External Examiner's Report: Thesis of Piotr Waluk

The thesis

1. The thesis consists of five sections and six appendices, a total of 94 pages.

2. The first section describes the problem addressed in the thesis and reviews some preliminary mathematical material needed later. The second section contains some general discussion of the problem of defining mass in General Relativity, and in particular reviews an approach due to Kijowski from 1985 to proving positivity of the ADM mass. The third section reviews some quasi-local definitions of mass, motivated by a Hamiltonian formulation of General Relativity and leading up to the Kijowski-Liu-Yau and Wang-Yau definitions. The Hawking definition, which is important later, is also reviewed.

   The exposition in this part of the thesis is competent and clear, and gives the reader confidence in the author. The use of English is overall very good. I will note below the few places where it could be improved.

3. I will raise here a couple of points about the material reviewed. These are for the candidate's consideration – they are *not* suggested corrections. In the discussion of Kijowski's approach to positivity of the ADM mass, a number of possible gauge conditions for a foliation of the Cauchy surface are introduced – this is around sections 2.3.1, 2.3.2 and 2.3.3. Evidently, to complete a proof of positivity of the ADM mass by Kijowski's method, one needs to know that foliations, or coordinates, exist satisfying these gauge conditions and I imagine this is quite hard to prove – one recalls the long time between Geroch suggesting the use of Inverse Mean Curvature Flow in a similar setting (1973), and Huisken and Ilmanen's successful proof based on the idea (2001). I'm not for a moment suggesting the candidate should have done this, but he might make it clearer that this problem is still open.

   The second point is about the 'rigid sphere' condition, section 3.3.4, which is in turn based on the definition of 'equilibrated spherical coordinates' in [29]. As I understand from [29] one defines equilibrated spherical coordinates on a topological 2-sphere by requiring

   $$X_i := \int_S \ell_i dS = 0 \text{ where } \ell_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \text{ and } dS = \Omega^2 \sin\theta d\theta d\phi,$$

   and the metric is

   $$g = \Omega^2(d\theta^2 + \sin^2\theta d\phi^2),$$

   so effectively the condition is

   $$\int_{S_0} \Omega^2 \ell_i \sin\theta d\theta d\phi = 0 \text{ for } i = 1, 2, 3.$$

   Then one adjusts Ω by a conformal transformation in (θ, ϕ) to achieve this – so this condition amounts to arranging Ω^2 to have no $\ell = 1$ part. That is fine but might one not use a different condition to constrain the coordinates? For example

   $$0 = \int_S \Omega^k \ell_i dS = \int_{S_0} \Omega^{k+2} \ell_i \sin\theta d\theta d\phi, \tag{0.1}$$

for other choices of k. The choice made in [29] is quite natural but there are reasons for considering other choices i.e. other values of k.

This freedom is significant since it will affect the definition of equilibrated spherical coordinates and therefore the definition of a 'rigid sphere'. I notice another ambiguity in defining rigid spheres: in 3.3.4, why choose $||H|| = 0$ rather than $(||H||)^2 = 0$? The former has a square root which might be thought to be less natural.

Of course, for linearisation about a standard 'round' sphere in Minkowski space these different definitions are likely (possibly even certain) to coincide. If $\Omega =$ constant $+O(\epsilon)$ then (0.1) gives the same condition to order ϵ for all k, and the different possible definitions of 'rigid sphere' probably also coincide. Since the proof of the existence of rigid spheres is presumably by an application of the Implicit Function Theorem, then existence probably also carries over to these other definitions too, at least near the standard examples. My point is that taking account of higher orders, the definitions of rigid sphere, and therefore the definition of mass, are likely to differ – the mass will not be unique if there is freedom in the definition of a rigid sphere.

4. In Section 4 we reach the original work in the thesis. This section is devoted to analysing linear perturbations of the 4-dimensional Kottler metric in order to find a Hamiltonian formulation. The technique is essentially to separate the perturbation equations in spherical harmonics and to introduce and work with gauge-invariant quantities. In a sense this has been done before, starting with Regge, Wheeler and Zerilli, but it's reasonable to rework the calculation in this slightly more general case (including the cosmological constant) to have the results in a formalism adapted to Section 5. These are substantial calculations and the outcome is the relatively simple expressions at the end of Section 4.

5. Section 5 contains the principal new result of the thesis: that the Hawking mass at a boundary 2-sphere is tied to the mass defined by the Hamiltonian of linear theory on a spanning 3-surface, provided the boundary spheres are subjected to constraints which are motivated by the earlier discussion of rigid spheres. This is an attractive result and in some sense a further vindication of the Hawking mass as a quasi-local definition of mass, at least when close to flat space or to standard spheres in spherical symmetry. It gives a consistent picture tying together the Hawking mass and the Hamiltonian definitions.

Recommendation

The Candidate has 5 articles on the arXiv, all jointly authored. Two of these are on a different set of problems from those considered in the thesis. The other 3 are related to the thesis. The first of these reviews the Kijowski approach to proving the positive mass theorem; the second contains material related to Section 4; the third contains material related to Section 5 and the principal new result of the thesis. The last two of the second set and one of the first set have been published in refereed journals. Thus two refereed publications have emerged from the thesis. This is a good rate of publication.

I think this is a good thesis, well-written and well-constructed, with good results which have led to refereed publications.Therefore I conclude that the presented dissertation meets the formal requirements for a Ph.D. thesis and recommend admission of the

Candidate to the subsequent stages of the procedure, including the public defense.

Minor errors

- There a few places where the English word used may not be the best choice. As examples: on p4 line 21 of the text 'obscure' might be better as 'unfamiliar'; on p6, third bullet point, 'big Latin indices' should be either 'capital' or 'upper case' Latin indices; on p15 section 2.3.1, does the author mean 'topologically flat'? 'flatness' is not a topological property – are these flat 2-planes? or toroidal?; on p25 line 18 of section 3.2 'variate' should just be 'vary; p26, 4 lines up, 'revolting' is a bit strong – how about 'unattractive' or 'undesirable'?; on p27, bottom line, 'tangible' doesn't seem like the right adjective – should it be one of 'definite', 'physically reasonable', 'precise' or 'concrete'?; on p30 just after (3.28) the boundary condition on ϕ looks like Dirichlet rather than Neumann; p35 line 32, 'in a general case' would be better as just 'in general'; p42 line 20 (point 3), 'requested' should be 'required'; p44 line 18 'instance' should be 'instant'; p46 bottom line, 'gravitational' should be 'cosmological'; p50 line 16, 'merit' should be 'virtue'; p63 top line 'in this approximation' is better than 'in its approximation'; not strictly use of English but on p70 and once or twice elsewhere there is reference to 'Kerr-Schild' metrics which I don't think is meant – probably this should be 'Kerr' or generalisations of Kerr to include Λ – Kerr-Schild metrics are something else; also 'Belinfante' is spelt 'Belifante' in several places.

- Simple errors I've spotted are: p8 line 9 'extrinsic curvature' should be 'extrinsic torsion'; still on p8, the definition of dip(A) isn't quite right – as defined there, dip(A) is a number (or a set of numbers) when it should be a linear combination of $\ell = 1$ spherical harmonics; in section 2.2 there are occurences of k in the text which should probably be k; p28 just before (3.18) 'a following' should be 'the following' (articles are always difficult I know); p30 line 19 'an information' should just be 'information'; p38 line 21, 'two two' should be 'two'; p44 line 14, 'Such tube' should be 'Such a tube'; p46 line 15, 'gauge freedom of the' should be 'gauge freedom of'; p46 bottom line 'include possible' should be 'include the possible'; p53 line 31 there is a 'that' which should be a 'than'; p92 ref [58], 'Stringcture' should be 'Structure'.

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