Review of the doctoral dissertation

Nowe zasady wariacyjne w Ogólnej Teorii Względności i ich związki z zasadą Hilberta

by Katarzyna Senger

For decades various generalizations of Einstein's theory of gravity based upon Lagrangians depending nonlinearly on the curvature tensor have been studied. The motivation for this line of research came both from efforts to construct renormalizable quantum gravity models and from the aim to build models of cosmology which, in particular, could possibly describe dark matter.

The important question whether such models yield genuine generalizations of Einstein's theory of gravity was answered by Kijowski and collaborators: such models are mathematically equivalent to Einstein's theory of gravity interacting with additional matter fields. This seems to be an almost trivial observation, because higher order differential equations may be subsequently reduced by treating higher order derivatives as additional variables, but – under some natural regularity assumptions on the Lagrangian – Kijowski and collaborators were able to additionally prove the following:

1. The reduced theory is covariant.
2. There is a well-defined variational principle and an associated Hamiltonian formulation after reduction.

In the doctoral dissertation under consideration, the author considers a broader class of generalized Lagrangians by allowing them to depend additionally on a finite number k of covariant derivatives of the curvature tensor. In such a theory, besides the standard Bianchi identities, there is a finite number of additional differential identities induced from the Bianchi identities and, thus, it is a nontrivial task to find a set of independent quantities. For this analysis, it is convenient to describe the curvature of the symmetric connection Γ under consideration locally by a curvature tensor field $K^{\lambda}_{\mu\nu\kappa}$ which is related to the standard Riemann tensor field $R^{\lambda}_{\mu\nu\kappa}$ as follows:

$$K^{\lambda}_{\mu\nu\kappa} = -\frac{1}{3}\left(R^{\lambda}_{\mu\nu\kappa} + R^{\lambda}_{\nu\mu\kappa}\right) .$$

Clearly, by this relation, K inherits Bianchi identities from R in an obvious way.

Equivalently, $K^\lambda_{\mu\nu\kappa}$ may be characterized as the tensor field whose components coincide with the partial derivatives $\Gamma^\lambda_{\mu\nu\kappa}$ of the Christoffel symbols of Γ in any inertial system of order 1, that is, in any local coordinate system at some $m \in M$ which is inertial at m and which fulfils $\Gamma^\lambda_{(\mu\nu\kappa)}(m) = 0$. This nice interpretation of the curvature in terms of local reference frames may be easily extended to higher order, that is, one defines the curvature tensor of order k by

$$K^\lambda_{\kappa\sigma\mu_1\ldots\mu_k} = \Gamma^\lambda_{\kappa\sigma\mu_1\ldots\mu_k} ,$$

in any coordinate system which is inertial of order k.

To use these quantities turns out to be the key technical idea of the dissertation under consideration. Now, viewing a given symmetric connection Γ in the standard way as a section of the affine bundle $R(M)$ of local reference frames over space time M, the kinematics of the k-th order theory can be elegantly described in terms of the k-th order jet bundle $J^kR(M)$. In detail, the author obtains the following:

1. Consider the symmetric connection Γ in a neighbourhood of $m \in M$, given in terms of its Christoffel symbols $\Gamma^\lambda_{\kappa\sigma}$. Then, there is a local coordinate transformation such that the quantities

$$\left(\Gamma^\lambda_{\kappa\sigma}, \Gamma^\lambda_{(\kappa\sigma\mu)}, \ldots, \Gamma^\lambda_{(\kappa\sigma\mu_1\ldots\mu_k)}\right)$$

take arbitrary chosen values.

2. At each point $m \in M$, the quantities

$$\left(K^\lambda_{\kappa\sigma\mu}, K^\lambda_{\kappa\sigma\mu_1\mu_2}, \ldots, K^\lambda_{\kappa\sigma\mu_1\ldots\mu_k}; \Gamma^\lambda_{\kappa\sigma}, \Gamma^\lambda_{(\kappa\sigma\mu)}, \ldots, \Gamma^\lambda_{(\kappa\sigma\mu_1\ldots\mu_k)}\right) ,$$

yield global coordinates in the fibre $J^k_mR(M)$. Thus, interpreting the coordinate transformations under point 1 as gauge transformations, every k-th jet of a connection Γ splits into gauge equivalence classes labeled by the curvature tensors $\left(K^\lambda_{\kappa\sigma\mu}, K^\lambda_{\kappa\sigma\mu_1\mu_2}, \ldots, K^\lambda_{\kappa\sigma\mu_1\ldots\mu_k}\right)$.

3. As a consequence, any tensor-valued function on $J^kR(M)$ can only depend on the curvature tensors $K^\lambda_{\kappa\sigma\mu_1\ldots\mu_n}$, $n = 0, 1, \ldots, k$. In particular, the generalized Lagrangian mentioned at the beginning takes the form

$$L = L\left(K^\lambda_{\kappa\sigma\mu}, K^\lambda_{\kappa\sigma\mu_1\mu_2}, \ldots, K^\lambda_{\kappa\sigma\mu_1\ldots\mu_k}\right) . \tag{1}$$

Now, by (1), the reduction of the theory may be accomplished in a gauge invariant way. The starting point is the variation of L given by

$$\delta L = Q_{\lambda}^{\mu\nu\sigma}\delta K^{\lambda}_{\mu\nu\sigma} + Q_{\lambda}^{\mu\nu\sigma_1\sigma_2}\delta K^{\lambda}_{\mu\nu\sigma_1\sigma_2} + \ldots + Q_{\lambda}^{\mu\nu\sigma_1\ldots\sigma_k}\delta K^{\lambda}_{\mu\nu\sigma_1\ldots\sigma_k} ,$$

where the momenta $Q^{\mu\nu\sigma_1\ldots\sigma_n}$ are invariant tensor densities. To proceed, the author needs a further technical input. She proves that the following recursion-type relations hold:

$$K^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_n} = S^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_n} + f\left(K^{\lambda}_{\kappa\sigma\mu}, K^{\lambda}_{\kappa\sigma\mu_1\mu_2}, \ldots, K^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_{n-1}}\right) ,$$

where S is a linear combination of the covariant derivatives $\nabla_{\mu_n}K^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_{n-1}}$ and f is a possibly nonlinear tensorial function. By this identity, one may replace $K^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_k}$ by $S^{\lambda}_{\kappa\sigma\mu_1\ldots\mu_k}$ in L and, therefore, accomplish the first step of the gauge invariant reduction procedure by performing the Legendre transformation between S and the highest order momentum. This way, the variational problem splits into the first-order variational problem for the highest order momentum (which becomes a new matter field) and a variational problem of order $(k-1)$. Repeating this procedure $(k-1)$ times leads to a variational problem of first order with a number of new matter fields given by the canonical momenta Q. By the above mentioned results of Kijowski and collaborators, such a theory is equivalent to Einstein's theory of gravity interacting with some matter fields.

I should like to add the following remarks:

1. In the Introduction, I would have expected at least some comments on why higher order Lagrangians are supposed to be relevant for describing cosmological phenomena like e.g. dark matter.

2. The statement in Chapter 6 that the dissertation presents a novel approach to connection theory is not true. It is well known that any connection on a fibre bundle $Y \to X$ may be viewed as a section of the affine bundle $J^1Y \to Y$. A detailed presentation of this point of view may be found e.g. in the monographs by Saunders on the geometry of jet bundles and by Kolar, Michor and Slovak on natural operations in differential geometry. The local geometric interpretation of the curvature used in this dissertation was, up to my knowledge, given first by Kijowski in his textbook on differential geometry as a tool in natural sciences. The extension of this interpretation to higher order curvature tensors by using inertial frames of higher order is novel and – as it turns out – it is very useful.

3. It would have been nice to see at least one example (based on a concrete higher order Lagrangian) illustrating the whole reduction procedure.

To summarize, the dissertation under consideration is an interesting contribution to the theory of gravity and cosmology written from a mathematical physics perspective. I assess it as 'good'.

I conclude that the presented dissertation meets the formal requirements for a Ph.D. thesis and recommend admission of the Candidate to the subsequent stages of the procedure, including the public defense.

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