} + \nabla_{(\nu}\nabla_{\mu)}K^{\lambda}_{\kappa\sigma\gamma}\right)\right.\\ &- \left(\nabla_{(\gamma}\nabla_{\sigma)}K^{\lambda}_{\mu\nu\kappa} + \nabla_{(\kappa}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\sigma} + \nabla_{(\mu}\nabla_{\sigma)}K^{\lambda}_{\nu\gamma\kappa}\right.\\ &\left.\left.+\nabla_{(\kappa}\nabla_{\mu)}K^{\lambda}_{\nu\gamma\sigma} + \nabla_{(\nu}\nabla_{\sigma)}K^{\lambda}_{\gamma\mu\kappa} + \nabla_{(\kappa}\nabla_{\nu)}K^{\lambda}_{\gamma\mu\sigma}\right)\right]\\ &- \frac{3}{80}\left[23\left(K^{\lambda}_{\kappa\alpha(\gamma}K^{\alpha}_{\mu\nu)\sigma} + K^{\lambda}_{\sigma\alpha(\gamma}K^{\alpha}_{\mu\nu)\kappa}\right) - 37K^{\alpha}_{\kappa\sigma(\mu}K^{\lambda}_{\nu\gamma)\alpha}\right.\\ &\left.-2\left(K^{\alpha}_{(\mu\nu)\kappa}K^{\lambda}_{(\gamma)\alpha\sigma} + K^{\alpha}_{(\mu\nu)\sigma}K^{\lambda}_{(\gamma)\alpha\kappa}\right)\right]\,. \end{aligned} \tag{41}$$

Using formula (B.16) (see Appendix B), the above identity can be proved by long but straightforward calculations.

6. Conclusions and discussion

The mathematical object introduced in this paper, namely the "local reference frame" realizes, in our opinion, the fundamental intuition of Albert Einstein concerning the active role of space and time in physics. Einstein himself repeated this idea many times, in many scientific papers and newspaper articles, using the only mathematical structure which was at that time available to him, namely the Levi-Civita metric connection or its slight generalizations due to Weyl or Cartan (cf. [4]). Nowadays, we understand that the connection is an autonomous, geometric object, and not necessarily a property of a metric structure. In mathematics, "linear connection" was rigorously defined by J. L. Koszul or, equivalently, by K. Nomizu as "connection in a principal fiber bundle" (inducing immediately linear connections in its associated bundles, cf. [15]). When applied to spacetime and its tangent or cotangent bundles, these definitions lead to the notion of a non-symmetric connection, which is not an irreducible object. This is due to the fact, that the tangent bundle (or, in general, any tensorial bundle) carries an additional structure, called the *solder form*. Due to this structure, the non-symmetric connection splits canonically into two different, irreducible objects: a symmetric connection and a certain tensorial field called *torsion*. In this paper we have defined symmetric connection as a field of *inertial frames*. This definition is, in our opinion, an adequate, mathematically rigorous formulation of the original idea of A. Einstein.

On the other hand, physical theories of gravitation based on non-symmetric connection have been considered already in 20'. They are known as the *Einstein–Cartan–Kibble theories* or theories "with torsion and spin". (These theories were later analyzed by Sciama, Kibble, Trautman and others, see [1] and the references herein). Our approach can be easily applied to such theories if we only acknowledge that variation of the Lagrangian with respect to a non-symmetric connection is equivalent to the variation with respect to the two independent objects: 1) the symmetric connection (i.e. with respect to the field of inertial frames) and 2) the tensor field representing torsion of the connection. These results will be published soon.

Acknowledgments

This research was supported in part by Narodowe Centrum Nauki (Poland) under Grant No. 2016/21/B/ST1/00940.

Appendix A. Second order curvature tensor

To show that formula (34) is correct, we claim that there exists a unique tensor $K^{\lambda}_{\kappa\sigma\mu\nu}$ fulfilling the following identity:

$$\nabla_{\mu}K^{\lambda}_{\kappa\sigma\nu} = \frac{1}{3}\left(2K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{\nu\sigma\kappa\mu} - K^{\lambda}_{\nu\kappa\sigma\mu}\right) , \tag{A.1}$$

and that this tensor is precisely the second order curvature tensor $K^{\lambda}_{\kappa\sigma\nu\mu}$, as defined by (34).

Proof. In order to prove uniqueness, observe that the symmetrization of (A.1) in the indices μ and ν gives us:

$$3\nabla_{\mu)}K^{\lambda}_{\kappa\sigma(\nu} = 2K^{\lambda}_{\kappa\sigma\nu\mu} - \frac{1}{2}K^{\lambda}_{\nu\sigma\kappa\mu} - \frac{1}{2}K^{\lambda}_{\nu\kappa\sigma\mu} - \frac{1}{2}K^{\lambda}_{\mu\sigma\kappa\nu} - \frac{1}{2}K^{\lambda}_{\mu\kappa\sigma\nu} . \tag{A.2}$$

But, taking into account the symmetry properties of K, we obtain:

$$0 = K^{\lambda}_{(\kappa\sigma\nu\mu)} = \frac{1}{24}\left(4K^{\lambda}_{\kappa\sigma\nu\mu} + 4K^{\lambda}_{\nu\mu\kappa\sigma} + 4K^{\lambda}_{\kappa\mu\sigma\nu} + 4K^{\lambda}_{\kappa\nu\sigma\mu} + 4K^{\lambda}_{\mu\sigma\kappa\nu} + 4K^{\lambda}_{\nu\sigma\kappa\mu}\right) ,$$

or, equivalently

$$K^{\lambda}_{\kappa\mu\sigma\nu} + K^{\lambda}_{\kappa\nu\sigma\mu} + K^{\lambda}_{\mu\sigma\kappa\nu} + K^{\lambda}_{\nu\sigma\kappa\mu} = -K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{\nu\mu\kappa\sigma} . \tag{A.3}$$

Plugging this into (A.2) we obtain

$$3\nabla_{\mu)}K^{\lambda}_{\kappa\sigma(\nu} = \frac{5}{2}K^{\lambda}_{\kappa\sigma\nu\mu} + \frac{1}{2}K^{\lambda}_{\nu\mu\kappa\sigma} . \tag{A.4}$$

Exchanging $(\kappa\sigma)$ with $(\nu\mu)$ we obtain:

$$3\nabla_{\kappa)}K^{\lambda}_{\mu\nu(\sigma} = \frac{5}{2}K^{\lambda}_{\nu\mu\kappa\sigma} + \frac{1}{2}K^{\lambda}_{\kappa\sigma\nu\mu} , \tag{A.5}$$

and, whence:

$$3\nabla_{\mu)}K^{\lambda}_{\kappa\sigma(\nu} - \frac{3}{5}\nabla_{\kappa)}K^{\lambda}_{\mu\nu(\sigma} = \left(\frac{5}{2} - \frac{1}{10}\right)K^{\lambda}_{\kappa\sigma\nu\mu} = \frac{12}{5}K^{\lambda}_{\kappa\sigma\nu\mu} , \tag{A.6}$$

which is precisely (34).

In order to prove existence, it suffices to show that the tensor

$$N^{\lambda}_{\kappa\sigma\nu\mu} := 3\nabla_{\mu}K^{\lambda}_{\kappa\sigma\nu} - \left(2K^{\lambda}_{\kappa\sigma\nu\mu} - K^{\lambda}_{\nu\sigma\kappa\mu} - K^{\lambda}_{\nu\kappa\sigma\mu}\right) , \tag{A.7}$$

where K is defined by (34), vanishes identically.

To this end, observe that (A.7) is symmetric in the first pair of indices: $N^{\lambda}_{(\kappa\sigma)\nu\mu} = N^{\lambda}_{\kappa\sigma\nu\mu}$, and fulfills Bianchi identities of the first and the second type

$$N^{\lambda}_{(\kappa\sigma\nu)\mu} = 0 \;\; ; \;\; N^{\lambda}_{\kappa[\sigma\nu\mu]} = 0 . \tag{A.8}$$

But we know already that the symmetric part $N^{\lambda}_{\kappa\sigma(\nu\mu)}$ of N vanishes. Hence, N is antisymmetric in the last two indices:

$$N^{\lambda}_{\kappa\sigma[\nu\mu]} = N^{\lambda}_{\kappa\sigma\nu\mu} .$$

We shall prove that N must vanish identically. Indeed, first Bianchi identity (due to symmetry) reads:

$$N^{\lambda}_{\kappa\sigma\nu\mu} + N^{\lambda}_{\nu\kappa\sigma\mu} + N^{\lambda}_{\sigma\nu\kappa\mu} = 0 ,$$

whereas second Bianchi identity (due to antisymmetry) reads:

$$N^{\lambda}_{\kappa\sigma\nu\mu} + N^{\lambda}_{\kappa\mu\sigma\nu} + N^{\lambda}_{\kappa\nu\mu\sigma} = 0 .$$

Their combination gives us:

$$N^{\lambda}_{\nu\kappa\sigma\mu} + N^{\lambda}_{\sigma\nu\kappa\mu} = N^{\lambda}_{\kappa\mu\sigma\nu} + N^{\lambda}_{\kappa\nu\mu\sigma} ,$$

or, due to symmetry in first two indices and antisymmetry in last indices,

$$N^{\lambda}_{\sigma\nu\kappa\mu} - N^{\lambda}_{\kappa\mu\sigma\nu} = 2N^{\lambda}_{\kappa\nu\mu\sigma} .$$

Exchanging $(\kappa \leftrightarrow \sigma)$ and $(\mu \leftrightarrow \nu)$ we obtain:

$$N^{\lambda}_{\kappa\mu\sigma\nu} - N^{\lambda}_{\sigma\nu\kappa\mu} = 2N^{\lambda}_{\sigma\mu\nu\kappa} ,$$

and the sum of these identities implies:

$$0 = N^{\lambda}_{\kappa\nu\mu\sigma} + N^{\lambda}_{\sigma\mu\nu\kappa} .$$

Finally, symmetrization in κ and ν gives us the thesis:

$$0 = N^{\lambda}_{(\kappa\nu)\mu\sigma} + N^{\lambda}_{\sigma\mu(\nu\kappa)} = N^{\lambda}_{\kappa\nu\mu\sigma} . \quad \square$$

Appendix B. Algebra of highest order derivatives of a connection

Proof of Lemma.

$$\begin{aligned}\nabla_{\mu_2}\cdots\nabla_{\mu_k}K^{\lambda}_{\kappa\sigma\mu_1} &= \partial_{\mu_2}\cdots\partial_{\mu_k}\left(\Gamma^{\lambda}_{\kappa\sigma\mu_1} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1)}\right) + f(l.o.d.\ \Gamma) \\ &= \Gamma^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} - \frac{1}{3}\left(\Gamma^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} + \Gamma^{\lambda}_{\mu_1\kappa\sigma\mu_2\cdots\mu_k} + \Gamma^{\lambda}_{\sigma\mu_1\kappa\mu_2\cdots\mu_k}\right) + f(l.o.d.\ \Gamma) \\ &= \frac{2}{3}\Gamma^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} - \frac{1}{3}\left(\Gamma^{\lambda}_{\kappa\mu_1\sigma\mu_2\cdots\mu_k} + \Gamma^{\lambda}_{\sigma\mu_1\kappa\mu_2\cdots\mu_k}\right) + f(l.o.d.\ \Gamma) ,\end{aligned}$$

where "*l.o.d.*" means "lower order derivatives of". Because the totally symmetric part of both sides vanishes, we can replace every $\Gamma^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k}$ by its "totally-symmetric-free" part, which we denote by:

$$\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} = \Gamma^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} - \Gamma^{\lambda}_{(\kappa\sigma\mu_1\mu_2\cdots\mu_k)} , \tag{B.1}$$

which gives us, *modulo* lower order terms on both sides:

$$\partial_{\mu_2}\cdots\partial_{\mu_k}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1} = \frac{2}{3}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} - \frac{1}{3}\left(\mathcal{K}^{\lambda}_{\kappa\mu_1\sigma\mu_2\cdots\mu_k} + \mathcal{K}^{\lambda}_{\sigma\mu_1\kappa\mu_2\cdots\mu_k}\right) . \tag{B.2}$$

We see that our problem has been reduced to the following linear problem: is it possible for $k > 1$ to solve above system of equations with respect to quantities $\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2\cdots\mu_k}$, if all the derivatives $\partial_{\mu_2}\cdots\partial_{\mu_k}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1}$ are known?

To prove solvability we define:

$$\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} := \mathrm{sym}_{\mu_1\cdots\mu_k}\left(\partial_{\mu_2}\cdots\partial_{\mu_k}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1}\right) = \mathcal{K}^{\lambda}_{\kappa\sigma(\mu_1,\,\mu_2\cdots\mu_k)} . \tag{B.3}$$

Observe that $\mathcal{L}$ and $\mathcal{K}$ satisfy the same identities as the curvature tensors do, i.e. (29) and (30). Combining the definition with (B.2) we obtain:

$$\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \frac{2}{3}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \frac{1}{3k}\left(\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\sigma} + \sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\kappa}\right) . \tag{B.4}$$

But, identity (30) reads:

$$0 = \frac{(k+2)(k+1)}{2}\mathcal{K}^{\lambda}_{(\kappa\sigma\mu_1\cdots\mu_k)} = \tag{B.5}$$

$$= \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\sigma} + \sum_{j=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\kappa} + \sum_{i<j}\mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} \tag{B.6}$$

and, consequently:

$$-\left(\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\sigma} + \sum_{j=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\kappa}\right) = \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} , \tag{B.7}$$

where $\mathcal{M}$ has been defined as

$$\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} := \sum_{i<j}\mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} . \tag{B.8}$$

Inserting (B.7) into (B.4) we obtain:

$$\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \left(\frac{2}{3} + \frac{1}{3k}\right)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \frac{1}{3k}\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} . \tag{B.9}$$

$$3k\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = (2k+1)\,\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} . \tag{B.10}$$

In the next step we are going to prove another identity for the same quantities, namely:

$$3k \sum_{i<j} \mathcal{L}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} = (k-1)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + (k+3)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} . \tag{B.11}$$

Comparing the two identities we can eliminate $\mathcal{M}$'s. This way we obtain the final result:

$$3k \left((k+3)\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \sum_{i<j} \mathcal{L}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} \right) = 2(k+1)(k+2)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} , \tag{B.12}$$

or, equivalently,

$$\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} = \frac{3k}{2(k+1)(k+2)} \left((k+3)\mathcal{L}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} - \sum_{i<j} \mathcal{L}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} \right) . \tag{B.13}$$

What remains now is the proof of identity (B.11). For this purpose we apply the procedure "$\sum_{i<j}$" to Eq. (B.10) and obtain:

$$3k \sum_{i<j} \mathcal{L}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} = (2k+1)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \sum_{i<j} \mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} . \tag{B.14}$$

We are going to prove that the last quantity is a combination of $\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$ and $\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$. Given two indices $i < j$, we have according to (B.8):

$$\begin{aligned}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} &= \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k} + \sum_{i\neq l\neq j} \mathcal{K}^{\lambda}_{\kappa\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\sigma} \\ &\quad + \sum_{i\neq l\neq j} \mathcal{K}^{\lambda}_{\sigma\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\kappa} + \sum_{\substack{l,n\notin\{i,j\}\\ l<n}} \mathcal{K}^{\lambda}_{\mu_l\mu_n\mu_1\cdots\underset{l}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} .\end{aligned}$$

But:

$$\sum_{i\neq l\neq j} \mathcal{K}^{\lambda}_{\kappa\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\sigma} = \sum_{l=1}^{k} \mathcal{K}^{\lambda}_{\kappa\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\sigma} - \mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\sigma} - \mathcal{K}^{\lambda}_{\kappa\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\sigma}$$

$$\sum_{i\neq l\neq j} \mathcal{K}^{\lambda}_{\sigma\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\kappa} = \sum_{l=1}^{k} \mathcal{K}^{\lambda}_{\sigma\mu_l\mu_1\cdots\underset{l}{\wedge}\cdots\mu_k\kappa} - \mathcal{K}^{\lambda}_{\sigma\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\kappa} - \mathcal{K}^{\lambda}_{\sigma\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\kappa}$$

Finally:

$$\begin{aligned}\sum_{\substack{l,n\notin\{i,j\}\\ l<n}} \mathcal{K}^{\lambda}_{\mu_l\mu_n\mu_1\cdots\underset{l}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} &= \sum_{l<n} \mathcal{K}^{\lambda}_{\mu_l\mu_n\mu_1\cdots\underset{l}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} \\ &\quad - \sum_{n\neq i} \mathcal{K}^{\lambda}_{\mu_i\mu_n\mu_1\cdots\underset{i}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} + \mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} \\ &\quad - \sum_{l\neq j} \mathcal{K}^{\lambda}_{\mu_l\mu_j\mu_1\cdots\underset{l}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} + \mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} \\ &\quad - \mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} ,\end{aligned}$$

and, whence:

$$\begin{aligned}\sum_{\substack{l,n\notin\{i,j\}\\ l<n}} \mathcal{K}^{\lambda}_{\mu_l\mu_n\mu_1\cdots\underset{l}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} &= \sum_{l<n} \mathcal{K}^{\lambda}_{\mu_l\mu_n\mu_1\cdots\underset{l}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} - \sum_{n\neq i} \mathcal{K}^{\lambda}_{\mu_i\mu_n\mu_1\cdots\underset{i}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} \\ &\quad - \sum_{l\neq j} \mathcal{K}^{\lambda}_{\mu_l\mu_j\mu_1\cdots\underset{l}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} + \mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} .\end{aligned}$$

Using identity (B.6) we obtain:

$$\begin{aligned}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} &= -\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\sigma} - \mathcal{K}^{\lambda}_{\kappa\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\sigma} \\ &\quad - \mathcal{K}^{\lambda}_{\sigma\mu_i\mu_1\cdots\underset{i}{\wedge}\cdots\mu_k\kappa} - \mathcal{K}^{\lambda}_{\sigma\mu_j\mu_1\cdots\underset{j}{\wedge}\cdots\mu_k\kappa} \\ &\quad - \sum_{n\neq i} \mathcal{K}^{\lambda}_{\mu_i\mu_n\mu_1\cdots\underset{i}{\wedge}\cdots\underset{n}{\wedge}\cdots\mu_k\kappa\sigma} - \sum_{l\neq j} \mathcal{K}^{\lambda}_{\mu_l\mu_j\mu_1\cdots\underset{l}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma} + \mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\underset{i}{\wedge}\cdots\underset{j}{\wedge}\cdots\mu_k\kappa\sigma}\end{aligned}$$

and, whence

$$\begin{aligned}\sum_{i<j}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma} &= \frac{1}{2}\sum_{i\neq j}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\\ &= -(k-1)\left(\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\hat{i}\cdots\mu_k\sigma}+\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_i\mu_1\cdots\hat{i}\cdots\mu_k\kappa}\right)\\ &\quad -(k-1)\sum_{n\neq i}\mathcal{K}^{\lambda}_{\mu_i\mu_n\mu_1\cdots\hat{i}\cdots\hat{n}\cdots\mu_k\kappa\sigma}+\sum_{i<j}\mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\\ &= -(k-1)\left(\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\hat{i}\cdots\mu_k\sigma}+\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_i\mu_1\cdots\hat{i}\cdots\mu_k\kappa}\right)\\ &\quad -(2k-3)\sum_{i<j}\mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\end{aligned}$$

Using identity (B.7):

$$-\left(\sum_{i=1}^{k}\mathcal{K}^{\lambda}_{\kappa\mu_i\mu_1\cdots\hat{i}\cdots\mu_k\sigma}+\sum_{j=1}^{k}\mathcal{K}^{\lambda}_{\sigma\mu_j\mu_1\cdots\hat{j}\cdots\mu_k\kappa}\right)=\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}+\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}$$

we obtain:

$$\begin{aligned}\sum_{i<j}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma} &= (k-1)\left(\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}+\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\right)-(2k-3)\sum_{i<j}\mathcal{K}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\\ &= (k-1)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}-(k-2)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\,.\end{aligned}$$

Putting this into formula (B.14) we obtain:

$$\begin{aligned}3k\sum_{i<j}\mathcal{L}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma} &= (2k+1)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}+\sum_{i<j}\mathcal{M}^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\\ &= (2k+1)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}+(k-1)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}-(k-2)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\\ &= (k-1)\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}+(k+3)\mathcal{M}^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}\,,\end{aligned}$$

which ends the proof of identity (B.11) and, consequently, of the final formula (B.13).

To complete formally proof of the lemma we define

$$L^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}:=\mathrm{sym}_{\mu_1\cdots\mu_k}\left(\nabla_{\mu_2}\cdots\nabla_{\mu_k}K^{\lambda}_{\kappa\sigma\mu_1}\right)\,, \tag{B.15}$$

analogous to (B.3) and

$$S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}=\frac{3k}{2(k+1)(k+2)}\left((k+3)L^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_k}-\sum_{i<j}L^{\lambda}_{\mu_i\mu_j\mu_1\cdots\hat{i}\cdots\hat{j}\cdots\mu_k\kappa\sigma}\right)\,, \tag{B.16}$$

analogous to the right hand side of (B.13), which implies the thesis (39).

Remark. Both formulae (34) for $k=2$ and (41) for $k=3$, follow from (B.16).

Appendix C. Polarization theorem for multilinear forms

Theorem. *Let $B: V^k \to \mathbb{R}$ be a totally symmetric, k-linear form on a vector space V. B vanishes on the diagonal if and only if it vanishes identically:*

$$B\underbrace{(u,u,\ldots,u)}_{k\ \text{times}}\equiv 0 \qquad\Longleftrightarrow\qquad B(u_1,u_2,\ldots,u_k)\equiv 0$$

for an arbitrary set of k vectors $u_1,u_2,\ldots,u_k$.

Example. For $k=3$ the thesis is implied by the following polarization formula:

$$\begin{aligned}6B(x,y,z) &= B(x+y+z,x+y+z,x+y+z)\\ &\quad -B(x+y,x+y,x+y)-B(x+z,x+z,x+z)-B(y+z,y+z,y+z)\\ &\quad +B(x,x,x)+B(y,y,y)+B(z,z,z)\,,\end{aligned}$$

in strict analogy with the algebraic identity:

$$6xyz = (x+y+z)^3 - (x+y)^3 - (x+z)^3 - (y+z)^3 + x^3 + y^3 + z^3 \ .$$

In case of an arbitrary k the proof follows the same pattern.

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Rozdział 7

Praca: On the remarkable universality of Einstein's gravity theory

International Journal of Geometric Methods in Modern Physics
Vol. 19, No. 7 (2022) 2250111 (24 pages)

DOI: 10.1142/S0219887822501110

On the remarkable universality of Einstein's gravity theory

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Received 24 December 2021
Accepted 7 March 2022
Published 7 April 2022

In this paper, we prove that the gravity theory based on a generalized Lagrangian density, which depends in a nonlinear way not only upon the entire curvature tensor, but also upon its covariant derivatives up to a fixed order n, is equivalent to the standard Einstein theory of gravitational field, maybe interacting with additional matter fields.

Keywords: Curvature; Riemann tensor; theory of connection; Jet theory; general relativity theory; variational principles.

Mathematics Subject Classification 2020: 53A45, 53A55, 83C40, 83D05

1. Introduction

To find an appropriate description of the "dark matter" and/or "dark energy", many authors consider recently generalizations of General Relativity Theory based on "nonlinear Lagrangians", i.e. Lagrangian functions depending in a nonlinear way upon the curvature tensor. The particular case of a Lagrangian L depending nonlinearly upon the scalar curvature R or even the complete Ricci tensor $R_{\mu\nu}$, but not depending upon the Weyl tensor, was considered by many authors (cf. [1] and [2]). Probably, the first, physically well-motivated proposal of such a theory was the nonlinear "R^2"-Lagrangian proposed by Sacharov (see [3]). It belongs to the family of "$f(R)$-theories", where $L = \sqrt{|g|}f(R)$ (see also [4]).

Mathematically, these theories are equivalent to conventional version of General Relativity Theory, interacting with extra matter fields. For $f(R)$-theories, this equivalence was proved already in 1987 (see [5] and [6]). Later on, this result was extended to Lagrangians depending upon the Ricci tensor (see [6] and [7]) and, finally, to those which depend upon the complete Riemann curvature $R^{\lambda}_{\kappa\sigma\mu}$ (see [8]).

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In this paper, we prove this remarkable universality of the standard General Relativity Theory in a much broader context, namely when the Lagrangian function depends also upon higher-order covariant derivatives $\nabla_{\mu_k} \cdots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}$ of the curvature

$$L = L(R^{\lambda}_{\kappa\sigma\nu}, \nabla_{\mu} R^{\lambda}_{\kappa\sigma\nu}, \nabla_{\mu_2}\nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}, \ldots, \nabla_{\mu_n} \ldots \nabla_{\mu_1} R^{\lambda}_{\kappa\sigma\nu}). \tag{1}$$

The idea of the proof is based on the so-called "Palatini method" of variation, where the metric g and the connection Γ are treated as *a priori* independent geometric objects. Unless in case of the classical Hilbert Lagrangian density, where variation with respect to Γ produces the metricity equation

$$\nabla_{\lambda} g_{\mu\nu} = 0, \tag{2}$$

as one of the Euler–Lagrange equations and, consequently, implies the simple relation between metric and connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_{\nu} g_{\kappa\mu} + \partial_{\mu} g_{\kappa\nu} - \partial_{\kappa} g_{\mu\nu}), \tag{3}$$

in case of a generic Lagrangian density L depending upon metric, curvature tensor and (possibly) its covariant derivatives, the zero on the right-hand side of (2) is replaced by a complicated combination of these quantities and, whence, connection Γ is no longer the Levi-Civita metric connection (3). Naively, one could conclude, that the resulting theory is not equivalent to the original theory, when metricity of the connection is assumed *a priori*, i.e. when Γ is nothing but a shortcut combination (3) of the metric g and its derivatives. Such a conclusion is, however, false. Indeed, decomposing the nonmetric connection Γ into the metric connection $\overset{\circ}{\Gamma}$ and the remaining tensor field

$$\Gamma^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + N^{\lambda}_{\mu\nu},$$

we can rewrite the entire theory in terms of the metric tensor interacting with the new "matter fields" described by the non-metricity tensor field N. Moreover, we have been able to prove that the interaction between metric and this new matter, implied by the resulting theory, is that of the conventional Einstein theory. It turns out that in case of the Sacharov Lagrangian density (i.e. containing the term $c \cdot R^2$), as well as in case of any other "$f(R)$" theory, the entire information about N is carried by a single scalar field (see [5] and [6]) and, whence, such a theory is equivalent to the conventional Einstein theory, where the metric field interacts with a scalar matter field. The only non-universal aspect of the theory is the specific matter Lagrangian density of this new scalar field, uniquely implied by the original Lagrangian $f(R)$.

The above construction has been well described for a generic Lagrangian density depending upon metric, the entire curvature tensor $R^{\lambda}_{\kappa\sigma\mu}$ and, possibly, some matter fields in paper [8]. What remains is, therefore, to prove that the higher-order theory (i.e. theory with a Lagrangian density depending upon derivatives of the curvature) can be equivalently rewritten as a first-order theory (i.e. theory with a Lagrangian density depending upon the curvature and — possibly — additional "matter fields", without dependence upon derivatives of the curvature). Such a procedure of "lowering the differential order" of the theory, where the higher derivatives are treated as new variables, is standard in the analysis of partial differential equations. What is, however, new and original in our present approach is that such a "lowering order procedure" can be performed in a way which respects: (1) the variational character and: (2) the diffeomorphism-invariance of the theory.

2. Variational Principle as Symplectic Control Theory

Consider a Lagrangian density $L = L(\varphi^K, \varphi^K_\lambda)$ depending upon a one-parameter-family of fields

$$\varphi^K = \varphi^K(x^\mu, \epsilon),$$

where (x^μ) are spacetime coordinates and

$$\varphi^K_\lambda := \partial_\lambda \varphi^K.$$

Traditionally, derivative with respect to the parameter ϵ is denoted by δ

$$\delta := \frac{\mathrm{d}}{\mathrm{d}\epsilon}. \tag{4}$$

Operator δ obviously commutes with spacetime derivatives

$$\delta\varphi^K_\mu = \partial_\mu \delta\varphi^K,$$

(this simple observation is sometimes upgraded to "the fundamental lemma of the calculus of variations"). However, calculus of variations is based on the following identity:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \varphi^K}\delta\varphi^K + \frac{\partial L}{\partial \varphi^K_\lambda}\delta\varphi^K_\lambda \\ &= \left(\frac{\partial L}{\partial \varphi^K} - \partial_\lambda \frac{\partial L}{\partial \varphi^K_\lambda}\right)\delta\varphi^K + \partial_\lambda\left(\frac{\partial L}{\partial \varphi^K_\lambda}\delta\varphi^K\right), \end{aligned} \tag{5}$$

the first term of which will be referred to as the *volume part* and the second one as the *boundary part*. Traditionally, one neglects the boundary part because, when integrated over a volume $\mathcal{O}$ and imposing boundary conditions on its boundary $\partial\mathcal{O}$, we have $\delta\varphi^K|_{\partial\mathcal{O}} = 0$. This way one derives the Euler–Lagrange equations (second-order partial differential equations in variables φ^K)

$$\frac{\partial L}{\partial \varphi^K} - \partial_\lambda \frac{\partial L}{\partial \varphi^K_\lambda} = 0, \tag{6}$$

as the necessary condition for the extremum of the functional

$$\mathcal{F} := \int_V L. \tag{7}$$

For example, Misner, Thorne and Wheeler in their monograph [9], otherwise excellent as an introduction to General Relativity Theory, calculate the volume part of the variation of the Hilbert Lagrangian density and then provide the following statement (see [9], p. 520, just above their formula 21.86): "Variation of the geometry interior to the boundary makes no difference in the value of the surface term. Therefore, it has no influence on the equations of motion to drop the term (21.85)".

The term which is dropped there is precisely the surface term, which was never calculated in this monograph (cf. also [10] for more details).

This procedure — working perfectly for elliptic problems, where one *really* needs to "optimize" something (i.e. the potential energy of a freely hanging chain) — is entirely false in case of dynamical (i.e. hyperbolic) theories. Generations of theoretical physicists learned already that there is no extremum, but only a "saddle point" in the hyperbolic case ("integration over trajectories", or "Feynman integral" quantization method, is based precisely on this observation, which implies that the classical trajectory — i.e. the stationary point — gives the main contribution to the Feynman integral). But the very difficulty lies elsewhere! Namely: imposing boundary conditions is forbidden in hyperbolic theories. This means, that there is no solution of Eq. (6) for a generic choice of boundary data! To convince oneself that this is the case, it is sufficient to consider wave equation in a two-dimensional spacetime.

The simplest method to give meaning to classical procedures used in field theory is to work "on shell", i.e. to restrict oneself only to those jets of the field φ^K which fulfill field equations (6). This means, that — instead of neglecting the boundary part of (5) — we neglect its volume part. This way, (5) is no longer an identity but becomes an equation

$$\begin{aligned}\delta L(\varphi^K, \varphi^K_\lambda) &= \partial_\lambda \left(\frac{\partial L}{\partial \varphi^K_\lambda} \delta\varphi^K \right) \\ &= (\partial_\lambda p_K^\lambda)\delta\varphi^K + p_K^\lambda \, \delta\varphi^K_\lambda, \qquad (8)\end{aligned}$$

where the canonical momentum p_K^λ has been introduced as a shortcut notation for the following expression:

$$p_K^\lambda := \frac{\partial L}{\partial \varphi^K_\lambda}. \qquad (9)$$

We see that the system of first-order partial differential equations (8) for the variables (φ^K, p_K^λ) is equivalent to the second-order Euler–Lagrange equation (6), written in the following form:

$$\partial_\lambda p_K^\lambda = \frac{\partial L}{\partial \varphi^K}, \qquad (10)$$

together with definition (9) of momenta. This way, at each spacetime point $\mathbf{m} = (x^\mu)$, field equations (8) can be considered as a symplectic relation (i.e. a Lagrangian submanifold) in a symplectic space $\mathcal{P}_\mathbf{m}$ parameterized by the following "generalized jets" of fields: $(\varphi^K, \varphi^K_\lambda, p^\lambda_K, j_K := \partial_\lambda p^\lambda_K)$. Mathematically, this approach was rigorously defined in [11] and [12], but its strength consists in the fact that it is very well adapted for practical calculations in the canonical and Hamiltonian formalism, together with description of constraints. Practically, it is based on splitting canonical field variables into two groups: the "control parameters" (those which appear under the sign δ — in case of (8) these are configuration variables φ^K and the "velocities" φ^K_λ) and the "response parameters" (in case of (8) these are momenta p^λ_K and the "currents" j^K). Field equations are then treated as the "control-response relation".

This technique was informally present already in classical texts by Lagrange, Caratheodory and other pioneers of the calculus of variations and classical thermodynamics. As an example, consider the classical formula

$$\delta U(V,S) = -p\delta V + T\delta S \Leftrightarrow p = -\frac{\partial U}{\partial V}; \quad T = \frac{\partial U}{\partial S}, \tag{11}$$

which selects the two-dimensional subspace of all physically admissible states of a simple thermodynamical body, as a Lagrangian submanifold within a four dimensional symplectic manifold parameterized by (V, S, p, T) (volume, entrophy, pressure, temperature) and equipped with a canonical symplectic form

$$\omega = -\delta p \wedge \delta V + \delta T \wedge \delta S. \tag{12}$$

Similarly, classical mechanics can be formulated as a symplectic relation

$$\delta L(q,\dot{q}) = \frac{\mathrm{d}}{\mathrm{d}t}(p\delta q) = \dot{p}\delta q + p\delta\dot{q} \Leftrightarrow \dot{p} = \frac{\partial L}{\partial q}; \quad p = \frac{\partial L}{\partial \dot{q}}, \tag{13}$$

with respect to the canonical symplectic form

$$\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\delta p \wedge \delta q) = \delta\dot{p} \wedge \delta q + \delta p \wedge \delta\dot{q}. \tag{14}$$

Classical Legendre transformations: Transition from the Lagrangian to the Hamiltonian picture in mechanics, and transition from adiabatic to the thermostatic insulation in thermodynamics, are simply described in this formalism as an exchange between control and response parameters (T versus S in (12) and p versus $\dot{q}$ in (14)).

3. Higher-Order Variational Principles

Higher-order Lagrangian density

$$L = L(\varphi^K, \varphi^K_\mu, \varphi^K_{\mu_1\mu_2} \ldots, \varphi^K_{\mu_1\cdots\mu_n}), \tag{15}$$

where

$$\varphi^K_{\mu_1\cdots\mu_k} := \partial_{\mu_1}\cdots\partial_{\mu_k}\varphi^K, \tag{16}$$

with $k = 0, 1, 2, \ldots, n$, gives rise to the nth-order variational principle which, naively, can be formulated as a first-order variational principle, with the $(n-1)$th jet of the field φ taken as the configuration variable. Hence, independent field variables are $\varphi^K_{\mu_1\cdots\mu_k}$ with $k = 0, 1, 2, \ldots, n-1$. Following the "on-shell" techniques presented in Sec. 2, field equations can be derived from the following symplectic relation:

$$\begin{aligned}\delta L(\varphi^K, \ldots, \varphi^K_{\mu_1\cdots\mu_n}) &= \partial_\lambda(p^\lambda_K \delta\varphi^K + p^{\mu\lambda}_K \delta\varphi^K_\mu + \cdots + p^{\mu_1\cdots\mu_{n-1}\lambda}_K \delta\varphi^K_{\mu_1\cdots\mu_{n-1}}) \\ &= (\partial_\lambda p^\lambda_K)\delta\varphi^K + (p^\mu_K + \partial_\lambda p^{\mu\lambda}_K)\delta\varphi^K_\mu \\ &\quad + \cdots + (p^{\mu_1\cdots\mu_{n-1}}_K + \partial_\lambda p^{\mu_1\cdots\mu_{n-1}\lambda}_K)\delta\varphi^K_{\mu_1\cdots\mu_{n-1}} \\ &\quad + p^{\mu_1\cdots\mu_n}_K \delta\varphi^K_{\mu_1\cdots\mu_n},\end{aligned} \tag{17}$$

or, equivalently

$$p^{\mu_1\cdots\mu_n}_K = \frac{\partial L}{\partial\varphi^K_{\mu_1\cdots\mu_n}} \tag{18}$$

$$p^{\mu_1\cdots\mu_{n-1}}_K + \partial_\lambda p^{\mu_1\cdots\mu_{n-1}\lambda}_K = \frac{\partial L}{\partial\varphi^K_{\mu_1\cdots\mu_{n-1}}} \tag{19}$$

$$\cdots = \cdots \tag{20}$$

$$p^\mu_K + \partial_\lambda p^{\mu\lambda}_K = \frac{\partial L}{\partial\varphi^K_\mu} \tag{21}$$

$$\partial_\lambda p^\lambda_K = \frac{\partial L}{\partial\varphi^K}. \tag{22}$$

Inserting successively first equation into the second one, second into the third, etc., we finally obtain the Euler–Lagrange equation

$$0 = \frac{\partial L}{\partial\varphi^K} - \partial_\mu \frac{\partial L}{\partial\varphi^K_\mu} + \partial_{\mu_1}\partial_{\mu_2}\frac{\partial L}{\partial\varphi^K_{\mu_1\mu_2}} + \cdots + (-1)^n \partial_{\mu_1}\cdots\partial_{\mu_n}\frac{\partial L}{\partial\varphi^K_{\mu_1\cdots\mu_n}}, \tag{23}$$

together with the definition of all the conjugate momenta $(p^\mu_K, p^{\mu_1\mu_2}_K, \ldots, p^{\mu_1\cdots\mu_n}_K)$.

There is, however, one subtle point is this formulation: the corresponding "symplectic" structure is not symplectic because it is degenerate. However, this degeneracy can be simply removed *via* an appropriate symplectic reduction. This construction was mathematically rigorously analyzed in paper [13]. It was proved that the simplest way to perform this reduction is to impose the total symmetry to all the momentum coefficients

$$p^{\mu_1\cdots\mu_k}_K = p^{(\mu_1\cdots\mu_k)}_K. \tag{24}$$

A priori, only the highest momentum (i.e. for $k = n$) fulfills the total symmetry condition, implied by Eq. (18), whereas lower order momenta can contain also a non-symmetric part, which does not influence the field evolution (i.e. Euler–Lagrange equation (23) was derived without any symmetry condition). Hence, symmetry condition (24) plays role of a gauge condition, which enables us to represent *uniquely* physical state of the field in terms of positions and momenta. Generating formula (17) together with the gauge condition (24) define *uniquely* the canonical (Hamiltonian) formulation of a field theory defined by a higher-order variational principle. (cf. [13] for details).

In the theory of gravity, the role of configuration variable is played by the connection: $\varphi^K = \Gamma^\lambda_{\kappa\sigma}$. Consequently, the corresponding momenta $P_\lambda^{\kappa\sigma\mu_1\cdots\mu_k}$ will be symmetric in indices $(\mu_1\cdots\mu_k)$.

4. Lowering Order of the Theory

Every system of partial differential equations can be treated as a system of first differential order if we treat higher-order derivatives as new field variables. This trivial observation can also be applied to variational principles. For this purpose, it is sufficient to perform the Legendre transformation between highest order derivative and the highest order momentum. For this purpose, generating formula (17) can be rewritten as follows:

$$\delta L = (\partial_\lambda p^\lambda_K)\delta\varphi^K + (p^\mu_K + \partial_\lambda p^{\mu\lambda}_K)\delta\varphi^K_\mu + \cdots + p_K^{\mu_1\cdots\mu_{n-1}}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}} + \partial_\lambda(p_K^{\mu_1\cdots\mu_{n-1}\lambda}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}), \tag{25}$$

and the last term can be transformed as

$$\partial_\lambda(p_K^{\mu_1\cdots\mu_{n-1}\lambda}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}) = \delta[\partial_\lambda(p_K^{\mu_1\cdots\mu_{n-1}\lambda}\varphi^K_{\mu_1\cdots\mu_{n-1}})] - \partial_\lambda(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda}).$$

The complete differential

$$\delta[\partial_\lambda(p_K^{\mu_1\cdots\mu_{n-1}\lambda}\varphi^K_{\mu_1\cdots\mu_{n-1}})] = \delta(p_K^{\mu_1\cdots\mu_n}\varphi^K_{\mu_1\cdots\mu_n}) + \delta[(\partial_\lambda p_K^{\mu_1\cdots\mu_{n-1}\lambda})\varphi^K_{\mu_1\cdots\mu_{n-1}}] \tag{26}$$

can be put on the left-hand side and this way we obtain the new Lagrangian density. It is, however, worthwhile to perform this operation in two subsequent steps:

Step 1: We define an auxiliary generating function

$$\widetilde{L}(\varphi^K, \ldots, \varphi^K_{\mu_1\cdots\mu_{n-1}}, p_K^{\mu_1\cdots\mu_n}) := L(\varphi^K, \ldots, \varphi^K_{\mu_1\cdots\mu_n}) - p_K^{\mu_1\cdots\mu_n}\varphi^K_{\mu_1\cdots\mu_n}, \tag{27}$$

where the highest order derivatives $\varphi^K_{\mu_1\cdots\mu_n}$ on the right-hand side are defined by Eq. (18) as implicit functions of remaining variables of the function $\widetilde{L}$. After this, first step equation (17) gets transformed to the following form:

$$\delta\widetilde{L}(\varphi^K, \ldots, \varphi^K_{\mu_1\cdots\mu_{n-1}}, p_K^{\mu_1\cdots\mu_n}) = (\partial_\lambda p^\lambda_K)\delta\varphi^K + (p^\mu_K + \partial_\lambda p^{\mu\lambda}_K)\delta\varphi^K_\mu + \cdots + (p_K^{\mu_1\cdots\mu_{n-1}} + \partial_\lambda p_K^{\mu_1\cdots\mu_{n-1}\lambda})\delta\varphi^K_{\mu_1\cdots\mu_{n-1}} - \varphi^K_{\mu_1\cdots\mu_n}\delta p_K^{\mu_1\cdots\mu_n}. \tag{28}$$

The last term of the above expression implies the following field equation:

$$-\varphi^K_{\mu_1\cdots\mu_n} = \frac{\partial \widetilde{L}}{\partial p_K^{\mu_1\cdots\mu_n}}, \tag{29}$$

dual (and equivalent) to the first field equation (18).

Exchanging "velocities" $\varphi^K_{\mu_1\cdots\mu_n}$ with momenta $p_K^{\mu_1\cdots\mu_n}$ is exactly the same Legendre transformation as the one when passing from the Lagrangian to the Hamiltonian picture in mechanics: $-H(q,p) := L(q,\dot{q}) - p\dot{q}$, where $\dot{q}$ on the right-hand side are defined as implicit functions of (q,p) by equation $p = \frac{\partial L}{\partial \dot{q}}$. The latter becomes then equivalent to $\dot{q} = \frac{\partial H}{\partial p}$.

Step 2: Putting the last term of (26) on the left-hand side, we obtain the final Lagrangian:

$$\mathcal{L}(\varphi^K, \ldots, \varphi^K_{\mu_1\cdots\mu_{n-1}}, p_K^{\mu_1\cdots\mu_n}, \partial p_K^{\mu_1\cdots\mu_n}) = \widetilde{L} - (\partial_\lambda p_K^{\mu_1\cdots\mu_{n-1}\lambda})\varphi^K_{\mu_1\cdots\mu_{n-1}}.$$

The generating formula (28) acquires now the following form:

$$\begin{aligned}\delta\mathcal{L} = (\partial_\lambda p_K^\lambda)\delta\varphi^K + (p_K^\mu + \partial_\lambda p_K^{\mu\lambda})\delta\varphi^K_\mu + \cdots + p_K^{\mu_1\cdots\mu_{n-1}}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\\ - \partial_\lambda(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda}).\end{aligned} \tag{30}$$

We see that the last term describes the variational formula for the variable $p_K^{\mu_1\cdots\mu_n}$. This variation is of the first differential order and the role of the conjugate momenta is assumed by the following object:

$$\Pi^{K\lambda}_{\mu_1\cdots\mu_n} := -\varphi^K_{(\mu_1\cdots\mu_{n-1}}\delta^\lambda_{\mu_n)} = \frac{\partial\mathcal{L}}{\partial(\partial_\lambda p_K^{\mu_1\cdots\mu_n})}, \tag{31}$$

according to the identity

$$-\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda} = \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta p_K^{\mu_1\cdots\mu_n}. \tag{32}$$

Consequently,

$$-\partial_\lambda(\varphi^K_{\mu_1\cdots\mu_{n-1}}\delta p_K^{\mu_1\cdots\mu_{n-1}\lambda}) = (\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n})\delta p_K^{\mu_1\cdots\mu_n} + \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta(\partial_\lambda p_K^{\mu_1\cdots\mu_n}). \tag{33}$$

Field equations derived from the new generating formula

$$\begin{aligned}\delta\mathcal{L} = (\partial_\lambda p_K^\lambda)\delta\varphi^K + (p_K^\mu + \partial_\lambda p_K^{\mu\lambda})\delta\varphi^K_\mu + \cdots + p_K^{\mu_1\cdots\mu_{n-1}}\delta\varphi^K_{\mu_1\cdots\mu_{n-1}}\\ + (\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n})\delta p_K^{\mu_1\cdots\mu_n} + \Pi^{K\lambda}_{\mu_1\cdots\mu_n}\delta(\partial_\lambda p_K^{\mu_1\cdots\mu_n}),\end{aligned} \tag{34}$$

together with definition (31) of the momentum Π, are equivalent to Eqs. (18)–(22). They assume now the following form:

$$\partial_\lambda\Pi^{K\lambda}_{\mu_1\cdots\mu_n} = -\varphi^K_{\mu_1\cdots\mu_n} = \frac{\partial\mathcal{L}}{\partial p_K^{\mu_1\cdots\mu_n}} = \frac{\partial\widetilde{L}}{\partial p_K^{\mu_1\cdots\mu_n}}, \tag{35}$$

$$p_K^{\mu_1\cdots\mu_{n-1}} = \frac{\partial\mathcal{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-1}}} = \frac{\partial\widetilde{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-1}}}, \tag{36}$$

$$p_K^{\mu_1\cdots\mu_{n-2}} + \partial_\lambda p_K^{\mu_1\cdots\mu_{n-2}\lambda} = \frac{\partial\mathcal{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-2}}} = \frac{\partial\widetilde{L}}{\partial\varphi^K_{\mu_1\cdots\mu_{n-2}}}, \tag{37}$$

$$\cdots = \cdots\,, \tag{38}$$

$$p_K^{\mu} + \partial_\lambda p_K^{\mu\lambda} = \frac{\partial\mathcal{L}}{\partial\varphi^K_{\mu}} = \frac{\partial\widetilde{L}}{\partial\varphi^K_{\mu}}, \tag{39}$$

$$\partial_\lambda p_K^{\lambda} = \frac{\partial\mathcal{L}}{\partial\varphi^K} = \frac{\partial\widetilde{L}}{\partial\varphi^K}. \tag{40}$$

Equation (35) (together with definition (31) of the momentum Π) is the Euler–Lagrange equation of the first-differential-order variational principle defined by the Lagrangian

$$\mathcal{L} = \mathcal{L}(j^{n-1}(\varphi), p_K^{\mu_1\cdots\mu_n}, \partial_\lambda p_K^{\mu_1\cdots\mu_n}),$$

whereas Eqs. (36)–(40) describe the $(n-1)$-differential order variational principle defined by the dependence of $\mathcal{L}$ upon the $(n-1)$th jet of the field configuration $x \to \varphi^K(x)$.

We conclude that the variational principle of order n for the field φ^K can be equivalently replaced by another variational principle, which is of order $(n-1)$ for the field φ^K and of order 1 for an auxiliary ("matter") field $p_K^{\mu_1\cdots\mu_n}$. The successive application of this theorem enables us to reduce the differential order of any higher-order variational principle to the order 1. The price which must be paid for such a reduction is the introduction of several additional "matter fields".

5. Higher-Order Curvature Tensors

As already declared in Introduction, the goal of this paper is to reduce the differential order of the variational principle (1). Because the curvature tensor $R^\lambda_{\kappa\sigma\nu}$ contains first-order derivatives of the connection coefficients $\Gamma^\lambda_{\kappa\sigma}$, the Lagrangian (1) is of the $(n+1)$th differential order with respect to the configuration variables $\Gamma^\lambda_{\kappa\sigma}$ (playing role of the field variables φ^K). Hence, to lower the differential order of (1), we can use *a priori* the method described in the previous section. Unfortunately, neither connection coefficients nor their derivatives are tensorial objects and, consequently, such a straightforward application of this method would not be coordinate invariant. To preserve coordinate invariance of the theory, we shall keep together derivatives of Γ's and Γ's themselves in such combinations, which correspond to the curvature tensor or its covariant derivatives. But, again, there is a problem here: there are many identities imposed on these objects (e.g. Bianchi I and Bianchi II, together with their covariant derivatives) and, consequently, it is difficult to select independent quantities among them, which would allow us to express any Lagrangian density (1) in terms of independent objects.

To solve this problem, the notion of *higher-order curvature tensors*, describing the coordinate-invariant part of *higher-order jets of the connection*, was introduced in paper [14]. For the convenience of the reader, we shall briefly sketch this construction below.

Lemma. *Given a symmetric connection* Γ *on a 4D manifold* $\mathcal{M}$ *and a local coordinate chart* (x^λ) *in a neighborhood of a point* $\mathcal{M} \ni \mathbf{m} = (0, \ldots, 0)$, *there is a higher-order correction to coordinates* (x^λ), *i.e.:*

$$y^\lambda = x^\lambda + \frac{1}{2} Q^\lambda_{\mu\nu} x^\mu x^\nu + \frac{1}{3!} W^\lambda_{\mu\nu\sigma} x^\mu x^\nu x^\sigma + \frac{1}{4!} S^\lambda_{\mu\nu\sigma\gamma} x^\mu x^\nu x^\sigma x^\gamma \cdots \tag{41}$$

such that, calculated in the corrected coordinates (y^λ), *connection coefficients fulfill the following symmetry conditions at* $\mathbf{m}$:

$$\Gamma^\lambda_{\kappa\sigma}(\mathbf{m}) = \Gamma^\lambda_{(\kappa\sigma\mu)}(\mathbf{m}) = \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2)}(\mathbf{m}) = \cdots = \Gamma^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)}(\mathbf{m}) = 0, \tag{42}$$

where we denote

$$\Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} := \partial_{\mu_1} \cdots \partial_{\mu_k} \Gamma^\lambda_{\kappa\sigma}. \tag{43}$$

Remark 1. Condition $\mathbf{m} = (0, \ldots, 0)$ does not constitute any restriction for the above statement. Indeed, in a generic case

$$\mathbf{m} = (c^1, \ldots, c^4),$$

the statement remains valid if we replace x^λ by $(x^\lambda - c^\lambda)$.

Remark 2. To fulfill condition (42) for a given $k \in \mathbb{N}$ it is sufficient (and necessary) to consider only corrections (41) of order $(k+2)$, because transformation laws for the connection coefficients contain the second derivatives of the new coordinates with respect to the old ones. Hence, kth derivatives of Γ's are affected by all the derivatives up to $(k+2)$ order and do not feel higher ones, because all of them vanish at $\mathbf{m}$.

Definition. Coordinates (y^λ) fulfilling (42) are called "inertial of degree k at $\mathbf{m}$". In particular "inertial of degree 0" means simply "inertial", i.e. such that the connection coefficients vanish at $\mathbf{m}$, i.e. $\Gamma^\lambda_{\kappa\sigma}(\mathbf{m}) = 0$.

Remark 3. The above definition is a straightforward generalization of the notion of the *global* inertial coordinates (introduce by Newton in his first law), i.e. such that the freely falling bodies move at a constant speed along a straight line, i.e. they satisfy the following equation of motion:

$$\ddot{y}^\lambda = 0. \tag{44}$$

Nowadays, following Einstein, we formulate laws of physics in terms of *local quantities*. This is precisely the case, because in a *local inertial frame* Eq. (44) is fulfilled only at a single point ($\mathbf{m}$)

$$\ddot{y}^\lambda(\mathbf{m}) = 0, \tag{45}$$

and a *global* inertial frame does not exist, unless the connection is globally flat!

Definition. The curvature tensor of order k is a tensor $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}$ such that, in a particular coordinate system which is inertial of order $(k+1)$, its components are equal to partial derivatives of the connection coefficients

$$K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} \stackrel{*}{=} \Gamma^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}. \tag{46}$$

Theorem 1. *The above definition is correct, i.e.* $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k}$ *exists and is unique. Moreover, in arbitrary (non necessarily inertial) coordinates, the components of* K *are given by the following formula:*

$$K^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} = \Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} - \Gamma^\lambda_{(\mu\nu\sigma_1\cdots\sigma_n)} + f \text{ (lower order derivatives of } \Gamma'\text{s)}. \tag{47}$$

In particular, for $k = 1$, *the universal (i.e. valid in any coordinate system) formula for* $K^\lambda_{\mu\nu\sigma}$ *reads*

$$K^\lambda_{\mu\nu\sigma} = \Gamma^\lambda_{\mu\nu\sigma} - \Gamma^\lambda_{(\mu\nu\sigma)} + \Gamma^\lambda_{\gamma\sigma}\Gamma^\gamma_{\mu\nu} - \Gamma^\lambda_{\gamma(\sigma}\Gamma^\gamma_{\mu\nu)}, \tag{48}$$

which, indeed, reduces to $K^\lambda_{\mu\nu\sigma} \stackrel{*}{=} \Gamma^\lambda_{\mu\nu\sigma}$ *in inertial coordinates of order* 1 *and to* $K^\lambda_{\mu\nu\sigma} \stackrel{*}{=} \Gamma^\lambda_{\mu\nu\sigma} - \Gamma^\lambda_{(\mu\nu\sigma)}$ *in inertial coordinates of order* 0.

Theorem 2. *The curvature tensors fulfill the symmetry identities*

$$K^\lambda_{\kappa\sigma\mu_1\cdots\mu_k} = K^\lambda_{(\kappa\sigma)\mu_1\cdots\mu_k} = K^\lambda_{\kappa\sigma(\mu_1\cdots\mu_k)}, \tag{49}$$

$$K^\lambda_{(\kappa\sigma\mu_1\cdots\mu_k)} = 0. \tag{50}$$

In particular, the first-order curvature fulfills

$$K^\lambda_{\mu\nu\kappa} = K^\lambda_{\nu\mu\kappa}; \quad K^\lambda_{(\mu\nu\kappa)} = 0. \tag{51}$$

Moreover, collection of curvature tensors up to order k *contains exactly the same information as the Riemann tensor and its covariant derivatives up to order* $(k-1)$. *In particular, the first-order curvature* $K^\lambda_{\mu\nu\kappa}$ *contains exactly the same information as the Riemann tensor* $R^\lambda_{\mu\nu\kappa}$, *according to the following identities:*

$$R^\lambda_{\mu\nu\kappa} = -2K^\lambda_{\mu[\nu\kappa]}; \quad K^\lambda_{\mu\nu\kappa} = -\frac{2}{3}R^\lambda_{(\mu\nu)\kappa}. \tag{52}$$

The above theorems have been proved in paper [14]. For our present purposes, however, we need an additional observation, which was differently formulated there:

Theorem 3. *Curvature tensor* $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n}$, $n > 1$, *can be expressed as a sum of two terms*: (1) *a linear combination of the first-order covariant derivatives* $\nabla_{\mu_n} K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_{n-1}}$ *and* (2) *a* (*possibly non-linear*) *tensorial function of lower curvature tensors* $K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_l}$, $l \le n-1$:

$$K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n} = S^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_n} + f(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\cdots\mu_{n-1}}), \tag{53}$$

where

$$S^{\lambda}_{\nu_1\nu_2\cdots\nu_{n+2}} = \sum_{\pi} c(\pi) \cdot \nabla_{\nu_{\pi(1)}} K^{\lambda}_{\nu_{\pi(2)}\nu_{\pi(3)}\cdots\nu_{\pi(n+2)}} \tag{54}$$

where π *is a permutation of* $(n+2)$ *elements and* $c(\pi)$ *are fixed coefficients.*

Proof. See Appendix A. □

Examples.

$$K^{\lambda}_{\kappa\sigma\mu\nu} = \frac{5}{8}\nabla_{\nu} K^{\lambda}_{\kappa\sigma\mu} + \frac{5}{8}\nabla_{\mu} K^{\lambda}_{\kappa\sigma\nu} - \frac{1}{8}\nabla_{\sigma} K^{\lambda}_{\mu\nu\kappa} - \frac{1}{8}\nabla_{\kappa} K^{\lambda}_{\mu\nu\sigma}\,. \tag{55}$$

Hence, for $n = 2$ we have $f = 0$ (an easy proof was given in [14]). But for $n = 3$ it is no longer the case, because the following identity will be proved in the Appendix:

$$\begin{aligned} K^{\lambda}_{\kappa\sigma\mu\nu\gamma} = {} & \frac{11}{10}\nabla_{\nu)} K^{\lambda}_{\kappa\sigma(\mu_1\mu_2} + \frac{1}{10}\nabla_{(\nu} K^{\lambda}_{\mu_1\mu_2)\kappa\sigma} - \frac{2}{5}\nabla_{\sigma)} K^{\lambda}_{(\mu_1\mu_2\nu)(\kappa} \\ & + \frac{3}{80}[-25(K^{\lambda}_{\kappa\alpha(\gamma} K^{\alpha}_{\mu\nu)\sigma} + K^{\lambda}_{\sigma\alpha(\gamma} K^{\alpha}_{\mu\nu)\kappa}) + 31 K^{\alpha}_{\kappa\sigma(\mu} K^{\lambda}_{\nu\gamma)\alpha} \\ & + 10(K^{\alpha}_{(\mu\nu)\kappa} K^{\lambda}_{(\gamma)\alpha\sigma} + K^{\alpha}_{(\mu\nu)\sigma} K^{\lambda}_{(\gamma)\alpha\kappa})]. \end{aligned} \tag{56}$$

6. Invariant Variational Principles in Theory of Gravity

Invariant Lagrangian cannot depend upon gravitational field Γ and its derivatives in an arbitrary way but only through invariant geometric objects: the curvature tensors. They contain the complete information about Riemann tensor and its covariant derivatives, whereas formulae (49) and (50) show how many independent objects are necessary to carry this information. According to the previous section, the Lagrangian (1) can, therefore, be rewritten in terms of the higher curvature tensor:

$$L = L(\varphi, \varphi_{\kappa}, K^{\lambda}_{\mu\nu\sigma}, \ldots, K^{\lambda}_{\mu\nu\sigma_1\cdots\sigma_n}), \tag{57}$$

where φ symbolize the matter fields (to simplify notation we skip their possible indices) interacting with gravitational field $\Gamma^{\lambda}_{\mu\nu}$. Variation with respect to matter fields produces obvious Euler–Lagrange equations *via* the standard generating formula (8)

$$\delta L(\varphi, \varphi_\kappa, \ldots) = \partial_\kappa (p^\kappa \delta\varphi) + \cdots , \tag{58}$$

whereas variation with respect to Γ produces the remaining part of the generating formula, according to (25), with variables φ^K being replaced by $\Gamma^\lambda_{\mu\nu}$:

$$\begin{aligned}\delta L &= \cdots + \partial_\kappa (P_\lambda^{\mu\nu\kappa} \delta\Gamma^\lambda_{\mu\nu} + P_\lambda^{\mu\nu\sigma\kappa} \delta\Gamma^\lambda_{\mu\nu\sigma} + \cdots + P_\lambda^{\mu\nu\sigma_1\cdots\sigma_n\kappa} \delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}) \\ &= \cdots + (\partial_\kappa P_\lambda^{\mu\nu\kappa}) \delta\Gamma^\lambda_{\mu\nu} + (P_\lambda^{\mu\nu\sigma} + \partial_\kappa P_\lambda^{\mu\nu\sigma\kappa}) \delta\Gamma^\lambda_{\mu\nu\sigma} + \cdots \\ &\quad + (P_\lambda^{\mu\nu\sigma_1\cdots\sigma_{n-1}} + \partial_\kappa P_\lambda^{\mu\nu\sigma_1\cdots\sigma_{n-1}\kappa}) \delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} + P_\lambda^{\mu\nu\sigma_1\cdots\sigma_n} \delta\Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} .\end{aligned} \tag{59}$$

To calculate the value of the consequent momenta $P_\lambda^{\mu\nu\sigma_1\cdots\sigma_k}$, we begin with the variation of the Lagrangian density (57) (here, we skip, for simplicity, the matter part of the formula):

$$\delta L = Q_\lambda^{\mu\nu\sigma} \delta K^\lambda_{\mu\nu\sigma} + Q_\lambda^{\mu\nu\sigma_1\sigma_2} \delta K^\lambda_{\mu\nu\sigma_1\sigma_2} + \cdots + Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n} \delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} , \tag{60}$$

where the objects Q are tensor-densities (in contrast with P's). Expressing each curvature tensor $K^\lambda_{\mu\nu\sigma_1\cdots\sigma_k}$, $k \le n$, in terms of the connection coefficients $\Gamma^\lambda_{\mu\nu}$ and their derivatives up to kth order we can finally express all the momenta $P_\lambda^{\mu\nu\sigma_1\cdots\sigma_k}$ in terms of derivatives of $\Gamma^\lambda_{\mu\nu}$ and, whence, derive the Euler–Lagrange equations of the theory. In particular, expression for the highest rank momentum is especially simple

$$P_\lambda^{\mu\nu\sigma_1\cdots\sigma_n} = Q_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}, \tag{61}$$

because the curvature tensor depends in a simple way upon the highest order derivatives of the connection — see formula (47).

7. Lowering Order of the Theory in an Invariant Way

Applying the procedure described in Sec. 4, we can immediately lower the differential order of the variational principle (59) with respect to Γ by upgrading the highest rank momentum $P_\lambda^{\mu\nu\sigma_1\cdots\sigma_n}$ to the level of matter fields. As already explained in Sec. 5, such a procedure is not satisfactory because the quantity

$$\mathcal{L} = L + \partial_\kappa (P_\lambda^{\mu\nu\sigma_1\cdots\sigma_{n-1}\kappa} \Gamma^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}})$$

is not an invariant scalar density. However, this procedure can be slightly modified in such a way, that the resulting Lagrangian density $\mathcal{L}$ is a genuine scalar density and, therefore, the new field equations are not coordinate dependent i.e. have an intrinsic geometric meaning. For this purpose we begin with the following.

Step 0: As a consequence of Theorem 3 from the previous section (see (53)), we may use $S^\lambda_{\kappa\sigma\mu_1\cdots\mu_n}$ instead of $K^\lambda_{\kappa\sigma\mu_1\cdots\mu_n}$ to parameterize the covariant nth jet of the connection Γ. Thus, inserting (53) into the last term of (60), we obtain (skipping again the matter part of the formula, because it does not change):

$$\begin{aligned}\delta L(K^\lambda_{\mu\nu\sigma}, \dots, K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}, S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}) \\ = \widetilde{Q}^{\mu\nu\sigma}_\lambda \delta K^\lambda_{\mu\nu\sigma} + \widetilde{Q}^{\mu\nu\sigma_1\sigma_2}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\sigma_2} \\ + \cdots + \widetilde{Q}^{\mu\nu\sigma_1\cdots\sigma_{n-1}}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} + Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda \delta S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}.\end{aligned} \tag{62}$$

The derivatives Q — with exception of the last one — have changed their value to $\widetilde{Q}$ because of the corrections coming from the derivatives of the function f in formula (53). But all of them are invariant tensor densities (derivatives of the scalar density L with respect to the tensors K).

Step 1: Now we perform the Legendre transformation between S and the highest order momentum $Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda$

$$Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda \delta S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} = \delta(Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}) - S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} \delta Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda. \tag{63}$$

Putting the total derivative on the left-hand side, we obtain the new generating function

$$\widetilde{L} = L(K^\lambda_{\mu\nu\sigma}, \dots, K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}, Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda) := L - Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}, \tag{64}$$

where the "velocity" $S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}$ on the right-hand side is treated as an implicit function of the remaining variables, defined by Eq. (62), i.e.

$$Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda = \frac{\partial L}{\partial S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}}. \tag{65}$$

As a result, we obtain the following formula for the differential of $\widetilde{L}$:

$$\begin{aligned}\delta \widetilde{L}(K^\lambda_{\mu\nu\sigma}, \dots, K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}, S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n}) \\ = \widetilde{Q}^{\mu\nu\sigma}_\lambda \delta K^\lambda_{\mu\nu\sigma} + \widetilde{Q}^{\mu\nu\sigma_1\sigma_2}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\sigma_2} \\ + \cdots + \widetilde{Q}^{\mu\nu\sigma_1\cdots\sigma_{n-1}}_\lambda \delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}} - S^\lambda_{\mu\nu\sigma_1\cdots\sigma_n} \delta Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda.\end{aligned} \tag{66}$$

Step 2: Using formula (54), we define the momentum canonically conjugate to $Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda$

$$\Pi^{\lambda\kappa}_{\nu_1\nu_2\cdots\nu_{n+2}} = -\sum_\pi c(\pi) \cdot \delta^\kappa_{\nu_{\pi(1)}} K^\lambda_{\nu_{\pi(2)}\nu_{\pi(3)}\cdots\nu_{\pi(n+2)}}, \tag{67}$$

fulfilling the obvious identity

$$\nabla_\kappa \Pi^{\lambda\kappa}_{\nu_1\nu_2\cdots\nu_{n+2}} = -S^\lambda_{\nu_1\nu_2\cdots\nu_{n+2}} \tag{68}$$

and add to both sides of Eq. (66) the quantity becomes

$$\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda . \tag{69}$$

This means that we define the new Lagrangian function

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(K^\lambda_{\mu\nu\sigma},\ldots,K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}},Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda,\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda)\\ &:= \widetilde{L}+\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}\cdot\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda .\end{aligned} \tag{70}$$

But

$$\begin{aligned}\delta(\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda) = \Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}\cdot\delta(\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda)\\ +(\nabla_\kappa Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda)\delta\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n} .\end{aligned} \tag{71}$$

This implies that formula (66) assumes now the following form:

$$\begin{aligned}\delta\mathcal{L} = \widetilde{Q}^{\mu\nu\sigma}_\lambda\delta K^\lambda_{\mu\nu\sigma}+\widetilde{Q}^{\mu\nu\sigma_1\sigma_2}_\lambda\delta K^\lambda_{\mu\nu\sigma_1\sigma_2}\\ +\cdots+\widetilde{\widetilde{Q}}{}^{\mu\nu\sigma_1\cdots\sigma_{n-1}}_\lambda\delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}+\nabla_\kappa(\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda),\end{aligned} \tag{72}$$

where $\widetilde{\widetilde{Q}}$ is equal to $\widetilde{Q}$ plus terms coming from the last term in Eq. (71) (remember that $\delta\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}$ is a superposition of terms $\delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}$).

We stress that all the quantities appearing in the above formula are tensors or tensor densities. In particular, the quantity $\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda$ in the last bracket is a vector density. Hence, its covariant divergence is equal to the coordinate divergence. We conclude that identity (72) can be written in a following, equivalent, way:

$$\begin{aligned}\delta\mathcal{L} = \widetilde{Q}^{\mu\nu\sigma}_\lambda\delta K^\lambda_{\mu\nu\sigma}+\widetilde{Q}^{\mu\nu\sigma_1\sigma_2}_\lambda\delta K^\lambda_{\mu\nu\sigma_1\sigma_2}\\ +\cdots+\widetilde{\widetilde{Q}}{}^{\mu\nu\sigma_1\cdots\sigma_{n-1}}_\lambda\delta K^\lambda_{\mu\nu\sigma_1\cdots\sigma_{n-1}}+\partial_\kappa(\Pi^{\lambda\kappa}_{\mu\nu\sigma_1\cdots\sigma_n}Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda).\end{aligned} \tag{73}$$

This is the first-order Lagrangian formula for the new matter field $Q^{\mu\nu\sigma_1\cdots\sigma_n}_\lambda$ plus the $(n-1)$th order variational formula for the gravitational field $\Gamma^\lambda_{\mu\nu}$, analogous to the nth order formula (60).

Applying this procedure $(n-1)$ times we can lower the order of the gravitational degrees of freedom contained in the Lagrangian to one. Such a theory is equivalent to the conventional Einstein theory, as proved in [8].

Acknowledgment

This research was supported by Narodowe Centrum Nauki (Poland): Grant No. 2016/21/B/ST1/00940.

Appendix A: Proof of Theorem 3

$$\begin{aligned}\nabla_\nu K^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n} &= \partial_\nu(\Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n)}) + f(l.o.d.\ \Gamma) \\ &= \Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n)\nu} + f(l.o.d.\ \Gamma),\end{aligned}$$

where "$l.o.d.\ \Gamma$ means lower order derivatives of Γ".

On the other hand, we have

$$K^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} = \Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu)} + f(l.o.d.\ \Gamma)\,.$$

Denote by $\mathcal{K}$ the following combination of derivatives of Γ:

$$\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} = \Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_k} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_k)}. \tag{A.1}$$

We have, therefore

$$\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} = \Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu)}, \tag{A.2}$$

$$\partial_\nu\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n} = \Gamma^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} - \Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n)\nu}. \tag{A.3}$$

We see that our problem has been reduced to the following algebraic, linear problem: is it possible to express the combination (A.2) of derivatives in terms of combinations (A.3)? To solve the problem, let us sum the left-hand side of (A.2) and (A.3) over all transpositions of the index ν with the remaining indices. Observe that this operation applied to the right-hand sides of both (A.2) and (A.3) gives the same result because the sum over these transposition of $\Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n)\nu}$ gives nothing but $(n+3)\Gamma^\lambda_{(\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu)}$ and, whence, the same result as the one coming from the last term of (A.2). This observation implies the following identity:

$$\begin{aligned}&(n+1)\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu} + \mathcal{K}^\lambda_{\kappa\nu\mu_1\mu_2\cdots\mu_n\sigma} + \mathcal{K}^\lambda_{\sigma\nu\mu_1\mu_2\cdots\mu_n\kappa} \\ &\quad = \partial_\nu\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n} + \partial_\kappa\mathcal{K}^\lambda_{\sigma\nu\mu_1\mu_2\cdots\mu_n} + \partial_\sigma\mathcal{K}^\lambda_{\kappa\nu\mu_1\mu_2\cdots\mu_n} + \sum_{k=1}^{n}\partial_{\mu_k}\mathcal{K}^\lambda_{\kappa\sigma\mu_1\cdots\underset{k}{\wedge}\cdots\mu_n\nu} \\ &\quad := \mathcal{L}^\lambda_{\kappa\sigma\nu\mu_1\mu_2\cdots\mu_n}.\end{aligned} \tag{A.4}$$

Hence, we have to solve the following equation with respect to $\mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2\cdots\mu_n\nu}$:

$$(n+1)\mathcal{K}^\lambda_{\kappa\sigma\nu\mu_1\mu_2\cdots\mu_n} + \mathcal{K}^\lambda_{\kappa\nu\sigma\mu_1\mu_2\cdots\mu_n} + \mathcal{K}^\lambda_{\sigma\nu\kappa\mu_1\mu_2\cdots\mu_n} = \mathcal{L}^\lambda_{\kappa\sigma\nu\mu_1\mu_2\cdots\mu_n}. \tag{A.5}$$

By virtue of (A.4) we see that the totally symmetric part of $\mathcal{L}$ vanishes identically:

$$\mathcal{L}^\lambda_{(\kappa\sigma\nu\mu_1\mu_2\cdots\mu_n)} = 0.$$

For every collection of indices $(\lambda, \mu_1, \ldots, \mu_n)$ we have to solve the same equation, which can be written in a symbolic way as

$$(n+1)\mathcal{K}_{\kappa\sigma\nu} + \mathcal{K}_{\kappa\nu\sigma} + \mathcal{K}_{\sigma\nu\kappa} = \mathcal{L}_{\kappa\sigma\nu}, \tag{A.6}$$

where both the unknown $\mathcal{K}_{\kappa\sigma\nu}$ and the known $\mathcal{L}_{\kappa\sigma\nu}$ are symmetric in first two indices. Consequently

$$(n+3)\mathcal{K}_{(\kappa\sigma\nu)} = \mathcal{L}_{(\kappa\sigma\nu)}. \tag{A.7}$$

Denoting

$$\mathcal{M}_{\kappa\sigma\nu} := \mathcal{K}_{\kappa\sigma\nu} - \mathcal{K}_{(\kappa\sigma\nu)} \tag{A.8}$$

and subtracting (A.7) from (A.6) we obtain

$$(n+1)\mathcal{M}_{\kappa\sigma\nu} + \mathcal{M}_{\kappa\nu\sigma} + \mathcal{M}_{\sigma\nu\kappa} = \mathcal{L}_{\kappa\sigma\nu} - \mathcal{L}_{(\kappa\sigma\nu)}. \tag{A.9}$$

But

$$0 = \mathcal{M}_{(\kappa\sigma\nu)} = \mathcal{M}_{\kappa\sigma\nu} + \mathcal{M}_{\kappa\nu\sigma} + \mathcal{M}_{\sigma\nu\kappa},$$

and, whence

$$\mathcal{M}_{\kappa\nu\sigma} + \mathcal{M}_{\sigma\nu\kappa} = -\mathcal{M}_{\kappa\sigma\nu}.$$

Inserting this into (A.9) we obtain

$$n \cdot \mathcal{M}_{\kappa\sigma\nu} = \mathcal{L}_{\kappa\sigma\nu} - \mathcal{L}_{(\kappa\sigma\nu)}, \tag{A.10}$$

or

$$n \cdot \mathcal{K}_{\kappa\sigma\nu} = \mathcal{L}_{\kappa\sigma\nu} - \mathcal{L}_{(\kappa\sigma\nu)} + n \cdot \mathcal{K}_{(\kappa\sigma\nu)} = \mathcal{L}_{\kappa\sigma\nu} - \left(1 - \frac{n}{n+3}\right)\mathcal{L}_{(\kappa\sigma\nu)}. \tag{A.11}$$

Finally, we have

$$\mathcal{K}_{\kappa\sigma\nu} = \frac{1}{n}\left\{\mathcal{L}_{\kappa\sigma\nu} - \frac{3}{n+3} \cdot \mathcal{L}_{(\kappa\sigma\nu)}\right\}. \tag{A.12}$$

When "dressed" with missing indices $(\lambda, \mu_1, \ldots, \mu_n)$, the formula reads

$$\begin{aligned}
&\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} \\
&\quad = \frac{1}{n}\left\{\mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} - \frac{3}{n+3} \cdot \mathcal{L}^{\lambda}_{(\kappa\sigma\nu)\mu_1,\ldots,\mu_n}\right\} \\
&\quad = \frac{1}{n}\left\{\mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} - \frac{1}{n+3}(\mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} + \mathcal{L}^{\lambda}_{\kappa\nu\sigma\mu_1,\ldots,\mu_n} + \mathcal{L}^{\lambda}_{\sigma\nu\kappa\mu_1,\ldots,\mu_n})\right\} \\
&\quad = \frac{1}{n}\left\{\frac{n+2}{n+3}\mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} - \frac{1}{n+3}(\mathcal{L}^{\lambda}_{\kappa\nu\sigma\mu_1,\ldots,\mu_n} + \mathcal{L}^{\lambda}_{\sigma\nu\kappa\mu_1,\ldots,\mu_n})\right\}.
\end{aligned} \tag{A.13}$$

This formula, however, is not fully correct because we did not pay attention to symmetries of the higher-order curvature tensor $\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n}$. Indeed, even if the symmetry in κ and σ and the vanishing of the totally symmetric part is assured by expression (A.13), the symmetry in indices $(\nu\mu_1, \ldots, \mu_n)$ is not! To obtain the fully correct expression, we must from the very beginning (i.e. from formula (A.5)) symmetrize all the formulae appearing in the above proof with respect to those indices. This is, however, equivalent, to the symmetrization of the final formula (A.13) with respect to $(\nu\mu_1, \ldots, \mu_n)$. We conclude, that the correct, final result reads

$$\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} = \frac{1}{n} \cdot \mathrm{sym}_{(\nu\mu_1,\ldots,\mu_n)} \left\{ \frac{n+2}{n+3} \mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1,\ldots,\mu_n} - \frac{1}{n+3} (\mathcal{L}^{\lambda}_{\kappa\nu\sigma\mu_1,\ldots,\mu_n} + \mathcal{L}^{\lambda}_{\sigma\nu\kappa\mu_1,\ldots,\mu_n}) \right\}. \tag{A.14}$$

This ends the proof because — modulo lower-order-derivatives-of-Gammas — the left-hand side is equal to the left-hand side of (53), whereas the right-hand side is a combination of covariant derivatives of the lower order curvature tensors.

In particular, for $n = 1$, we have

$$\begin{aligned}
\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu} &= \mathrm{sym}_{(\nu\mu)} \left\{ \frac{3}{4} \mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu} - \frac{1}{4} (\mathcal{L}^{\lambda}_{\kappa\nu\sigma\mu} + \mathcal{L}^{\lambda}_{\sigma\nu\kappa\mu}) \right\} \\
&= \mathrm{sym}_{(\nu\mu)} \left\{ \frac{3}{4} (\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu} + \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu} + \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu}) \right. \\
&\quad - \frac{1}{4} (\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu} + \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu} + \partial_\mu \mathcal{K}^{\lambda}_{\kappa\nu\sigma}) \\
&\quad \left. - \frac{1}{4} (\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu} + \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu} + \partial_\mu \mathcal{K}^{\lambda}_{\nu\sigma\kappa}) \right\} \\
&= \mathrm{sym}_{(\nu\mu)} \left\{ \frac{1}{4} \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \frac{1}{4} \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu} + \frac{1}{4} \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu} + \frac{3}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu} \right. \\
&\quad \left. - \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\nu\sigma} - \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\nu\sigma\kappa} \right\} \\
&= \frac{1}{2} \left(\frac{1}{4} \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu} + \frac{1}{4} \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu} + \frac{1}{4} \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\mu\nu} + \frac{1}{4} \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu} \right. \\
&\quad + \frac{1}{4} \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\mu\nu} + \frac{3}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu} + \frac{3}{4} \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} - \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\nu\sigma} \\
&\quad \left. - \frac{1}{4} \partial_\nu \mathcal{K}^{\lambda}_{\kappa\mu\sigma} - \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\nu\sigma\kappa} - \frac{1}{4} \partial_\nu \mathcal{K}^{\lambda}_{\mu\sigma\kappa} \right) \\
&= \frac{1}{2} \left(\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} + \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu} - \frac{1}{4} \partial_\kappa \mathcal{K}^{\lambda}_{\mu\nu\sigma} - \frac{1}{4} \partial_\sigma \mathcal{K}^{\lambda}_{\mu\nu\kappa} + \frac{1}{4} \partial_\mu \mathcal{K}^{\lambda}_{\kappa\sigma\nu} + \frac{1}{4} \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu} \right) \\
&= \frac{5}{4} \mathcal{K}^{\lambda}_{\kappa\sigma(\mu;\nu)} - \frac{1}{4} \mathcal{K}^{\lambda}_{\mu\nu(\kappa;\sigma)},
\end{aligned} \tag{A.15}$$

in agreement with formula (55).

For $n = 2$, we have calculated in [14] the value of $\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1\mu_2}$ in terms of the $\mathcal{K}^{\lambda}_{\kappa\sigma\nu}$ and its second derivatives (see formula (41) in paper [14]). But, in this paper, we need a slightly modified version of this result, namely second-order derivatives of the first-order curvature tensor have to be replaced by the first-order derivatives of the second-order curvature tensor. In what follows, we calculate it independently

$$\begin{aligned}
\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1\mu_2} &= \frac{1}{2}\cdot \mathrm{sym}_{(\nu\mu_1\mu_2)}\left\{\frac{4}{5}\mathcal{L}^{\lambda}_{\kappa\sigma\nu\mu_1\mu_2} - \frac{1}{5}\left(\mathcal{L}^{\lambda}_{\nu\kappa\sigma\mu_1\mu_2} + \mathcal{L}^{\lambda}_{\sigma\nu\kappa\mu_1\mu_2}\right)\right\} \\
&= \frac{1}{2}\cdot \mathrm{sym}_{(\nu\mu_1\mu_2)}\left\{\frac{4}{5}(\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} + \partial_\kappa \mathcal{K}^{\lambda}_{\nu\sigma\mu_1\mu_2} + \partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu_1\mu_2}\right. \\
&\quad + \partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\nu}) - \frac{1}{5}(\partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu_1\mu_2} + \partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu_1\mu_2} + \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} \\
&\quad + \partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\nu\sigma\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\nu\mu_1\sigma}) - \frac{1}{5}(\partial_\kappa \mathcal{K}^{\lambda}_{\nu\sigma\mu_1\mu_2} + \partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} \\
&\quad \left. + \partial_\sigma \mathcal{K}^{\lambda}_{\nu\kappa\mu_1\mu_2} + \partial_{\mu_1}\mathcal{K}^{\lambda}_{\nu\sigma\kappa\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\nu\sigma\mu_1\kappa})\right\} \\
&= \frac{1}{2}\cdot \mathrm{sym}_{(\nu\mu_1\mu_2)}\left\{\frac{2}{5}\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} + \frac{2}{5}\partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu_1\mu_2} + \frac{2}{5}\partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu_1\mu_2}\right. \\
&\quad + \frac{4}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_2} + \frac{4}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\nu} - \frac{1}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\nu\sigma\mu_2} - \frac{1}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\nu\sigma\mu_1} \\
&\quad \left. - \frac{1}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\sigma\nu\kappa\mu_2} - \frac{1}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\sigma\nu\kappa\mu_1}\right\}. \qquad \text{(A.16)}
\end{aligned}$$

The symmetrization with respect to indices $(\nu\mu_1\mu_2)$ gives us

$$\begin{aligned}
\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_1\mu_2} &= \frac{1}{2}\cdot\frac{1}{3}\left[\frac{2}{5}\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} + \frac{2}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_2} + \frac{2}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\nu} + \frac{2}{5}\partial_\kappa \mathcal{K}^{\lambda}_{\sigma\nu\mu_1\mu_2}\right. \\
&\quad + \frac{2}{5}\partial_\kappa \mathcal{K}^{\lambda}_{\sigma\mu_1\nu\mu_2} + \frac{2}{5}\partial_\kappa \mathcal{K}^{\lambda}_{\sigma\mu_2\mu_1\nu} + \frac{2}{5}\partial_\sigma \mathcal{K}^{\lambda}_{\kappa\nu\mu_1\mu_2} + \frac{2}{5}\partial_\sigma \mathcal{K}^{\lambda}_{\kappa\mu_1\nu\mu_2} \\
&\quad + \frac{2}{5}\partial_\sigma \mathcal{K}^{\lambda}_{\kappa\mu_2\mu_1\nu} + \frac{4}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_2} + \frac{4}{5}\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\mu_2} + \frac{4}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\nu} \\
&\quad + \frac{4}{5}\partial_{\mu_2}\mathcal{K}^{\lambda}_{\kappa\sigma\mu_1\nu} + \frac{4}{5}\partial_\nu \mathcal{K}^{\lambda}_{\kappa\sigma\mu_2\mu_1} + \frac{4}{5}\partial_{\mu_1}\mathcal{K}^{\lambda}_{\kappa\sigma\nu\mu_2} \\
&\quad - \frac{1}{5}(\partial_{\mu_1}\mathcal{K}^{\lambda}_{\nu\kappa\sigma\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\nu\kappa\sigma\mu_1}) - \frac{1}{5}\left(\partial_\nu \mathcal{K}^{\lambda}_{\mu_1\kappa\sigma\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\mu_1\kappa\sigma\nu}\right) \\
&\quad - \frac{1}{5}(\partial_{\mu_1}\mathcal{K}^{\lambda}_{\mu_2\kappa\sigma\nu} + \partial_\nu \mathcal{K}^{\lambda}_{\mu_2\kappa\sigma\mu_1}) - \frac{1}{5}\left(\partial_{\mu_1}\mathcal{K}^{\lambda}_{\nu\sigma\kappa\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\nu\sigma\kappa\mu_1}\right) \\
&\quad \left. - \frac{1}{5}(\partial_\nu \mathcal{K}^{\lambda}_{\mu_1\sigma\kappa\mu_2} + \partial_{\mu_2}\mathcal{K}^{\lambda}_{\mu_1\sigma\kappa\nu}) - \frac{1}{5}(\partial_{\mu_1}\mathcal{K}^{\lambda}_{\mu_2\sigma\kappa\nu} + \partial_\nu \mathcal{K}^{\lambda}_{\mu_2\sigma\kappa\mu_1})\right]. \\
&\qquad \text{(A.17)}
\end{aligned}$$

Using the property of vanishing of the totally symmetric part of the curvature tensor $\mathcal{K}^{\lambda}_{\kappa\sigma\mu\nu}$

$$\mathcal{K}^{\lambda}_{\kappa\sigma\mu\nu} + \mathcal{K}^{\lambda}_{\mu\nu\kappa\sigma} + \mathcal{K}^{\lambda}_{\kappa\mu\sigma\nu} + \mathcal{K}^{\lambda}_{\kappa\nu\sigma\mu} + \mathcal{K}^{\lambda}_{\sigma\mu\kappa\nu} + \mathcal{K}^{\lambda}_{\sigma\nu\kappa\mu} = 0,$$

we obtain

$$
\begin{aligned}
\mathcal{K}^\lambda_{\kappa\sigma\nu\mu_1\mu_2} = \frac{1}{2}\cdot\frac{1}{3}\Big[& 2\partial_\nu \mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2} + 2\partial_{\mu_1}\mathcal{K}^\lambda_{\kappa\sigma\nu\mu_2} + 2\partial_{\mu_2}\mathcal{K}^\lambda_{\kappa\sigma\mu_1\nu} - \frac{2}{5}\partial_\kappa \mathcal{K}^\lambda_{\nu\mu_1\mu_2\sigma} \\
& - \frac{2}{5}\partial_\kappa \mathcal{K}^\lambda_{\mu_2\mu_1\nu\sigma} - \frac{2}{5}\partial_\kappa \mathcal{K}^\lambda_{\nu\mu_2\mu_1\sigma} - \frac{2}{5}\partial_\sigma \mathcal{K}^\lambda_{\nu\mu_1\mu_2\kappa} - \frac{2}{5}\partial_\sigma \mathcal{K}^\lambda_{\mu_2\mu_1\nu\kappa} \\
& - \frac{2}{5}\partial_\sigma \mathcal{K}^\lambda_{\nu\mu_2\mu_1\kappa} - \frac{1}{5}\partial_{\mu_1}(\mathcal{K}^\lambda_{\kappa\nu\sigma\mu_2} + \mathcal{K}^\lambda_{\sigma\nu\kappa\mu_2} + \mathcal{K}^\lambda_{\kappa\mu_2\sigma\nu} + \mathcal{K}^\lambda_{\sigma\mu_2\kappa\nu}) \\
& - \frac{1}{5}\partial_{\mu_2}(\mathcal{K}^\lambda_{\kappa\nu\sigma\mu_1} + \mathcal{K}^\lambda_{\sigma\nu\kappa\mu_1} + \mathcal{K}^\lambda_{\kappa\mu_1\sigma\nu} + \mathcal{K}^\lambda_{\sigma\mu_1\kappa\nu}) \\
& - \frac{1}{5}\partial_\nu(\mathcal{K}^\lambda_{\kappa\mu_1\sigma\mu_2} + \mathcal{K}^\lambda_{\sigma\mu_1\kappa\mu_2} + \mathcal{K}^\lambda_{\kappa\mu_2\sigma\mu_1} + \mathcal{K}^\lambda_{\sigma\mu_2\kappa\mu_1})\Big] \\
= \frac{1}{2}\cdot\frac{1}{3}\Bigg(& \frac{11}{5}\partial_\nu \mathcal{K}^\lambda_{\kappa\sigma\mu_1\mu_2} + \frac{11}{5}\partial_{\mu_1}\mathcal{K}^\lambda_{\kappa\sigma\nu\mu_2} + \frac{11}{5}\partial_{\mu_2}\mathcal{K}^\lambda_{\kappa\sigma\mu_1\nu} + \frac{1}{5}\partial_{\mu_1}\mathcal{K}^\lambda_{\mu_2\nu\kappa\sigma} \\
& + \frac{1}{5}\partial_{\mu_2}\mathcal{K}^\lambda_{\mu_1\nu\kappa\sigma} + \frac{1}{5}\partial_\nu \mathcal{K}^\lambda_{\mu_1\mu_2\kappa\sigma} - \frac{6}{5}\partial_\kappa \mathcal{K}^\lambda_{(\nu\mu_1\mu_2)\sigma} - \frac{6}{5}\partial_\sigma \mathcal{K}^\lambda_{(\nu\mu_1\mu_2)\kappa}\Bigg) \\
= \frac{11}{10} & \mathcal{K}^\lambda_{\kappa\sigma(\mu_1\mu_2;\nu)} + \frac{1}{10}\mathcal{K}^\lambda_{\mu_1\mu_2)\kappa\sigma(;\nu} - \frac{2}{5}\mathcal{K}^\lambda_{(\mu_1\mu_2\nu)(\kappa;\sigma)},
\end{aligned}
\tag{A.18}
$$

and, whence, we have

$$
\begin{aligned}
K^\lambda_{\kappa\sigma\nu\mu_1\mu_2} = & \frac{11}{10}\nabla_{\nu)} K^\lambda_{\kappa\sigma(\mu_1\mu_2} + \frac{1}{10}\nabla_{(\nu} K^\lambda_{\mu_1\mu_2)\kappa\sigma} \\
& - \frac{2}{5}\nabla_{\sigma)} K^\lambda_{(\mu_1\mu_2\nu)(\kappa} + f(l.o.d.\ \Gamma).
\end{aligned}
\tag{A.19}
$$

To calculate the value of $f(l.o.d.\ \Gamma)$ we have to: (1) insert formula (55) into (A.19), and then to: (2) calculate the second-order covariant derivatives according to

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta K^\lambda_{\kappa\sigma\mu} &= \nabla_{(\alpha}\nabla_{\beta)} K^\lambda_{\kappa\sigma\mu} + \nabla_{[\alpha}\nabla_{\beta]} K^\lambda_{\kappa\sigma\mu} \\
&= \nabla_{(\alpha}\nabla_{\beta)} K^\lambda_{\kappa\sigma\mu} + R^\lambda_{\nu\alpha\beta} K^\nu_{\kappa\sigma\mu} + R^\nu_{\kappa\alpha\beta} K^\lambda_{\nu\sigma\mu} \\
&\quad + R^\nu_{\sigma\alpha\beta} K^\lambda_{\kappa\nu\mu} + R^\nu_{\mu\alpha\beta} K^\lambda_{\kappa\sigma\nu}.
\end{aligned}
\tag{A.20}
$$

Finally, we have to: (3) express the Riemann tensor R in terms of K, according to formula (52):

$$
R^\lambda_{\mu\nu\kappa} = -2K^\lambda_{\mu[\nu\kappa]} = K^\lambda_{\mu\kappa\nu} - K^\lambda_{\mu\nu\kappa}. \tag{A.21}
$$

Let us rewrite (A.19) in the form

$$K^{\lambda}_{\kappa\sigma\nu\mu\gamma} = \underbrace{\frac{11}{30}(\nabla_\mu K^{\lambda}_{\kappa\sigma\nu\gamma} + \nabla_\nu K^{\lambda}_{\kappa\sigma\mu\gamma} + \nabla_\gamma K^{\lambda}_{\kappa\sigma\mu\nu})}_{(1)} + \underbrace{\frac{1}{30}(\nabla_\mu K^{\lambda}_{\nu\gamma\kappa\sigma} + \nabla_\nu K^{\lambda}_{\mu\gamma\kappa\sigma} + \nabla_\gamma K^{\lambda}_{\mu\nu\kappa\sigma})}_{(2)} - \underbrace{\frac{2}{30}(\nabla_\sigma K^{\lambda}_{\mu\nu\gamma\kappa} + \nabla_\sigma K^{\lambda}_{\gamma\mu\nu\kappa} + \nabla_\sigma K^{\lambda}_{\nu\gamma\mu\kappa} + \nabla_\kappa K^{\lambda}_{\mu\nu\gamma\sigma} + \nabla_\kappa K^{\lambda}_{\gamma\mu\nu\sigma} + \nabla_\kappa K^{\lambda}_{\nu\gamma\mu\sigma})}_{(3)} \tag{A.22}$$

Now, using (55) for each covariant derivative of the second-order curvature tensor, we get the following:

$$\begin{aligned}(1) = \frac{11}{30}\Big[&\frac{5}{8}\nabla_\mu\nabla_\nu K^{\lambda}_{\kappa\sigma\gamma} + \frac{5}{8}\nabla_\mu\nabla_\gamma K^{\lambda}_{\kappa\sigma\nu} - \frac{1}{8}\nabla_\mu\nabla_\kappa K^{\lambda}_{\nu\gamma\sigma} - \frac{1}{8}\nabla_\mu\nabla_\sigma K^{\lambda}_{\nu\gamma\kappa} \\ &+ \frac{5}{8}\nabla_\nu\nabla_\mu K^{\lambda}_{\kappa\sigma\gamma} + \frac{5}{8}\nabla_\nu\nabla_\gamma K^{\lambda}_{\kappa\sigma\mu} - \frac{1}{8}\nabla_\nu\nabla_\kappa K^{\lambda}_{\mu\gamma\sigma} - \frac{1}{8}\nabla_\nu\nabla_\sigma K^{\lambda}_{\mu\gamma\kappa} \\ &+ \frac{5}{8}\nabla_\gamma\nabla_\mu K^{\lambda}_{\kappa\sigma\nu} + \frac{5}{8}\nabla_\gamma\nabla_\nu K^{\lambda}_{\kappa\sigma\mu} - \frac{1}{8}\nabla_\gamma\nabla_\kappa K^{\lambda}_{\mu\nu\sigma} - \frac{1}{8}\nabla_\gamma\nabla_\sigma K^{\lambda}_{\mu\nu\kappa}\Big] \\ = \frac{11}{30}\Bigg[&\frac{5}{4}\nabla_{(\mu}\nabla_{\nu)} K^{\lambda}_{\kappa\sigma\gamma} + \frac{5}{4}\nabla_{(\mu}\nabla_{\gamma)} K^{\lambda}_{\kappa\sigma\nu} + \frac{5}{4}\nabla_{(\gamma}\nabla_{\nu)} K^{\lambda}_{\kappa\sigma\mu} - \frac{1}{8}\nabla_{(\mu}\nabla_{\kappa)} K^{\lambda}_{\nu\gamma\sigma} \\ &- \frac{1}{8}\nabla_{(\mu}\nabla_{\sigma)} K^{\lambda}_{\nu\gamma\kappa} - \frac{1}{8}\nabla_{(\nu}\nabla_{\kappa)} K^{\lambda}_{\mu\gamma\sigma} - \frac{1}{8}\nabla_{(\nu}\nabla_{\sigma)} K^{\lambda}_{\mu\gamma\kappa} \\ &- \frac{1}{8}\nabla_{(\gamma}\nabla_{\kappa)} K^{\lambda}_{\mu\nu\sigma} - \frac{1}{8}\nabla_{(\gamma}\nabla_{\sigma)} K^{\lambda}_{\mu\nu\kappa} - \frac{1}{8}\sum_{(1)} K\cdot K\Bigg],\end{aligned} \tag{A.23}$$

$$\begin{aligned}(2) = \frac{1}{30}\Big[&\frac{5}{8}\nabla_\mu\nabla_\sigma K^{\lambda}_{\nu\gamma\kappa} + \frac{5}{8}\nabla_\mu\nabla_\kappa K^{\lambda}_{\nu\gamma\sigma} - \frac{1}{8}\nabla_\mu\nabla_\gamma K^{\lambda}_{\kappa\sigma\nu} - \frac{1}{8}\nabla_\mu\nabla_\nu K^{\lambda}_{\kappa\sigma\gamma} \\ &+ \frac{5}{8}\nabla_\nu\nabla_\sigma K^{\lambda}_{\mu\gamma\kappa} + \frac{5}{8}\nabla_\nu\nabla_\kappa K^{\lambda}_{\mu\gamma\sigma} - \frac{1}{8}\nabla_\nu\nabla_\gamma K^{\lambda}_{\kappa\sigma\mu} - \frac{1}{8}\nabla_\nu\nabla_\mu K^{\lambda}_{\kappa\sigma\gamma} \\ &+ \frac{5}{8}\nabla_\gamma\nabla_\sigma K^{\lambda}_{\mu\nu\kappa} + \frac{5}{8}\nabla_\gamma\nabla_\kappa K^{\lambda}_{\mu\nu\sigma} - \frac{1}{8}\nabla_\gamma\nabla_\nu K^{\lambda}_{\kappa\sigma\mu} - \frac{1}{8}\nabla_\gamma\nabla_\mu K^{\lambda}_{\kappa\sigma\nu}\Big] \\ = \frac{1}{30}\Big[&\frac{5}{8}\nabla_{(\mu}\nabla_{\sigma)} K^{\lambda}_{\nu\gamma\kappa} + \frac{5}{8}\nabla_{(\mu}\nabla_{\kappa)} K^{\lambda}_{\nu\gamma\sigma} + \frac{5}{8}\nabla_{(\nu}\nabla_{\sigma)} K^{\lambda}_{\mu\gamma\kappa} + \frac{5}{8}\nabla_{(\nu}\nabla_{\kappa)} K^{\lambda}_{\mu\gamma\sigma}\end{aligned}$$

$$+\frac{5}{8}\nabla_{(\gamma}\nabla_{\sigma)}K^{\lambda}_{\mu\nu\kappa}+\frac{5}{8}\nabla_{(\gamma}\nabla_{\kappa)}K^{\lambda}_{\mu\nu\sigma}+\frac{5}{8}\sum_{(2)}K\cdot K$$

$$\left.-\frac{1}{4}\nabla_{(\mu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\nu}-\frac{1}{4}\nabla_{(\mu}\nabla_{\nu)}K^{\lambda}_{\kappa\sigma\gamma}-\frac{1}{4}\nabla_{(\nu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\mu}\right], \tag{A.24}$$

$$\begin{aligned}(3)=&-\frac{2}{30}\left[\frac{5}{8}\nabla_{\sigma}\nabla_{\kappa}K^{\lambda}_{\mu\nu\gamma}+\frac{5}{8}\nabla_{\sigma}\nabla_{\gamma}K^{\lambda}_{\mu\nu\kappa}-\frac{1}{8}\nabla_{\sigma}\nabla_{\nu}K^{\lambda}_{\gamma\kappa\mu}-\frac{1}{8}\nabla_{\sigma}\nabla_{\mu}K^{\lambda}_{\gamma\kappa\nu}\right.\\
&+\frac{5}{8}\nabla_{\sigma}\nabla_{\kappa}K^{\lambda}_{\mu\gamma\nu}+\frac{5}{8}\nabla_{\sigma}\nabla_{\nu}K^{\lambda}_{\mu\gamma\kappa}-\frac{1}{8}\nabla_{\sigma}\nabla_{\gamma}K^{\lambda}_{\nu\kappa\mu}-\frac{1}{8}\nabla_{\sigma}\nabla_{\mu}K^{\lambda}_{\nu\kappa\gamma}\\
&+\frac{5}{8}\nabla_{\sigma}\nabla_{\kappa}K^{\lambda}_{\nu\gamma\mu}+\frac{5}{8}\nabla_{\sigma}\nabla_{\mu}K^{\lambda}_{\nu\gamma\kappa}-\frac{1}{8}\nabla_{\sigma}\nabla_{\gamma}K^{\lambda}_{\mu\kappa\nu}-\frac{1}{8}\nabla_{\sigma}\nabla_{\nu}K^{\lambda}_{\mu\kappa\gamma}\\
&+\frac{5}{8}\nabla_{\kappa}\nabla_{\sigma}K^{\lambda}_{\mu\nu\gamma}+\frac{5}{8}\nabla_{\kappa}\nabla_{\gamma}K^{\lambda}_{\mu\nu\sigma}-\frac{1}{8}\nabla_{\kappa}\nabla_{\nu}K^{\lambda}_{\gamma\sigma\mu}-\frac{1}{8}\nabla_{\kappa}\nabla_{\mu}K^{\lambda}_{\gamma\sigma\nu}\\
&+\frac{5}{8}\nabla_{\kappa}\nabla_{\sigma}K^{\lambda}_{\mu\gamma\nu}+\frac{5}{8}\nabla_{\kappa}\nabla_{\nu}K^{\lambda}_{\mu\gamma\sigma}-\frac{1}{8}\nabla_{\kappa}\nabla_{\gamma}K^{\lambda}_{\nu\sigma\mu}-\frac{1}{8}\nabla_{\kappa}\nabla_{\mu}K^{\lambda}_{\nu\sigma\gamma}\\
&\left.+\frac{5}{8}\nabla_{\kappa}\nabla_{\sigma}K^{\lambda}_{\nu\gamma\mu}+\frac{5}{8}\nabla_{\kappa}\nabla_{\mu}K^{\lambda}_{\nu\gamma\sigma}-\frac{1}{8}\nabla_{\kappa}\nabla_{\gamma}K^{\lambda}_{\mu\sigma\nu}-\frac{1}{8}\nabla_{\kappa}\nabla_{\nu}K^{\lambda}_{\mu\sigma\gamma}\right]\\
=&-\frac{2}{30}\left[\frac{5}{4}\nabla_{(\kappa}\nabla_{\sigma)}\left(K^{\lambda}_{\mu\nu\gamma}+K^{\lambda}_{\mu\gamma\nu}+K^{\lambda}_{\nu\gamma\mu}\right)+\frac{6}{8}\nabla_{\sigma}\nabla_{\nu}K^{\lambda}_{\mu\gamma\kappa}+\frac{6}{8}\nabla_{\sigma}\nabla_{\mu}K^{\lambda}_{\mu\nu\kappa}\right.\\
&\left.+\frac{6}{8}\nabla_{\sigma}\nabla_{\gamma}K^{\lambda}_{\mu\nu\kappa}+\frac{6}{8}\nabla_{\kappa}\nabla_{\nu}K^{\lambda}_{\mu\gamma\sigma}+\frac{6}{8}\nabla_{\kappa}\nabla_{\mu}K^{\lambda}_{\mu\nu\sigma}+\frac{6}{8}\nabla_{\kappa}\nabla_{\gamma}K^{\lambda}_{\mu\nu\sigma}\right]\\
=&-\frac{2}{30}\left[\frac{6}{8}\nabla_{(\sigma}\nabla_{\nu)}K^{\lambda}_{\mu\gamma\kappa}+\frac{6}{8}\nabla_{(\sigma}\nabla_{\mu)}K^{\lambda}_{\mu\nu\kappa}+\frac{6}{8}\nabla_{(\sigma}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\kappa}+\frac{6}{8}\nabla_{(\kappa}\nabla_{\nu)}K^{\lambda}_{\mu\gamma\sigma}\right.\\
&\left.+\frac{6}{8}\nabla_{(\kappa}\nabla_{\mu)}K^{\lambda}_{\mu\nu\sigma}+\frac{6}{8}\nabla_{(\kappa}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\sigma}+\frac{6}{8}\sum_{(3)}K\cdot K\right].\end{aligned} \tag{A.25}$$

Gathering all coefficients multiplying second-order covariant derivatives terms, we obtain

$$\begin{aligned}K^{\lambda}_{\kappa\sigma\mu\nu\gamma}=&\frac{3}{40}[6(\nabla_{(\mu}\nabla_{\nu)}K^{\lambda}_{\kappa\sigma\gamma}+\nabla_{(\mu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\nu}+\nabla_{(\nu}\nabla_{\gamma)}K^{\lambda}_{\kappa\sigma\mu})\\
&-(\nabla_{(\sigma}\nabla_{\mu)}K^{\lambda}_{\nu\gamma\kappa}+\nabla_{(\sigma}\nabla_{\nu)}K^{\lambda}_{\mu\gamma\kappa}+\nabla_{(\sigma}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\kappa}\\
&+\nabla_{(\kappa}\nabla_{\mu)}K^{\lambda}_{\nu\gamma\sigma}+\nabla_{(\kappa}\nabla_{\nu)}K^{\lambda}_{\mu\gamma\sigma}+\nabla_{(\kappa}\nabla_{\gamma)}K^{\lambda}_{\mu\nu\sigma})]\\
&+\left(-\frac{11}{30}\cdot\frac{1}{8}\right)\sum_{(1)}K\cdot K+\left(\frac{1}{30}\cdot\frac{5}{8}\right)\sum_{(2)}K\cdot K+\left(-\frac{2}{30}\cdot\frac{1}{8}\right)\sum_{(3)}K\cdot K.\end{aligned} \tag{A.26}$$

The quadratic terms $K \cdot K$ follow from the following formula:

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta K^\lambda_{\kappa\sigma\mu} &= \nabla_{(\alpha}\nabla_{\beta)} K^\lambda_{\kappa\sigma\mu} + \nabla_{[\alpha}\nabla_{\beta]} K^\lambda_{\kappa\sigma\mu} \\
&= \nabla_{(\alpha}\nabla_{\beta)} K^\lambda_{\kappa\sigma\mu} + R^\lambda_{\delta\alpha\beta} K^\delta_{\kappa\sigma\mu} + R^\delta_{\kappa\alpha\beta} K^\lambda_{\delta\sigma\mu} + R^\delta_{\sigma\alpha\beta} K^\lambda_{\kappa\delta\mu} + R^\delta_{\mu\alpha\beta} K^\lambda_{\kappa\sigma\delta} \\
&= \nabla_{(\alpha}\nabla_{\beta)} K^\lambda_{\kappa\sigma\mu} + (-K^\lambda_{\delta\alpha\beta} + K^\lambda_{\delta\beta\alpha}) K^\delta_{\kappa\sigma\mu} + (-K^\delta_{\kappa\alpha\beta} + K^\delta_{\kappa\beta\alpha}) K^\lambda_{\delta\sigma\mu} \\
&\quad + (-K^\delta_{\sigma\alpha\beta} + K^\delta_{\sigma\beta\alpha}) K^\lambda_{\kappa\delta\mu} + (-K^\delta_{\mu\alpha\beta} + K^\delta_{\mu\beta\alpha}) K^\lambda_{\kappa\sigma\delta}. \qquad \text{(A.27)}
\end{aligned}
$$

Calculating the quadratic terms for all three parts and grouping them together, we get the following expression:

$$
\begin{aligned}
\sum_{\text{total}} K \cdot K &= \frac{1}{40}[-12(K^\alpha_{(\mu\nu)\sigma} K^\lambda_{(\gamma)\alpha\kappa} + K^\alpha_{(\mu\nu)\kappa} K^\lambda_{(\gamma)\alpha\sigma}) \\
&\quad + 3(K^\alpha_{\mu\nu)\sigma} K^\lambda_{\kappa\alpha(\gamma} + K^\alpha_{(\mu\nu)\kappa} K^\lambda_{\sigma\alpha(\gamma)}) + 9K^\alpha_{\kappa\sigma(\mu} K^\lambda_{\nu\gamma)\alpha}] \\
&= \frac{3}{40}[-4(K^\alpha_{(\mu\nu)\sigma} K^\lambda_{(\gamma)\alpha\kappa} + K^\alpha_{(\mu\nu)\kappa} K^\lambda_{(\gamma)\alpha\sigma}) \\
&\quad + (K^\alpha_{\mu\nu)\sigma} K^\lambda_{\kappa\alpha(\gamma} + K^\alpha_{(\mu\nu)\kappa} K^\lambda_{\sigma\alpha(\gamma)}) + 3K^\alpha_{\kappa\sigma(\mu} K^\lambda_{\nu\gamma)\alpha}]. \qquad \text{(A.28)}
\end{aligned}
$$

Until now, for simplicity, we omitted the $f(l.o.d.\ \Gamma)$ part in calculating $K^\lambda_{\kappa\sigma\mu\nu\gamma}$. To obtain its complete form, we have to compare this K-quadratic expression with its counterpart in the formula (41) in paper [14], which reads

$$
\begin{aligned}
K^\lambda_{\kappa\sigma\mu\nu\gamma} &= \frac{3}{40}[6(\nabla_{(\gamma}\nabla_{\nu)} K^\lambda_{\kappa\sigma\mu} + \nabla_{(\mu}\nabla_{\gamma)} K^\lambda_{\kappa\sigma\nu} + \nabla_{(\nu}\nabla_{\mu)} K^\lambda_{\kappa\sigma\gamma}) \\
&\quad - (\nabla_{(\gamma}\nabla_{\sigma)} K^\lambda_{\mu\nu\kappa} + \nabla_{(\kappa}\nabla_{\gamma)} K^\lambda_{\mu\nu\sigma} + \nabla_{(\mu}\nabla_{\sigma)} K^\lambda_{\nu\gamma\kappa} \\
&\quad + \nabla_{(\kappa}\nabla_{\mu)} K^\lambda_{\nu\gamma\sigma} + \nabla_{(\nu}\nabla_{\sigma)} K^\lambda_{\gamma\mu\kappa} + \nabla_{(\kappa}\nabla_{\nu)} K^\lambda_{\gamma\mu\sigma})] \\
&\quad - \frac{3}{80}[23(K^\lambda_{\kappa\alpha(\gamma} K^\alpha_{\mu\nu)\sigma} + K^\lambda_{\sigma\alpha(\gamma} K^\alpha_{\mu\nu)\kappa}) - 37K^\alpha_{\kappa\sigma(\mu} K^\lambda_{\nu\gamma)\alpha} \\
&\quad - 2(K^\alpha_{(\mu\nu)\kappa} K^\lambda_{(\gamma)\alpha\sigma} + K^\alpha_{(\mu\nu)\sigma} K^\lambda_{(\gamma)\alpha\kappa})]. \qquad \text{(A.29)}
\end{aligned}
$$

So, the $f(l.o.d.\ \Gamma)$ term equals

$$
\begin{aligned}
f(l.o.d.\ \Gamma) &= \frac{3}{80}[-25(K^\lambda_{\kappa\alpha(\gamma} K^\alpha_{\mu\nu)\sigma} + K^\lambda_{\sigma\alpha(\gamma} K^\alpha_{\mu\nu)\kappa}) + 31K^\alpha_{\kappa\sigma(\mu} K^\lambda_{\nu\gamma)\alpha} \\
&\quad + 10(K^\alpha_{(\mu\nu)\kappa} K^\lambda_{(\gamma)\alpha\sigma} + K^\alpha_{(\mu\nu)\sigma} K^\lambda_{(\gamma)\alpha\kappa})]. \qquad \text{(A.30)}
\end{aligned}
$$

Finally, expressed in terms of first derivatives $\nabla_\gamma K^\lambda_{\kappa\sigma\mu\nu}$ of the curvature tensor $K^\lambda_{\kappa\sigma\mu\nu}$, the quantity $K^\lambda_{\kappa\sigma\mu\nu\gamma}$, assumes the form (56).

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