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Topology of configuration spaces for particles on graphs

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I dedicate this thesis to my mother.

Abstract

We study non-abelian quantum statistics on graphs via certain topological invariants, which are the homology groups of configuration spaces. In the first part of this thesis, we formulate a general framework for studying quantum statistics of particles constrained to move in a topological space X. The framework involves the study of flat complex vector bundles over the space of unordered tuples of points from X, known as the configuration space of X. In the second part, we apply this methodology for configuration spaces of graphs. In particular, we use discrete models of graph configuration spaces, which are due to Świątkowski and Abrams. The discrete models are CW complexes, who carry all information about the topology of graph configuration spaces. This allows us to use the tools from algebraic topology to compute the homology groups of graph configuration spaces for some families of graphs. These families are i) tree graphs, ii) wheel graphs, iii) complete bipartite graphs $K_{2,p}$ and $K_{3,3}$. We also describe the generators of the second homology group of simple graphs. Moreover, we compute the homology groups of graph configuration spaces for some small canonical graphs via the discrete Morse theory. As a conclusion, we provide families of graphs, which are good candidates for simplified models in the further study of quantum statistical phenomena and as such may find use for example in anyonic quantum computations.

Abstract in Polish

W poniższej pracy zajmujemy się problemem opisu nieabelowych statystyk kwantowych przy użyciu pewnych topologicznych niezmienników, którymi są grupy homologii przestrzeni konfiguracyjnych. W pierwszej części pracy formułujemy ogólną metodologię opisu statystyk kwantowych dle cząstek, których ruch ograniczony jest do przestrzeni topologicznej X. Metodologia ta korzysta z konstrukcji płaskich zespolonych wiązek wektorowych nad przestrzenią nieuporządkowanych krotek punktów z przestrzeni X, zwaną przestrzenią konfiguracyjną przestrzeni X. W drugiej części pracy stosujemy powyższą metodologię do przestrzeni konfiguracyjnych na grafach. W szczególności, wykorzystujemy dyskretne modele grafowych przestrzeni konifguracyjnych wprowadzone przez Świątkowskiego oraz Abramsa. Modele dyskretne to CW kompleksy, które niosą całą informację o topologii grafowych przestrzeni konfiguracyjnych. Pozwalają one na użycie narzędzi z topologii algebraicznej przy obliczaniu grup homologii. W efekcie, podajemy pełen opis grup homologii przestrzeni konfiguracyjnych dla grafów będących drzewami, grafami kołowymi oraz grafami dwudzielnymi $K_{2,p}$ i $K_{3,3}$. Podajemy również generatory drugiej grupy homologii przestrzeni konfiguracyjnych na grafach prostych. Ponadto, używając dyskretnej teorii Morse'a, obliczamy niektóre grupy homologii przestrzeni konfiguracyjnych na wybranych małych grafach. Na podstawie uzyskanych wyników podajemy rodziny grafów, które są dobrymi kandydatami do dalszych badań zjawisk kwantowo statystycznych i jako takie mogą znaleźć zastosowanie na przykład w anyonowych obliczeniach kwantowych.

The title of the thesis in Polish

Topologia przestrzeni konfiguracyjnych dla cząstek na grafach

List of publications

Publications related to the content of the thesis

1. Maciążek, T., Sawicki, A., *Homology groups for particles on one-connected graphs*, J. Math. Phys., Vol 58, no 6, 2017

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3. Maciążek, T., Tsanov, V., *Quantum marginals from pure doubly excited states*, J. Phys. A: Math. Theor. 50 465304, 2017

4. Maciążek, T., Wojtkiewicz, J., *On the phase diagram of the anisotropic XY chain in transverse magnetic field*, Physica A: Statistical Mechanics and its Applications Vol. 441, pp 131-140, 2016

5. Maciążek, T., Joyner, C., H., Smilansky, U., *The probability distribution of spectral moments for the Gaussian beta-ensembles*, Acta Physica Polonica A, i. 6, vol. 128, p. 983-995, 2015

6. Maciążek, T., Sawicki, A., *Critical points of the linear entropy for pure L-qubit states*, J. Phys. A: Math. Theor. 48 045305, 2015

7. Maciążek, T., Oszmaniec, M., Sawicki, A., *How many invariant polynomials are needed to decide local unitary equivalence of qubit states?*, J. Math. Phys. 54, 092201, 2013

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Chapter 1

Introduction

The first part of this thesis (chapters 1-3) contains general considerations about the connections between topology of configuration spaces and the existence of different types of quantum statistics. This is an attempt to collect and organise some of the results that are partially a folklore knowledge in mathematical physics. The underlying idea is to compare the approach to quantum statistics that dates back to the works of Souriau [1] and Leinaas and Myrheim [2] with the modern mathematical methods concerning the classification of complex vector bundles. We especially emphasise the important role of nontrivial flat vector bundles. One of our main goals is to formulate our work in a possibly general context, so that the considered topological spaces do not necessarily have to be differential manifolds. All the results formulated in this thesis are suitable for topological spaces that have the homotopy type of a finite CW-complex. Our work results with a number of open problems indicating the possibility of the existence of some new quantum statistical phenomena. The general methods that we lay down in the first three chapters of this thesis, are applied to a special class of configuration spaces of particles on graphs, which may serve as simple models for studying quantum statistical phenomena. In particular, we compute certain topological invariants of graph configuration spaces, which are the homology groups. We review the known methods of computing the homology groups of graph configuration spaces and develop new computational tools.

Configuration space of n indistinguishable particles confined in a topological space X is defined as the quotient space

$$C_n(X) := (X^{\times n} - \Delta_n)/S_n,$$

where $\Delta_n := \{(x_1, \ldots, x_n) \in X^{\times n} : \ \exists_{i \neq j} \ x_i = x_j\}$ and S_n is the permutation group that acts on $X^{\times n}$ by permuting coordinates. Such spaces appear in an alternative description of physical phenomena that are related to the indistinguishability of particles, where the property of particles being indistinguishable is imposed already at the level of their configurations. Such an approach dates back to the works of Souriau [1], Leinaas and Myrheim [2] (see also [3]), who proved the emergence of quantum statistics by considering wave functions and Schrödinger operators on $C_n(X)$. A systematic way to do this is to consider a proper quantisation scheme for configuration spaces. Quantisation is done in two steps.

1. Defining *quantum kinematics*, i.e. defining the space of wave functions and deriving momentum operators that satisfy the canonical commutation rules.

2. Defining *quantum dynamics*, i.e. deriving the Schrödinger equation that describes the evolution of wave functions.

In this thesis, we mainly focus on the problem of defining and classifying quantum kinematics in the case, when X is a finite graph. We formulate the results in analogy to the well-studied case, where X is a smooth manifold.

1.1. Quantum kinematics on smooth manifolds

A quantisation procedure for configuration spaces, where X is a smooth manifold, known under the name of *Borel quantisation*, has been formulated by H.D. Doebner et. al. and formalised in a series of papers [4, 5, 6, 8, 9]. Borel quantisation on smooth manifolds can be also viewed as a version of the geometric quantisation [7]. The main point of Borel quantisation is the fact that the possible quantum kinematics on $C_n(X)$ are in a one-to-one correspondence with conjugacy classes of unitary representations of the fundamental group of the configuration space. We denote this fact by

$$QKin_k(C_n(X)) \cong {}^{\mathrm{Hom}(\pi_1(C_n(X)),U(k))}/_{U(k)},$$

where $QKin_k$ are the quantum kinematics of rank k. i.e. kinematics, where wave functions have values in $\mathbb{C}^k$ and π_1 is the fundamental group. Let us next briefly describe the main ideas standing behind the Borel quantisation, which will be the starting point for building an analogous theory for indistinguishable particles on graphs.

In Borel quantisation on smooth manifolds, wave functions are viewed as square-integrable sections of hermitian vector bundles (Fig. 1.1).

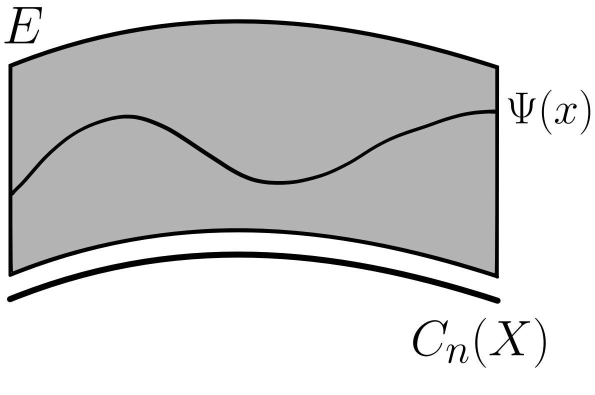


Figure 1.1: Wave function as a square-integrable section of a hermitian vector bundle E over a configuration space.

For a fixed hermitian vector bundle, the momentum operators are constructed by assigning a self-adjoint operator $\hat{p}_A$ acting on sections of E to a vector field A that is tangent to $C_n(X)$ in the way that respects the Lie algebra structure of tangent vector fields. Namely, we require the standard commutation rule for momenta, i.e.

$$[\hat{p}_A, \hat{p}_B] = \imath \hat{p}_{[A,B]}, \ \ A, B \in TC_n(X). \tag{1.1}$$

Moreover, for the position operator that acts on sections as multiplication by smooth functions

$$\hat{q}_f(\sigma) := f\sigma, \ \ f \in C^\infty(C_n(X)), \ \ \sigma \in Sec(E),$$

we require the remaining standard commutation rules, i.e.

$$[\hat{p}_A, \hat{q}_f] = \hat{q}_{Af}. \tag{1.2}$$

It turns out that such a requirement implies the form of the momentum operator, which is well-known form the minimal coupling principle, namely

$$\hat{p}_A = \iota \nabla_A + \frac{\iota}{2}\mathrm{div}(A), \tag{1.3}$$

where ∇_A is a covariant derivative in the direction of A, that is compatible with the hermitian structure. Moreover, commutation rule (1.1) implies, that ∇_A is necessarily the covariant derivative stemming from a flat connection. Flat hermitian connections of rank k are classified by conjugacy classes of $U(k)$ representations of $\pi_1(C_n(X))$ (see [72]). Representatives of these classes can be picked by specifying the holonomy on a fixed set of loops generating the fundamental group. In order to illustrate these concepts, consider the following example of one particle restricted to move on the plane and its scalar wave functions.

Example 1.1. Quantum kinematics of rank 1 for a single particle on the plane. The momentum has two components that are given by (1.3) for $A = \partial_x =: \partial_1$ and $A = \partial_y =: \partial_2$.

$$\hat{p}_1 := \hat{p}_{\partial_x} = \frac{1}{\iota}\partial_x - \alpha_1,\ \hat{p}_2 := \hat{p}_{\partial_y} = \frac{1}{\iota}\partial_y - \alpha_2.$$

By a straightforward calculation, one can check that commutation rule (1.2) is satisfied.

$$\forall_\Psi \ \ [\hat{p}_i, \hat{q}_f]\Psi = -\iota\Psi\partial_i f = \hat{q}_{-\iota\partial_i f}\Psi.$$

However, commutation rule (1.1) requires $[\hat{p}_1, \hat{p}_2] = 0$. The commutator reads

$$\forall_\Psi \ \ [\hat{p}_1, \hat{p}_2]\Psi = \iota\Psi(\partial_1\alpha_2 - \partial_2\alpha_1).$$

Therefore, in order to satisfy the momentum commutation rule, we need $\partial_1\alpha_2 - \partial_2\alpha_1 = 0$. This is precisely the condition for the connection form $\Gamma := \alpha_1 dx + \alpha_2 dy$ to have zero curvature, i.e. $d\Gamma = 0$. The plane is a contractible space, hence the problem of classifying flat connections is trivial and there are no topological effects in the quantum kinematics. However, we can make the problem nontrivial by considering the situation, where a particle is moving on a plane without a point, i.e. $X = \mathbb{R}^2 - \{*\}$. Then, $\pi_1(X) = \mathbb{Z}$ generated by a circle around $\{*\}$ travelled clockwise. Let us denote such a loop by γ. The parallel transport of Ψ around γ gives

$$\hat{T}_\gamma\Psi = e^{\iota\int_\gamma\Gamma}\Psi.$$

The phase factor $e^{\iota\int_\gamma\Gamma}$ does not depend on the choice of the circle. In order to see this, choose a different circle γ' that contains γ. Denote by D the area between the circles. We have $\partial D = \gamma' - \gamma$. Hence, by the Stokes theorem

$$0 = \int_D dxdy(d\Gamma) = \int_{\partial D}\Gamma = \int_{\gamma'}\Gamma - \int_\gamma\Gamma.$$

Hence, all $U(1)$ representations of $\pi_1(X)$ are isomorphic to the representations that assign a phase factor $e^{i\phi}$ to a chosen non-contractible loop. Physically, these representations can be realised as the Aharonov-Bohm effect and phase ϕ is the magnetic flux through point $*$ that is perpendicular to the plane.

Let us next review two scenarios that originally appeared in the paper by Leinaas and Myrheim [2] and that led to a topological explanation of the existence of bosons, fermions and anyons. These are the scenarios of two particles in $\mathbb{R}^2$ and $\mathbb{R}^3$. In both cases, the configuration space can be parametrised by the centre of mass coordinate R and the relative position r. In terms of the positions of particles, we have

$$R = \frac{1}{2}(x_1 + x_2),\ r = x_2 - x_1,\ x_i \in \mathbb{R}^m.$$

Then, $C_2(\mathbb{R}^m) = \{(R, r) : R \in \mathbb{R}^m,\ r \in \mathbb{R}^m - 0\}/S_2$. Permutation of particles results with changing r to $-r$, while R remains unchanged, hence

$$C_2(\mathbb{R}^m) = \mathbb{R}^m \times ((\mathbb{R}^m - 0)\,/\sim) \cong \mathbb{R}^m \times \mathbf{RP}^{m-1}.$$

In the above formula, $\mathbf{RP}^{m-1} := S^{m-1}/\sim$ is the real projective space that is constructed by identifying pairs of opposite points of the sphere. Space $(\mathbb{R}^m - 0)\,/\sim$ can be deformation retracted to $\mathbf{RP}^{m-1}$ by contracting all vectors so that they have length 1. In the case, when $m = 2$, $\mathbf{RP}^1$ is topologically a circle. Equivalently, $(\mathbb{R}^2 - 0)\,/\sim$ is a cone. Hence, we have

$$\pi_1(C_2(\mathbb{R}^2)) = \mathbb{Z},$$

so similarly to Example 1.1, there is a continuum of $U(1)$-representations of the fundamental group that assign an arbitrary phase factor to the wave function when transported around a non-contractible loop. Note that a loop in the configuration space corresponds to an exchange of particles (see Fig. 1.2).

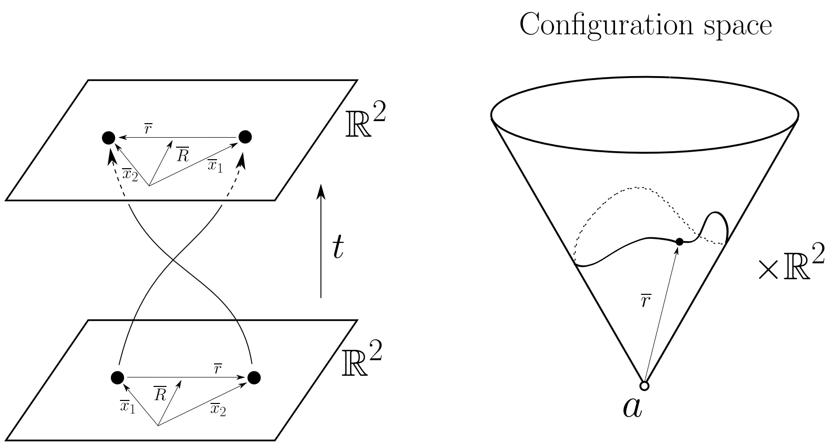


Figure 1.2: Exchange of two particles on the plane and the resulting loop in $C_2(\mathbb{R}^2)$.

The case of two particles moving in $\mathbb{R}^3$ has an important difference when compared to the other cases analysed in this thesis so far. Namely, there are two non-isomorphic hermitian vector bundles of rank 1 that admit a flat connection. In all previous cases there was only one such vector bundle, which was isomorphic to the trivial vector bundle $E_0 \cong C_n(X) \times \mathbb{C}$. For $m = 3$, there is one more flat hermitian vector bundle, which we denote by E'. Neglecting the $\mathbb{R}^3$ - component of $C_2(\mathbb{R}^3)$, which is contractible, bundles E_0 and E' can be constructed from a trivial vector bundle on S^2 in the following way.

$$E_0 = \left(S^2 \times \mathbb{C}\right)/\sim,\ (r, z) \sim (-r, z) \cong \mathrm{RP}^2 \times \mathbb{C},$$
$$E' = \left(S^2 \times \mathbb{C}\right)/\sim',\ (r, z) \sim' (-r, -z).$$

Intuitively, nontrivial bundle E' is constructed from the trivial vector bundle on S^2 by twisting fibres over antipodal points. In order to determine the statistical properties corresponding to each bundle, we consider $U(1)$ representations of the fundamental group for each vector bundle. The choice of statistical properties for each vector bundle is a consequence of a general construction of flat vector bundles, which we describe in more detail in section 3.3. The fundamental group reads

$$\pi_1(C_2(\mathbb{R}^3)) \cong \pi_1(\mathrm{RP}^2) \cong \mathbb{Z}_2.$$

There are two types of loops, the contractible ones and the non-contractible ones, which become contractible when composed twice (see Fig. 1.3).

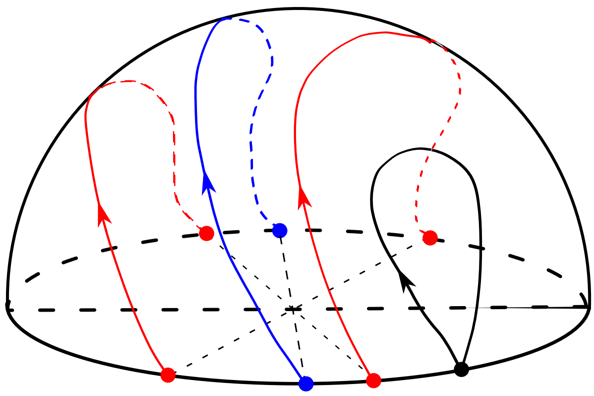


Figure 1.3: Two types of loops in RP^2 pictured as a half-sphere with the opposite points on the circumference of the base identified. Black loop and red loop are contractible, while blue loop is non-cntractible. Blue loop becomes homotopy equivalent to the red loop when crossed twice.

Bundle E_0 corresponds to the trivial representation of π_1, while E' corresponds to the alternating representation that acts with multiplication by a phase factor $e^{i\pi}$. Consequently, the holonomy group changes the sign of the wave function from E' when transported along a non-contractible loop, while the transport of a wave function from the trivial bundle results with the identity transformation. Therefore, bundle E_0 is called bosonic bundle, whereas bundle E' is called the fermionic bundle.

In general, when particles are constrained to move in a topological space X, which is paracompact and Hausdorff [1], we aim to realise the following programme.

Classification scheme for quantum kinematics of rank k on a topological space X

1. Topological classification of wave functions. Classify isomorphism classes of flat hermitian vector bundles of rank k over $C_n(X)$.

2. Classification of statistical properties. If X is a manifold, for each flat hermitian vector bundle, classify the flat connections. The parallel transport around

[1] Paracompactness of a space is a property of its open covers. For every open cover $\{U_\alpha : \alpha \in A\}$ (A being the set of indices), one can form a refinement, which is another cover $\{V_\beta : \beta \in B\}$ such, that for every α there exists β such, that $V_\beta \subset U_\alpha$. Space X is paracompact, when its every open cover has a refinement, which is locally finite. We assume X to be paracompact, because it makes the problem of classification of quantum kinematics more tractable. All topological spaces considered in this thesis are paracompact.

loops in $C_n(X)$ determines the statistical properties. For general paracompact X, this point can be phrased as classification of the $U(k)$ - representations of the fundamental group.

As we have seen in the above examples, there is a fundamental difference between anyons in $\mathbb{R}^2$ and bosons and fermions in $\mathbb{R}^3$. Anyons emerge as different flat connections on the trivial line bundle over $C_2(\mathbb{R}^2)$, while fermions and bosons emerge as flat connections on non isomorphic line bundles over $C_2(\mathbb{R}^3)$. As we explain in chapter 3, these results generalise to arbitrary numbers of particles.

In this thesis, we approach the problem of classifying complex vector bundles by computing the cohomology groups of configuration spaces over integers. Such strategy has also been used used in [4] to partially classify vector bundles over configuration spaces of distinguishable particles in $\mathbb{R}^m$. To this end, we combine the following results concerning the structure of $\mathrm{Vect}(B)$, the set of complex vector bundles over a paracompact base space B.

1. **Classification of complex vector bundles by maps $f:\ B \to Gr_k(\mathbb{C}^\infty)$ and Chern classes.** Any complex vector bundle over a paracompact base space B can be obtained as a pullback of the universal vector bundle over the infinite Grassmannian $Gr_k(\mathbb{C}^\infty)$. The homotopy classes of maps from B to $Gr_k(\mathbb{C}^\infty)$ are in a one-to-one correspondence with the isomorphism classes of vector bundles of rank k. Chern classes are defined as pullbacks of generators of the cohomology ring of $Gr_k(\mathbb{C}^\infty)$ to B. Chern classes of isomorphic vector bundles are necessarily the same. We describe all the above notions in section 3.1.

2. **Classification of vector bundles of rank 1 by the second cohomology group.** The set of line bundles equipped with the tensor product, $\mathrm{Vect}_1(B)$, is an abelian group. This group is isomorphic to the second cohomology group of B over integers.
$$\mathrm{Vect}_1(B) \cong H^2(B, \mathbb{Z}).$$

3. **Classification of vector bundles in the stable range.** We use the notion or stable equivalence of vector bundles. Two vector bundles are stable equivalent if they become isomorphic after taking a direct sum with a trivial bundle. Stable equivalence classes of vector bundles under the operation of fiberwise addition form a group, which is the reduced Grothendieck group $\tilde{K}(B)$. We use the result of Atiyah and Hirzebruch [11] (see also [12]) which states that $\tilde{K}(B)\otimes\mathbb{Q}$ is equal to the direct sum of even cohomology groups of B over rationals. If B has the homotopy type of a finite CW-complex and $H^*(B, \mathbb{Z})$ is torsion-free, $\tilde{K}(B)$ is the direct sum of even homology groups of B over integers. In the stable range, i.e. for $k \geq \frac{1}{2}\mathrm{dim}B$, stable equivalence of vector bundles implies their equivalence, hence in the case of torsion-free cohomology, we have
$$\mathrm{Vect}_k(B) \cong \bigoplus_{i=1}^{\infty} H^{2i}(B, \mathbb{Z}),\ \text{for}\ k \geq \frac{1}{2}\mathrm{dim}B.$$
For more details, see section 3.2.

As we explain in chapter 4, the homotopical dimension of a graph configuration space for a sufficiently large number of particles is equal to the number of essential vertices of the considered graph. Hence, the stable range for vector bundles over $C_n(\Gamma)$ and n large reads

$$k \geq \frac{1}{2}|\{v \in V(\Gamma) : \ d(v) \geq 3\}|.$$

A possible source of new signatures of topology in quantum mechanics would be the existence of non-trivial vector bundles that admit a flat connection. These bundles are detected by the Chern classes, which for flat bundles belong to torsion components of $H^{2i}(B,\mathbb{Z})$. We explain this fact and its relation with quantum statistics in section 3.3.

The formulation of quantum dynamics of rank 1 for n identical particles in $\mathbb{R}^m$ is rather straightforward. By the minimal coupling principle, we consider hamiltonians that describe a single particle in $\mathbb{R}^{mn}$, i.e.

$$\hat{H} = V(x_1^1,\ldots,x_m^1,x_1^2,\ldots,x_m^n) + \frac{1}{2}\sum_{\substack{i=1,\ldots,m\\ a=1,\ldots,n}} (\hat{p}_i^a)^2, \ \hat{p}_i^a = -i\partial_{x_i^a} - \alpha_i^a,$$

where α_i^a are the coefficients of some closed 1-form $\Gamma = \sum_{i,a}\alpha_i^a dx_i^a$ (the connection 1-form). Writing down all hamiltonians is equivalent to the task of classifying all flat connections over $C_n(\mathbb{R}^m)$. For particles in $\mathbb{R}^2$, such an approach leads one to the well-known anyonic hamiltonian [70]. If the one-particle configuration space X is a smooth manifold, the kinetic part of the hamiltonian is the lift of the Laplace-Beltrami operator to the considered vector bundle. For details of this construction, see [5]. For X being a graph, one is led to considering self-adjoint extensions of some symmetric operators, see chapter 1.2.

1.2. Quantum kinematics on graphs

Configuration spaces of indistinguishable particles on graphs are defined as

$$C_n(\Gamma) := (\Gamma^{\times n} - \Delta_n)/S_n,$$

where $\Delta_n = \{(x_1,\ldots,x_n) \in \Gamma^{\times n} : \ \exists_{i\neq j} \ x_i = x_j\}$ and graph Γ is regarded as a 1-dimensional cell complex. Let us take a closer look at the structure of $C_n(\Gamma)$. Note first that $\Gamma^{\times n}$ is a CW-complex. More specifically, it is a cube complex, i.e. a CW-complex, where all d-cells are cubes and the gluing maps are injective. The d-cells of $\Gamma^{\times n}$ are products of d edges and a number of vertices of Γ. The set of d-cells is denoted by Σ^d.

$$\Sigma^d(\Gamma^{\times n}) = \{\sigma_1 \times \cdots \times \sigma_n : \ \sigma_i \in E(\Gamma)\cup V(\Gamma), \ |\{i : \ \sigma_i \in E(\Gamma)\}| = d\}.$$

In particular, the n-cells in $\Gamma^{\times n}$ are of the form

$$\Sigma^n(\Gamma^{\times n}) = \{e_1 \times \cdots \times e_n : \ e_i \in E(\Gamma)\}.$$

The corresponding chain complex is a bigraded $\mathbb{Z}$-module with the canonical basis $\bigcup_{d=0}^{\infty}\Sigma^d$, which has the structure of a monoid with respect to the multiplication by edges and vertices of Γ (multiplication is associative, but not commutative). The degrees of the components are

$$|v| = (1,0), \ |e| = (1,1), \ v \in V(\Gamma), \ e \in E(\Gamma),$$

where the first index denotes the number of particles and the second index denotes the dimension. The boundary map is constructed by assigning an orientation to each edge of Γ, so that one can distinguish its initial vertex $\iota(e)$ and terminal vertex $\tau(e)$. Then, we have

$$\partial v = 0,\ \partial e = \tau(e) - \iota(e).$$

The boundary map for elements of a higher degree is determined by the Eilenberg-Zilber theorem:

$$\partial(\chi \otimes \eta) = (\partial\chi) \otimes \eta + (-1)^d \chi\partial\eta$$

for d-chain χ. Hence, the boundaries of two d-cells have a non-empty intersection of dimension $d-1$ iff the cells are composed of the same elements except one and the distinctive elements are edges in Γ that share a vertex. Moreover, a d-cell $\sigma_1 \times \cdots \times \sigma_n$ has a non-empty intersection with Δ_n iff $\sigma_i \cap \sigma_j \neq \emptyset$ for some $i \neq j$. Under the quotient by S_n cells that have an empty intersection with Δ_n, become sets of distinct elements of Γ, as different permutations of elements are identified. Cells that have a non-empty intersection with Δ_n are affected by the quotient map so that closures (as subsets of $\Gamma^{\times n}$) of their images have the form of simplices or cubes.

Example 1.2. Configuration space of two particles on graph Y. In $Y \times Y$ there are 9 two-cells. Six of them are products of distinct (but not disjoint) edges of Y. Their intersect with Δ_2 is a single point, which we denote by $(2,2)$. The three remaining two-cells are of the form $e \times e$. They have the form of squares, which intersect Δ_2 along the diagonal. Graph Y and space $C_2(Y)$ are shown on Fig. 1.4.

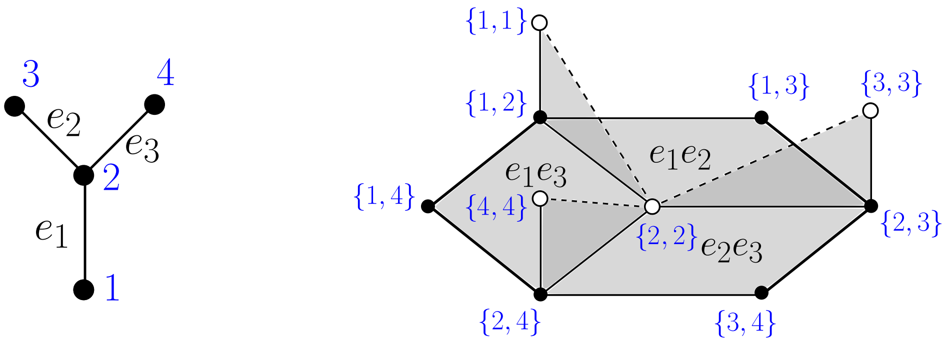


Figure 1.4: Graph Y and its two-particle configuration space. White dots and dashed lined denote the diagonal Δ_2.

The fact that $C_n(\Gamma)$ is composed of pieces that are locally isomorphic to $\mathbb{R}^n$ is the key property that allows one to define quantum kinematics as gluing the local quantum kinematics on $\mathbb{R}^n$. Namely, the momentum operator on $(e_1 \times e_2 \times \cdots \times e_n - \Delta_n)/S_n$ has n components that are defined as

$$\hat{p}_i = -\iota\partial_i - \alpha_i,\ i = 1, \ldots, n.$$

We may define orthonormal coordinates and connection coefficients on each n-cell separately. For each n-cell we require that the connection 1-form $\Gamma = \sum_{i=1}^n \alpha_i$ is closed, hence locally the connection is flat. In order to impose global flatness of the considered bundle, we require that the parallel transport does not depend on the homotopic deformations of curves that cross different pieces of $C_n(\Gamma)$. This requirement imposes conditions on the parallel transport operators along certain edges (1-dimensional cells) of $C_n(\Gamma)$. To see this, we need the following lemma by Abrams [28].

Lemma 1.3. *Fix n – the number of particles. If Γ has the following properties: i) each path between distinct vertices of degree not equal to 2 passes through at least $n-1$ edges, ii) each nontrivial loop passes through at least $n+1$ edges, then $C_n(\Gamma)$ deformation retracts to a CW-complex $D_n(\Gamma)$, which is a subspace of $C_n(\Gamma)$ and consists of the n-fold products of disjoint cells of Γ.*

Complex $D_n(\Gamma)$ is called Abram's discrete configuration space and we elaborate on its construction in chapter 4. For the construction of quantum kinematics, we only need the existence of the deformation retraction. This is because under this deformation, every loop in $C_n(\Gamma)$ can be deformed to a loop in $D_n(\Gamma) \subset C_n(\Gamma)$, which has a nicer structure of a CW-complex. Therefore, we only need to consider the parallel transport along loops in $D_n(\Gamma)$. Furthermore, every loop in $D_n(\Gamma)$ can be deformed homotopically to a loop contained in the one-skeleton of $D_n(\Gamma)$. The problem of gluing connections between different pieces of $C_n(\Gamma)$ becomes now discretised. Namely, we require that the unitary operators that describe parallel transport along the edges of $D_n(\Gamma)$ compose to the identity operator whenever the corresponding edges form a contractible loop. In other words,

$$U_{\sigma_1} U_{\sigma_2} \dots U_{\sigma_l} = \mathbb{1} \text{ if } \sigma_1 \to \sigma_2 \to \dots \to \sigma_l \text{ is a contractible loop in } D_n(\Gamma).$$

By $\sigma_1 \to \sigma_2 \to \dots \to \sigma_l$ we denote the path constructed by travelling along 1-cells σ_i in $D_n(\Gamma)$. This is a closed path whenever $\sigma_l \cap \sigma_1 \neq \emptyset$.

More formally, we classify all homomorphisms $\rho \in \mathrm{Hom}(\pi_1(C_n(\Gamma)), U(k))$ and consider the vector bundles that are induced by the action of ρ on the trivial principal $U(k)$-bundle over the universal cover of $C_n(\Gamma)$. For more details, see chapter 3.

Therefore, the classification quantum kinematics of rank k on $C_n(\Gamma)$ is equivalent to the classification of the $U(k)$ representations of $\pi_1(D_n(\Gamma))$. The described method of classification of quantum kinematics in the case of rank 1 becomes equivalent to the classification of discrete gauge potentials on $C_n(\Gamma)$ that were described in [15].

Example 1.4. Quantum kinematics of rank 1 of two particles on graph Y. The two-particle discrete configuration space of graph Y consists of 6 edges that form a circle (Fig. 1.5). Therefore, any non-contractible loop in $C_2(Y)$ is homeotopic with $D_2(Y)$.

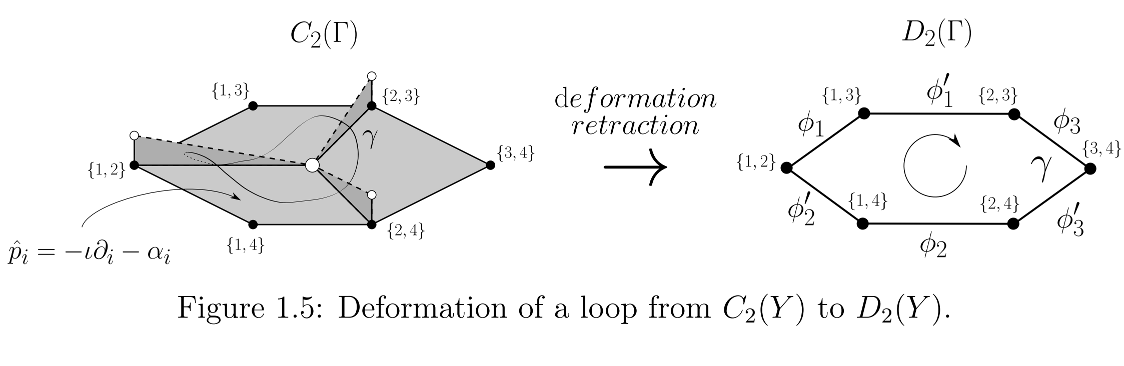


Figure 1.5: Deformation of a loop from $C_2(Y)$ to $D_2(Y)$.

The classification of kinematics of rank 1 boils down to writing down the consistency relation for $U(1)$ operators arising from the parallel transport along the edges in $D_2(Y)$. These operators are just phase factors

$$U_\sigma = e^{-i\phi_\sigma}, \; \phi_\sigma = \int_\sigma \alpha_1.$$

The parallel transport of a wave function results with

$$\hat{T}_\gamma \Psi = e^{-i\phi_0}\Psi,\ \phi_0 = \phi_1 + \phi_1' + \phi_2 + \phi_2' + \phi_3 + \phi_3'.$$

This is reflected in the fact that $\pi_1(C_2(Y)) = \mathbb{Z}$.

Quantum dynamics on graphs

We define quantum dynamics on graphs by studying self-adjoint extensions of hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{n}(-\iota\partial_i - \alpha_i)^2 \tag{1.4}$$

on $C_n(\Gamma)$, where the wave functions are defined piecewise on cells of the topological closure of $C_n(\Gamma)$ regarded as a subset of $\Gamma^{\times n}$. Namely, denote by $\Sigma^{(n)}(C_n(\Gamma))$ the set of n-cells of the topological closure of $C_n(\Gamma)$. Then, the wave functions are the square-integrable functions on $\Sigma^{(n)}(C_n(\Gamma))$

$$\Psi \in \bigoplus_{\sigma\in\Sigma^{(n)}(C_n(\Gamma))} L^2(\sigma).$$

A similar approach has been used in [17, 18, 19] to describe all self-adjoint extensions of the multi particle hamiltonian on networks

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{n}(-\iota\partial_i)^2 \tag{1.5}$$

with contact delta-interaction, which does not allow the particles to collide. There, the (scalar) wave functions are defined piecewise on cells of $\Gamma^{\times n}$ and the conditions for $\hat{H}$ from equation (1.5) to be self-adjoint take the form of certain boundary conditions for the wave functions and their derivatives on the boundaries of cubes from $\Gamma^{\times n}$. This approach is different form the approach proposed in this thesis, as the symmetry of wave functions with respect to the particle exchange must be imposed in the standard way as suitable symmetry of wave functions on $\Gamma^{\times n}$. Paper [16] shows a way of constructing some self adjoint extensions of hamiltonian (1.5) directly on $C_n(\Gamma)$ for two particles on a lasso graph and the graph being the wedge sum of two circles (graph of the shape of figure eight). Interestingly, some of the boundary conditions can be inferred just from the knowledge of the unitary representations of the fundamental group of $C_n(\Gamma)$. Hamiltonian (1.5) does not take into account the possibility of the existence of some non-trivial connection, which, as we have seen, is also a source of topological phenomena. The problem of describing the self-adjoint extensions of hamiltonian (1.4) is an open problem. A possible way of tackling this problem would be to join the methodology known from studying self adjoint extensions of hamiltonian (1.5) on $C_n(\Gamma)$ or on $\Gamma^{\times n}$ and the recent results concerning the unitary representations of $\pi_1(C_n(\Gamma))$ [20, 22].

Chapter 2

CW-complexes

In this chapter, we introduce a family of topological spaces that constitutes a common ground for all configuration spaces that are considered in this paper. These topological spaces are called CW complexes, and the reasons why they are important are the following two theorems.

Theorem 2.1. *The configuration space of any graph Γ can be deformation retracted to a finite cube complex.*

The above theorem has been already mentioned in section 1.2. A cube complex is a special kind of a CW complex, where the building blocks are cubes. We will return to the notion of a cube complex later in this chapter. This result is due to Świątkowski and Abrams, who introduced two different ways of derofming $C_n(\Gamma)$ to a cube complex [28, 14]. The following theorem regards a CW-complex structure of $C_n(\mathbb{R}^k)$ [55, 56].

Theorem 2.2. *The configuration space of n particles in $\mathbb{R}^n$ has the homotopy type of a finite CW-complex.*

As we explain in this chapter, using the structure of a CW-complex makes some computational problems more tractable. This is especially useful, while computing the homology groups of graph configuration spaces, because the corresponding CW-complexes have a simple, explicit form.

Let us next give the formal definition of a CW-complex.

Definition 2.1. *A CW-complex is a pair (X, Σ) that consists of a Hausdorff space X together with a partition of X into open cells that satisfies the following axioms.*

1. *For each n-dimensional cell $\sigma \in \Sigma$ there exists a continuous map $\Phi_\sigma : D^n \to X$, called the characteristic map, that maps the interior of n-dimensional disc D^n homeomorphically to σ and $\partial D^n \cong S^{n-1}$ to a finite sum of cells of dimension less than n.*

2. *The closure $\mathrm{cl}(\sigma)$ of every $\sigma \in \Sigma$ intersects a finite number of cells.*

3. *Subset $A \subset X$ is closed iff $A \cap \mathrm{cl}(\sigma)$ is closed for all σ.*

Axiom 2 is called the axiom of closure finiteness, and axiom 3 is called the weak topology axiom. The name "CW-complex" stands for these two properties. However, axioms 2 and 3 become relevant only, when X consists of an infinite number of cells.

For finite complexes, they are satisfied trivially. We denote by $\Sigma^{(n)}$ the set of all n-cells and by $X^{(n)}$ the n-skeleton of X, which is the sum of all cells in X of dimension $\leq n$.

$$X^{(n)} := \bigcup_{\sigma \in \Sigma^{(0)} \cup \cdots \cup \Sigma^{(n)}} \Phi_\sigma(D^{\dim \sigma}).$$

For every closed n-cell $\mathrm{cl}(\sigma) = \Phi_\sigma(\mathrm{D^n}) \subset \mathrm{X}$, we distinguish its interior and boundary, which are defined as[1]

$$\mathring{\sigma} := \mathrm{int}(\sigma) = \Phi_\sigma(D^n - \partial D^n), \; \dot{\sigma} := \mathrm{cl}(\sigma) - \mathrm{int}(\sigma) = \Phi_\sigma(\partial \mathrm{D^n}).$$

By the above definitions, we have, that $\dot{\sigma} \subset X^{(n-1)}$ and $\mathring{\sigma} \subset X^{(n)} - X^{(n-1)}$.

There is a universal inductive way of forming CW-complexes, which relies on gluing points from the boundary of D^n to the $(n-1)$-skeleton. More specifically, consider a continuous map $\phi_\sigma : \; \partial D^n \cong S^{n-1} \to X^{(n-1)}$. The quotient

$$X \cup_{\phi_\sigma} D^n =: (X \sqcup D^n) \, / \sim, \; x \sim \phi_\sigma(x)$$

is a new CW-complex that has one n-cell more than complex X. The characteristic map of the new cell is the composition of inclusion and quotient map

$$\Phi_\sigma : \; D^n \hookrightarrow X \sqcup D^n \to X \cup_{\phi_\sigma} D^n.$$

The boundary of the new cell is the set $\phi_\sigma(S^{n-1})$, which is a continuous, but not necessarily a homeomorphic image of S^{n-1} (see Fig. 2.1).

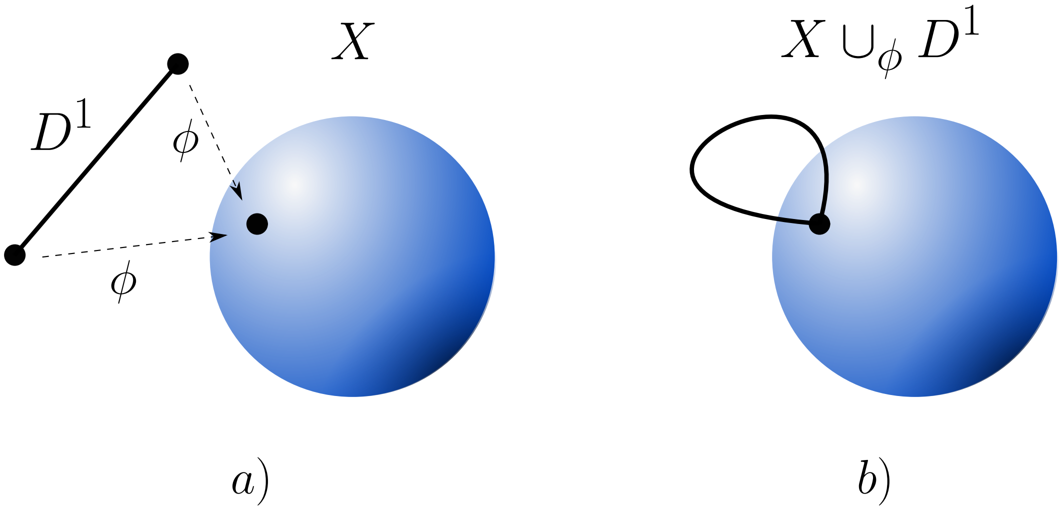


Figure 2.1: Attaching a 1-dimensional cell to complex X. Complex X is a 2-dimensional sphere that has one 0-cell and one 2-cell. Figure a) shows the prepared space $X \sqcup D^1$, and figure b) shows D^1 attached to X via gluing map $\phi : \; S^0 \to X^{(0)}$ that identifies the endpoints of D^1 with the one-cell of X.

The definition of a CW-complex allows for the existence of different kinds of irregularities. For example, the dimension of X can vary locally (as on Fig. 2.1) or the gluing maps need not be injective - they may glue some of the points of S^n. However, while studying the discrete models of graph configuration spaces we meet a certain class of CW-complexes that possess some additional regularity properties. The relevant class of CW-complexes is the class of *cube complexes*.

[1] We borrow the following notation from [59].

Definition 2.2. *A cube complex is a CW-complex, where*

- *the attached cells are cubes* $[0,1]^n \cong D^n$*,*
- *the characteristic maps are injective,*
- *the gluing maps identify faces of cubes of the same dimension by a homeomorphism.*

The above definition means, that each n-cell σ in X is homeomorphic to $[0,1]^n$, i.e. $\sigma \cap X$ contains $2n$ cells of dimension $n-1$ (faces of the cube), $4\binom{n}{2}$ cells of dimension $(n-2)$ that are $(n-2)$- dimensional facets of the cube, and so on. However, two cells can share more than one face (see Fig. 2.2).

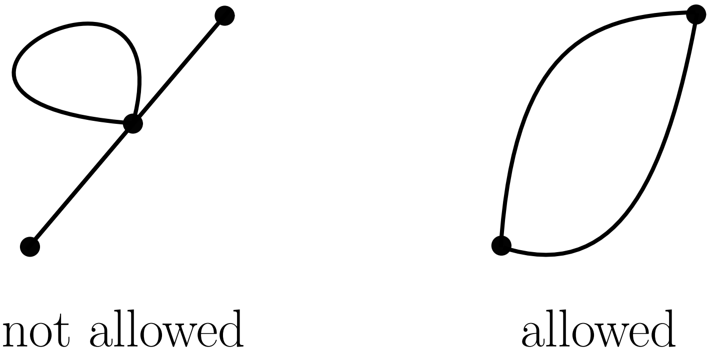


Figure 2.2: Illustration of the definition of a cube complex. Picture on the left hand side is not a cube complex, because one of the characteristic maps is not injective.

2.1. Homotopy

In this section, we explain homotopy - a notion underlying the "topological invariance" of different objects.

Definition 2.3. *Two continuous maps between topological spaces* $f,\ g:\ X \to Y$ *are homotopic,* $f \cong g$*, iff there exists a continuous map* $h:\ X \times [0,1] \to Y$ *such that* $h(x,0) = f(x)$ *and* $h(x,1) = g(x)$ *for all* X*. Map* h *is called a homotopy.*

The relation of homotopy is an equivalence relation. The set of equivalence classes of maps between X an Y with respect to this relation is denoted by $[X,Y]$. The homotopy classes of certain maps plays a key role in the classification of vector bundles, see chapter 3. The homotopy of maps yields the notion of homotopy equivalence of topological spaces.

Definition 2.4. *Topological space* X *is homotopy equivalent to topological space* Y *iff there exist continuous maps* $h:\ X \to Y$ *and* $g:\ Y \to X$ *such that* $g \circ f \cong Id_X$ *and* $f \circ g \cong Id_Y$*.*

The precise meaning of an object (a map, a functional) being "topologically invariant" is that such an object does not change, when evaluated on spaces from the same homotopy class. Using the definition of homotopy directly is usually technically difficult. A more tractable approach to the problem of deciding homotopic equivalence of topological spaces is via studying *deformation retracts*.

Definition 2.5. *Subspace A of a topological space X is a deformation retract of X iff there exists a continuous map $\rho: X \to A$ such that $\rho\big|_A = Id_A$ and $\rho \cong Id_X$. Such a map is called a deformation retraction. If, additionally, there exists a homotopy between ρ and Id_X such that*

$$h_t(a) = a \text{ for all } a \in A,\ t \in [0,1],$$

then A is a strong deformation retract and ρ is a strong deformation retraction.

A deformation retraction ρ together with inclusion map $i_A: A \hookrightarrow X$ are the maps required by definition 2.4 for A and X to be homotopically equivalent. Intuitively, one can view a strong deformation retraction as contracting space X to A along the lines given by $h_t(x)$, varying $t \in [0,1]$. The technique of strong deformation retracts has been used in [28, 14] to prove homotopy equivalence of graph configuration spaces and their discrete models.

Another notion that we extensively explore in this thesis is the homotopy of paths and homotopy of loops, that gives rise to the notion of fundamental group. A path γ is a continuous map from $[0,1]$ to X

$$\gamma: [0,1] \to X.$$

A loop is a path, whose endpoints meet. Hence, we can view it as a continuous map from the circle to X

$$\gamma: S^1 \to X.$$

Two paths γ and γ', that have common endpoints, are homotopically equivalent, iff there exists a homotopy $h: [0,1] \times [0,1] \to X$ that fixes the endpoints, i.e.

$$\forall_s\ h_0(s) = \gamma(s),\ h_1(s) = \gamma'(s),\ \forall_t\ h_t(0) = \gamma(0) = \gamma'(0),\ h_t(1) = \gamma(1) = \gamma'(1).$$

Intuitively, functions h_t form a continuous family of intermediate paths between γ and γ'. For paths that share at least one endpoint, one can define an operation of composition, which intuitively means travelling through path γ and γ' in a consecutive order. More formally, path $\gamma\cdot\gamma'$ is defined as

$$(\gamma\cdot\gamma')(s) = \begin{cases} \gamma(2s) & 0 \leq s \leq \frac{1}{2}, \\ \gamma'(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases} \tag{2.1}$$

Consequently, equation (2.1) defines composition of loops that are attached to the same base point $x_0 \in X$. For loops, one can also define inverse elements as $\gamma^{-1}: s \to \gamma(1-s)$, so that $\gamma\cdot\gamma^{-1}$ is a trivial loop (a point). The composition of loops descends to the set of homotopy classes of loops, which acquires the structure of a group

$$[\gamma]\cdot[\gamma'] := [\gamma\cdot\gamma'].$$

with the identity element being the homotopy class of contractible loops. Such a group is called the fundamental group of space (X, x_0) and denoted by $\pi_1(X, x_0)$. If X is a path-connected space (any pair of its points can be connected by a path), we often do not specify the base point, as for such a space, the fundamental groups at different base points are isomorphic. The following fact [57] asserts, that for spaces that are considered in this thesis, the fundamental group has a finite number of generators.

Fact 2.1. *The fundamental group of a finite CW complex is finitely generated.*

This means, that in all scenarios that are relevant in this thesis, the fundamental group can be described by choosing a finite set of generators $a_1, \dots, a_r$ and considering all combinations of generators and their inverses, subject to certain relations

$$\pi_1(X) = \langle a_1, a_2, \dots, a_r : \; W_1(a_1, \dots, a_r) = e, \dots, \; W_R(a_1, \dots, a_r) = e \rangle.$$

Relations $\{W_i\}$ have the form of words in $a_1, \dots, a_r$, i.e. can be written as

$$W_i(a_1, \dots, a_r) = a_{i_1}^{p_1} a_{i_2}^{p_2} \dots a_{i_k}^{p_k}, \; p_i \in \mathbb{Z}.$$

For path-connected spaces, one can use the language of homotopy classes of maps in defining the fundamental group

$$\pi_1(X) = [S^1, X].$$

One can extend this definition and consider homotopy classes of maps from higher dimensional spheres to X, which give rise to higher homotopy groups

$$\pi_k(X) := [S^k, X].$$

As the last part of this section, we would like to mention a special family of spaces, that configuration spaces often belong to. These are the Eilenberg-MacLane spaces. They are defined as spaces that have one single nontrivial homology group and are denoted by $K(G, k)$. Eilenberg-MacLane space of type $K(G, k)$ has the following properties

$$\pi_k(X) = G, \; \pi_i(X) = 0 \text{ for } i \neq k.$$

In particular, graph configuration spaces and $C_n(\mathbb{R}^2)$ are Eilenberg-MacLane spaces of type $K(G, 1)$, which are also called the *aspherical* spaces.

2.2. Homology and cohomology of CW-complexes

In this section we introduce the construction of homology groups of CW-complexes. Computation of homology groups for graph configuration spaces is the main subject of this thesis. The main references for this section are books by A. Hatcher [58] and E. Spanier [40].

2.2.1. Homology of chain complexes

We start with the definition of an abstract chain complex, then specify this construction for the case of CW-complexes.

Definition 2.6. *A chain complex $\mathfrak{C} = \{\mathfrak{C}_d\}$ is a graded abelian module over ring R, equipped with a boundary map, i.e. a homomorphism $\partial : \; \mathfrak{C}_d \to \mathfrak{C}_{d-1}$, that satisfies $\partial \circ \partial = 0$.*

We denote a basis of $\mathfrak{C}_d$ by $\Sigma^{(d)}$. The above definition means, that elements of $\mathfrak{C}_d$ are formal linear combinations of basis elements with coefficients in R

$$\sum_{\sigma \in \Sigma^{(d)}} a_\sigma \sigma, \; a_\sigma \in R,$$

and the allowed operations are addition of chains and scalar multiplication by elements of R. The homology groups of chain $\mathfrak{C}$ in coefficients in R are defined as the following quotients of abelian modules

$$H_d(\mathfrak{C}, R) := \ker \partial_d / \mathrm{im}\partial_{d+1}.$$

In this thesis, we only consider chain complexes as modules over the integers, $R = \mathbb{Z}$, or rationals, $R = \mathbb{Q}$. If $R = \mathbb{Z}$, the corresponding chain complex $\mathfrak{C}_d$ is actually an abelian group isomorphic to a direct sum of copies of $\mathbb{Z}$. Moreover, when the number of basis elements is finite for every d, we have the following theorem about the structure of homology groups.

Theorem 2.3. *If $\mathfrak{C}$ is a chain complex of finite dimension with coefficients in $\mathbb{Z}$, then the homology groups are finitely generated abelian groups, i.e. have the following form*

$$H_d(\mathfrak{C}, \mathbb{Z}) = \mathbb{Z}^K \oplus \bigoplus_{i=1}^{L} \mathbb{Z}_{p_i},$$

where $K, L \in \mathbb{N}$, and p_i divides p_{i+1}.

Number K is called the rank of $H_d(\mathfrak{C}, \mathbb{Z})$, and is equal to the nth Betti number of complex X.

$$K = \mathrm{rk}(H_d(\mathfrak{C}, \mathbb{Z})) = \beta_n(X).$$

The cyclic part of $H_d(\mathfrak{C}, \mathbb{Z})$ is called the torsion part and denoted by $T(H_d(\mathfrak{C}, \mathbb{Z}))$ or $T_d(\mathfrak{C}, \mathbb{Z})$. Let us next show how this theorem follows from theorem 2.4 about the existence of the Smith normal form of a matrix, as it will be relevant for chapter 5, where the computational results for graph configuration spaces are presented.

Theorem 2.4. *Every $m \times n$ matrix A with integer entries can be written as*

$$A = UDV, \tag{2.2}$$

where

- *matrix D is diagonal, i.e. $D_{i,j} = 0$ for $i \neq j$, and $D_{i,i}$ are natural numbers such, that $D_{i,i}$ divides $D_{i+1,i+1}$ for all i,*
- *matrices U and V are respectively $m \times m$ and $n \times n$ integer matrices, whose determinants are equal to ± 1.*

Matrix D from theorem 2.4 is called the Smith normal form of matrix A and $D_{i,i}$ are called elementary divisors of matrix A.

Proof. (of theorem 2.3) Consider the following sequence of maps

$$\mathfrak{C}_{d+1} \xrightarrow{\partial_{d+1}} \mathfrak{C}_d \xrightarrow{\partial_d} \mathfrak{C}_{d-1}, \tag{2.3}$$

where $\partial_k := \partial\big|_{\mathfrak{C}_k}$. Chain complex $\mathfrak{C}$ is of finite dimension, hence both boundary maps can be written as finite matrices with integer coefficients, whose dimensions correspond to the dimensions of the respective chain subcomplexes. Hence, sequence (2.3) is isomorphic to

$$\mathbb{Z}^k \xrightarrow{\partial_{d+1}} \mathbb{Z}^m \xrightarrow{\partial_d} \mathbb{Z}^n.$$

Because $\partial \circ \partial = 0$, we have $\text{im}\partial_{d+1} \subset \ker \partial_d$. Therefore, in computing quotient $\ker \partial_d / \text{im}\partial_{d+1}$, we can firstly restrict to considering a smaller matrix $\overline{\partial}_{d+1} : \mathbb{Z}^k \to \mathbb{Z}^s$, where $s = \dim(\ker \overline{\partial}_d)$. Denote by $[\partial]$ the matrix of the boundary map, written in the standard basis. Using theorem 2.4 for matrix $[\overline{\partial}_{d+1}]$, we show, that

$$\text{coker}\overline{\partial}_{d+1} := \mathbb{Z}^s / \text{im}\overline{\partial}_{d+1} = \mathbb{Z}^{s-r} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}_{p_i},$$

where r is the rank of $\overline{\partial}_{d+1}$. Indeed, because matrices V and U from formula (2.2) describe injective morphisms, we have $\text{coker}\overline{\partial}_{d+1} \cong \text{coker}D$. In general, matrix D has the form

$$D = \text{diag}(p_1, \ldots, p_r, 0, \ldots, 0).$$

Moving to the basis of $\mathbb{Z}^s$, in which matrix D is diagonal, the image of D is spanned by elements $p_1 a_1, \ldots, p_r a_r$, where $a_1, \ldots, a_r, a_{r+1}, \ldots, a_s$ are the basis elements of $\mathbb{Z}^s$. In the quotient space, element $a_i p_i$ gives rise to a $\mathbb{Z}_{p_i}$-direct summand, whenever $p_1 > 1$. □

If $R = \mathbb{Q}$, we simply have $H_d(\mathfrak{C}, \mathbb{Q}) = \mathbb{Q}^K$ for some $K \in \mathbb{N}$.

Realising the above constructions for *CW*-complexes poses two main problems. Firstly, one has to properly define the set of generators of $\mathfrak{C}$, which are in a one-to-one correspondence with cells of the considered *CW* complex. Secondly, one has to define a boundary map. The theory that deals with these issues and with the computation of the related homology groups is known under the name of cellular homology. Let us next review its main aspects. The first step in defining a chain complex over R, $\mathfrak{C}(X, R)$, that corresponds to a *CW* complex X, is choosing an *orientation* for each cell of X. This goes as follows (see [59] for more details). Every characteristic map $\Phi_\sigma : D^n \to X$ induces a homomorphism $H_\sigma : D^n/\partial D^n \to X^{(n)}/(X^{(n)} - \dot{\sigma})$, where the slash denotes the quotient by a subspace. Note, that both $D^n/\partial D^n$ and $X^{(n)}/(X^{(n)} - \dot{\sigma})$ are homeomorphic to S^n. Hence, map H_σ can be viewed as a homeomorphism $S^n \to S^n$. Orientation of n-cell σ is defined as the homotopy class of the corresponding homeomorphism $H_\sigma : S^n \to S^n$. There are two homotopy classes of such homeomorphisms, which we denote by ± 1. After choosing the ± 1 orientation for each cell, we obtain an oriented *CW* complex. Chain complex corresponding to X is the complex generated by oriented cells of X

$$\mathfrak{C}(X, R) = \bigoplus_{\sigma \in \Sigma} R.$$

In our notation, we omit the fact, that the cells are oriented, however when talking about chain complex $\mathfrak{C}(X, R)$ we keep the orientation in mind. The boundary of cell σ is a linear combination of cells, who belong to $\dot{\sigma}$ with proper coefficients. The coefficients are called the incidence numbers and are denoted by $[\sigma : \tau]$ for $\tau \subset \dot{\sigma}$. The boundary map is given by

$$\partial \sigma = \sum_{\tau \in \dot{\sigma}} [\sigma : \tau] \tau. \tag{2.4}$$

In general, determining the incidence numbers is a subtle problem. However, for *CW* complexes, whose gluing maps are homeomorphisms, the incidence numbers are determined by comparing the orientation of τ and the orientation induced by σ on τ. The value of $[\sigma : \tau]$ is simply $+1$ if the orientation induced by σ on τ and the orientation of τ agree, and -1 if they are opposite. If $\tau \cap \dot{\sigma} = \emptyset$, we put $[\sigma : \tau] = 0$.

Example 2.5. Möbius band as a regular CW-complex.

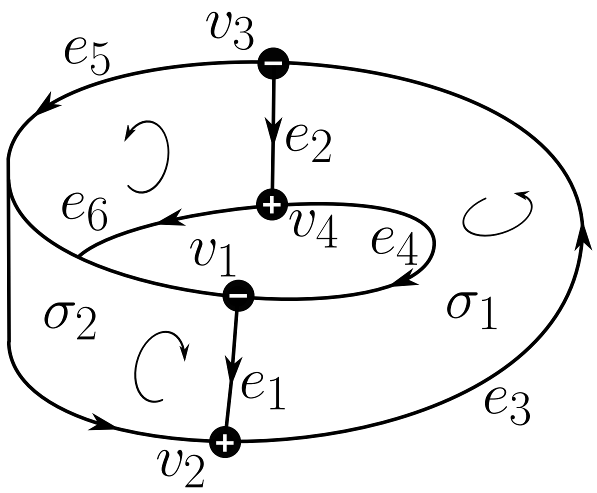


Figure 2.3: Möbius band as a regular oriented CW complex. It consists of four 0-dimensional cells v_i, six 1-cells e_i, and two 2-cells σ_i. The orientation of 2-cells is denoted by circles, the orientation of 1 cells by arrows and the orientation of 0-cells by $\pm$. The induced orientation is along the direction of circles and by the inflow $(+)$ and outflow $(-)$ of arrows for 1- and 0-cells respectively.

According to equation (2.4) and the discussion about regular complexes, the boundary map reads

$$\partial\sigma_1 = e_1 + e_2 + e_3 + e_4,\ \partial\sigma_2 = e_1 - e_2 + e_5 - e_6.$$

Clearly, the kernel of ∂_2 is trivial, hence $H_2(X,\mathbb{Z}) = 0$. The boundaries of 1-cells read

$$\partial e_1 = v_1 + v_2,\ \partial e_2 = v_3 + v_4,\ \partial e_3 = -v_2 - v_3,\ \partial e_4 = -v_1 - v_4,$$
$$\partial e_5 = -v_1 + v_3,\ \partial e_6 = v_2 - v_4.$$

By a straightforward calculation, we check that

$$\ker\partial_1 = \langle e_1 + e_2 + e_3 + e_4, e_1 + e_3 + e_5, e_2 + e_3 + e_6\rangle \cong \mathbb{Z}^3.$$

Note, that the first element of the basis of $\ker\partial_1$ is equal to $\partial\sigma_1$, while the last two elements are related by $(e_1 + e_3 + e_5) - (e_2 + e_3 + e_6) = e_1 - e_2 + e_5 - e_6 = \partial\sigma_2$. Hence, in the quotient by $\mathrm{im}\partial_2$ there is only one degree of freedom. By computing the Smith normal form of matrix ∂_2, we get, that there are no elementary divisors greater than 1, hence $H_1(X,\mathbb{Z}) = \mathbb{Z}$. Similarly, we obtain, that $H_0(X,\mathbb{Z}) = \mathbb{Z}$.

As different choices of orientation yield homotopy equivalent oriented CW-complexes [59], the homology groups are independent on the choice of orientation.

2.2.2. Homological exact sequences and Künneth formula

In this subsection, we review the basic algebraic tools, which we use in computing homology groups of discrete models of graph configuration spaces. The first tool is the Künneth formula, which describes the homology group of the Cartesian product of two complexes in terms of homology groups of the constituents. The second kind of tools are the homological exact sequences, which we use in this thesis in two main contexts.

Firstly, we explain the basic method of deriving homological exact sequences from short exact sequences of chain complexes. We use this general method to describe the effect of different manipulations on graphs, like removing a vertex or disjoining an edge, on the structure of their configuration spaces. We borrow this proof strategy from [61]. Secondly, we utilise the Mayer-Vietoris sequences, when we compute homology groups of a large complex, using the knowledge of homology groups of its smaller constituents (section 5.3).

The starting point for all the algebraic methods we use is a situation, where we have three chain complexes $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ that are related by a short exact sequence.

$$0 \to \mathfrak{A} \xrightarrow{f} \mathfrak{B} \xrightarrow{g} \mathfrak{C} \to 0. \tag{2.5}$$

The exactness of the above sequence means, that each map (denoted by an arrow) is a homomorphism, whose image is the kernel of the succeeding homomorphism. Therefore, map f in the above sequence is injective and map g is surjective. Moreover, we have the standard boundary maps in complexes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, that we denote by a common letter ∂. If maps f and g are chain maps, i.e. they commute with the boundary map, sequence (2.5) implies a long exact sequence of homology groups [58]

$$\cdots \to H_d(\mathfrak{A}) \xrightarrow{f_*} H_d(\mathfrak{B}) \xrightarrow{g_*} H_d(\mathfrak{C}) \xrightarrow{\delta} H_{d-1}(\mathfrak{A}) \xrightarrow{f_*} H_{d-1}(\mathfrak{B}) \to \dots.$$

The 'boundary map' δ is defined as follows. Let $c \in \ker \partial$ represent an element of $H_d(\mathfrak{C})$. By surjectivity of g, we have $c = g(b)$ for some $b \in \mathfrak{B}_d$. Because g is a chain map, we have $0 = \partial c = g(\partial b)$, i.e. $\partial b \in \ker(g : \mathfrak{B}_{d-1} \to \mathfrak{C}_{d-1})$. By exactness of (2.5), $\partial b \in \mathrm{im}(f : \mathfrak{A}_{d-1} \to \mathfrak{B}_{d-1})$. Because map f is injective, there is a unique $a \in \mathfrak{A}_{d-1}$ defined by $a = f^{-1}(\partial b)$. Note, that a is a cycle, because $0 = \partial\partial b = \partial f(a) = f(\partial a)$, where $\ker f = 0$. Map δ maps $[c] \in H_d(\mathfrak{C})$ to $[a] \in H_{d-1}(\mathfrak{A})$. It is a matter of straightforward calculation to check, that δ is well-defined, i.e. i) a is uniquely determined by the choice of ∂b, ii) chainging b to $b + b'$ with $b' \in \ker g$ yields a cycle, which is homologous to a, iii) changing c to $c + \partial c'$ leaves a unchanged.

Another useful technique is breaking a long exact sequence into a collection of short exact sequences. This means, that a long exact sequence of abelian groups

$$\cdots \to A_{d+2} \xrightarrow{f_{d+2}} A_{d+1} \xrightarrow{f_{d+1}} A_d \xrightarrow{f_d} A_{d-1} \xrightarrow{f_{d-1}} A_{d-2} \to \dots,$$

can be broken up into short exact sequences

$$0 \to \mathrm{coker} f_{d+2} \to A_d \to \ker f_{d-1} \to 0.$$

Map $\mathrm{coker} f_{d+2} \to A_d$ induced by f_{d+1} is injective, since $\mathrm{coker} f_{d+2} = A_{d+1}/\mathrm{im} f_{d+2} \cong A_{d+1}/\ker f_{d+1}$. Map $A_d \to \ker f_{d-1}$ is surjective by exactness – $\mathrm{im} f_d = \ker f_{d-1}$, as any map is surjective on its image. We often use the following isomorphisms, that follow from exactness of the long sequence

$$\ker f_{d-1} = \mathrm{im} f_d \cong \mathrm{coker} f_{d+1}.$$

Similarly, one can consider short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of abelian groups A, B, C. We are often interested in cases, when such a short exact sequence splits. This means, that there is a canonical isomorphism $B \cong A \oplus C$. Equivalently, i) there is a homomorphism $p : B \to A$ such that $p \circ f = id_A$, or ii) there is a homomorphism $s : C \to B$ such that $s \circ g = id_C$. The above three equivalent conditions for the exact sequence to split are satisfied if group C is free abelian (has no torsion), as one can choose a basis $\{c_i\}$ of C and define the desired homomorphism s as $c_i \to b_i$ for $\{b_i\}$ such that $g(b_i) = c_i$.

A special case of the above considerations are the Mayer-Vietoris sequences. Let X be a CW-complex, and A, B its subcomplexes such, that $X = A \cup B$. The Mayer-Vietoris sequence for $X = A \cup B$ reads [58]

$$\dots \to H_d(A \cap B) \xrightarrow{\Phi} H_d(A) \oplus H_2(B) \xrightarrow{\Psi} H_d(X) \xrightarrow{\delta} H_{d-1}(A \cap B) \xrightarrow{\Phi} \dots \tag{2.6}$$

The Mayer-Vietoris sequence can be derived from the following short exact sequence of chain complexes.

$$0 \to \mathfrak{C}_d(A \cap B) \xrightarrow{\phi} \mathfrak{C}_d(A) \oplus \mathfrak{C}_d(B) \xrightarrow{\psi} \mathfrak{C}_d(A \cup B) \to 0.$$

Map ϕ acts on d-chains from $A \cap B$ by assigning the same chain to each summand in the image, i.e. for $x \in \mathfrak{C}_d(A \cap B)$, $\phi(x) = (x, -x)$. Map ψ sums of chains, i.e. $\psi(x, y) = x + y$. It is straightforward to check, that $\mathrm{im}\phi = \ker \psi$. The boundary map δ (also called the connecting homomorphism) acts as follows. Every d-cycle, z, from $A \cup B$ can be written as a sum of d-chains form A and B respectively

$$z = x + y, \ x \in \mathfrak{C}_d(A), \ y \in \mathfrak{C}_d(B).$$

Because $\partial z = 0$, we have $\partial x = -\partial y$. Chains ∂x and ∂y are $(d-1)$-cycles, since $\partial\partial = 0$. Moreover, these cycles represent the same element of $H_{d-1}(A \cap B)$. In other words, $\delta[z] := [\partial x] = [-\partial y]$. Note that class $[\partial x]$ does not depend on the chosen decomposition of z. Long exact sequence (2.6) implies the short exact sequence

$$0 \to \mathrm{coker}(\Phi) \to H_d(X) \to \mathrm{coker}(\Psi) \to 0.$$

If $\mathrm{coker}(\Psi)$ is free abelian, the sequence splits, i.e.

$$H_d(X) = \mathrm{coker}(\Phi) \oplus \mathrm{coker}(\Psi) = \mathrm{coker}(\Phi) \oplus \mathrm{im}\delta.$$

In the last equality we used the fact that $\mathrm{im}\delta$ is isomorphic of $\mathrm{coker}(\Phi)$ by exactness of (2.6). If $X = A \cup B$ are general topological spaces (not necessarily CW complexes), for the Mayer-Vietoris sequence to work, it is necessary for X to be the union of interiors of A and B.

As the last point of this subsection, we review the Künneth formula, which describes the homology of products of topological spaces. This theorem will be often used in computation of homology groups of configuration spaces, as such spaces are locally Cartesian products. We begin with the observation, that when $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ are cellular chains corresponding to two finite CW complexes, then the set of d-dimesional cells of $\mathfrak{C}(X \times Y)$ is given by tensor products of cells from $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ of proper dimensions.

$$\Sigma^{(d)}(\mathfrak{C}(X \times Y)) = \bigsqcup_{k+l=d} \Sigma^{(k)}(X) \times \Sigma^{(l)}(Y).$$

Using natural isomorphisms $R[\Sigma \times \Sigma'] \cong R[\Sigma] \otimes R[\Sigma']$ and $R[\Sigma \sqcup \Sigma'] \cong R[\Sigma] \oplus R[\Sigma']$, we obtain, that

$$\mathfrak{C}_d(X \times Y) \cong (\mathfrak{C}(X) \otimes \mathfrak{C}(Y))_d \cong \bigoplus_{k+l=d} \mathfrak{C}_k(X) \otimes \mathfrak{C}_l(Y).$$

The boundary map in $\mathfrak{C}(X \times Y)$ is described via its action on the tensor products of chains in the following way.

$$\partial(c \otimes c') = (\partial c) \otimes c' + (-1)^{\dim c} c \otimes (\partial c').$$

By considering tensor products of cycles, we obtain a homomorphism

$$\bigoplus_{k+l=d} H_k(X) \otimes H_l(Y) \xrightarrow{h} H_d(X \otimes Y). \tag{2.7}$$

If all homology groups of the considered spaces are free, then homomorphism (2.7) is an isomorpism. In general, the kernel of homomorphism (2.7) is described by the so-called Tor-product of $H_i(X)$ and $H_{d-i-1}(Y)$ for $i = 0, \ldots d-1$,

$$\ker h \cong \bigoplus_{i=0}^{d-1} \operatorname{Tor}\left(H_i(X), H_{d-i-1}(Y)\right).$$

The Tor-product of finitely generated abelian groups can be calculated using the following properties [58].

- $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(B, A)$.
- $\operatorname{Tor}\left(\oplus_i A_i, B\right) \cong \oplus_i \operatorname{Tor}(A_i, B)$.
- $\operatorname{Tor}(A, B) = 0$ if A or B is free.
- $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(T(A), B)$.
- $\operatorname{Tor}(\mathbb{Z}_p, A) \cong \ker(A \xrightarrow{p} A)$, where $p: \; x \to px$.

Hence, if we are interested in computing the homology groups over $\mathbb{Z}$, it is enough to compute $\operatorname{Tor}(\mathbb{Z}_p, \mathbb{Z}_q)$. The result is $\operatorname{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_s$, where s is the greatest common divisor of p and q, which we denote by $GCD(p, q)$. This happens to be the same as the tensor product of $\mathbb{Z}_p$ and $\mathbb{Z}_q$, hence, we have the simple formula

$$\operatorname{Tor}(A, B) \cong T(A) \otimes T(B).$$

The above discussion can be summed up by the following version of the Künneth theorem.

Theorem 2.6. *Let X and Y be finite CW complexes. Then,*

$$H_d(X \times Y, \mathbb{Q}) \cong \bigoplus_{k+l=d} H_k(X, \mathbb{Q}) \otimes H_l(Y, \mathbb{Q}).$$

For homology groups with coefficients in $\mathbb{Z}$, we have

$$H_d(X \times Y, \mathbb{Z}) \cong \left(\bigoplus_{k+l=d} H_k(X, \mathbb{Z}) \otimes H_l(Y, \mathbb{Z}) \right) \oplus \left(\bigoplus_{k+l=d-1} T(H_k(X, \mathbb{Z})) \otimes T(H_l(Y, \mathbb{Z})) \right).$$

2.2.3. Cohomology and its ring structure

The cohomology of CW complexes is a notion dual to homology. Namely, in cohomology one considers cellular d-cochains, which are elements of $\mathrm{Hom}(\mathfrak{C}_d(X,R),R)$. There is a natural coboundary map ∂^*, that maps d-cochains to $(d+1)$-cochains via

$$\partial^*(g) := g \circ \partial,\ \ g \in \mathrm{Hom}(\mathfrak{C}_d(X,R),R).$$

Further on, we will identify cochains with chains via the standard pairing,

$$F(g) = \sum_{\sigma \in \Sigma^{(d)}} g(\sigma)\sigma,$$

which boils down to identifying generators of $\mathfrak{C}_d(X,R)$ with their duals. We denote the composition $F \circ \partial^* \circ F^{-1} :\ \mathfrak{C}_d(X,R) \to \mathfrak{C}_{d+1}(X,R)$ by Δ_d and sometimes omit the subscript, when the dimension of the domain is clear, or when Δ acts collectively on cochains of different dimensions. The action of this map reads

$$\Delta(\sigma) = \sum_{\tau \in \Sigma^{(d+1)}:\ \sigma \dot{\subset} \tau} [\tau : \sigma]\tau. \tag{2.8}$$

One can check, that $\Delta \circ \Delta = 0$. Consequently, we define cohomology groups of complex X as

$$H^d(X,R) = \ker \Delta_d / \mathrm{im}\Delta_{d-1}.$$

As in the case of homology, groups $H^k(X,\mathbb{Z})$ are again finitely generated and abelian. One of the main structural differences between homology and cohomology is that cohomology has the structure of a graded ring $H^*(X,R) = \bigoplus_n H^d(X,R)$, where the multiplication of cohomology classes is defined via the (associative and distributive) cup product $\smile$: $H^k(X,R) \times H^l(X,R) \to H^{k+l}(X,R)$. The cup product is also skew-commutative, i.e.

$$u \smile v = (-1)^{kl} v \smile u,\ \ u \in H^k(X,R),\ \ v \in H^l(X,R).$$

In order to define the cup product explicitly for any CW complex, one has to refer to a more general concept than the cellular cohomology, namely to the singular cohomology [58]. However, for cube complexes it is possible to define the cup product algorithmically, only in terms of cellular (co)chains [60]. We do not go into the details of this construction, as in this thesis we will only use the fact, that the cup product exists. We only mention, that the construction utilises the fact, that tensor products of (co)chains span the chain complex that corresponds to the Cartesian product $X \times X$. By defining a proper chain map from $\mathfrak{C}(X \times X, R) \to \mathfrak{C}(X,R)$, one obtains the multiplicative structure on the space of cochains.

In the remaining part of this subsection, we review the universal coefficient theorem, which relates homology to cohomology with different coefficients. For finite CW complexes and their (co)homology over $\mathbb{Z}$, the universal coefficient theorem boils down to the following well-known result, which can be proved using elementary methods.

Theorem 2.7. *The ranks of $H^k(X,\mathbb{Z})$ and $H_k(X,\mathbb{Z})$ are equal and the torsion of $H^k(X,\mathbb{Z})$ is equal to the torsion of $H_{k-1}(X,\mathbb{Z})$.*

Proof. The proof is along the same lines, as the proof of theorem 2.3 with keeping track of the dimensions of matrices. We consider two finite chain complexes, which are dual to each other.

$$0 \to \mathfrak{C}_N \xrightarrow{\partial_N} \ldots \to \mathfrak{C}_{d+1} \xrightarrow{\partial_{d+1}} \mathfrak{C}_d \xrightarrow{\partial_d} \mathfrak{C}_{d-1} \to \ldots \xrightarrow{\partial_1} \mathfrak{C}_0 \to 0, \tag{2.9}$$
$$0 \leftarrow \mathfrak{C}_N \xleftarrow{\Delta_{N-1}} \cdots \leftarrow \mathfrak{C}_{d+1} \xleftarrow{\Delta_d} \mathfrak{C}_d \xleftarrow{\Delta_{d-1}} \mathfrak{C}_{d-1} \leftarrow \ldots \xleftarrow{\Delta_0} \mathfrak{C}_0 \leftarrow 0,$$

By definition, $\Delta_d = (\partial_{d+1})^*$. Both operators are finite matrices over integers, hence we have $[\Delta_d] = [\partial_{d+1}]^T$. By the definition of (co)homology groups, we have

$$\mathrm{rk}(H^d(X,\mathbb{Z})) = \dim\ker[\Delta_d] - \mathrm{rk}[\Delta_{d-1}] = \dim\ker[\partial_{d+1}]^T - \mathrm{rk}[\partial_d]^T,$$
$$\mathrm{rk}(H_d(X,\mathbb{Z})) = \dim\ker[\partial_d] - \mathrm{rk}[\partial_{d+1}].$$

Denote the dimension of $\mathfrak{C}_d$ by d, i.e. $\mathfrak{C}_d \cong \mathbb{Z}^m$, and the ranks of boundary maps by $\mathrm{rk}[\partial_d] = p$, $\mathrm{rk}[\partial_{d+1}] = r$. Then, using the fact, that $\mathrm{rk}A = \mathrm{rk}A^T$, by direct inspection of sequences (2.9), we have, that $\dim\ker[\partial_d] = m - p$, $\dim\ker[\partial_{d+1}]^T = m - r$. Hence,

$$\mathrm{rk}(H^d(X,\mathbb{Z})) = \mathrm{rk}(H_d(X,\mathbb{Z})) = m - p - r.$$

To see the inheritance of torsion, compare $H_d(X,\mathbb{Z}) = \ker\partial_d/\mathrm{im}\partial_{d+1}$ with $H^{d+1}(X,\mathbb{Z}) = \ker(\partial_{d+2})^*/\mathrm{im}(\partial_{d+1})^*$. Groups $\ker\partial_d$ and $\ker(\partial_{d+1})^*$ are free abelian, hence the torsion in (co)homology groups comes from the quotient. The elementary divisors of matrix A and of its transposition are the same. This fact applied to matrix $[\partial_{d+1}]$ yields the result. □

The complete information about the relation between homology and cohomology is encoded in the following exact sequence [58].

Theorem 2.8. *If chain complex $\mathfrak{C}$ has homology groups $H_d(\mathfrak{C},\mathbb{Z})$, then the cohomology groups of the corresponding cochain complex are determined by the following split exact sequence.*

$$0 \to \mathrm{Ext}(H_{d-1}(\mathfrak{C},\mathbb{Z}),\mathbb{Z}) \to H^d(\mathfrak{C},\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_d(\mathfrak{C},\mathbb{Z}),\mathbb{Z}) \to 0.$$

The natural homomorphism h is defined in the following way. Elements of $H^d(\mathfrak{C},\mathbb{Z})$ are represented by cocycles, i.e. homomorphisms $g \in \mathrm{Hom}(\mathfrak{C}_d,\mathbb{Z})$ such, that $g \circ \partial = 0$. In other words, such homomorphisms vanish on d-dimensional boundaries $B_d = \mathrm{im}\partial_{d+1}$. Hence, when evaluated on elements of $Z_n = \ker\partial_d$ they descend to the quotients $\overline{g}: Z_d/B_d \to \mathbb{Z}$. Recall, that $Z_d/B_d = H_d$. Map $h : Z_d/B_d = H^d \to \mathrm{Hom}(H_d,\mathbb{Z})$ assigns to g the quotient homomorphism of its restriction to Z_n, which we denoted by $\overline{g}$. In fact, every homomorphism from H_d to $\mathbb{Z}$ can be obtained in this way. To see this,decompose the space of d-chains as $\mathfrak{C}_d \cong Z_d \oplus (\mathfrak{C}_d/Z_d) \cong Z_d \oplus B_{d-1}$. This is possible, since all the appearing groups are free abelian. This allows us to define a map, that takes $\overline{g}: Z_d/B_d \to \mathbb{Z}$ and assigns an element of H^d to it. Denote by $\overline{g}_*$ the pullback of $\overline{g}$ to Z_d. Any element of $\mathrm{Hom}(Z_d,\mathbb{Z})$ that vanishes on $B_n \subset Z_n$ can be extended to an element of $\mathrm{Hom}(\mathfrak{C}_d,\mathbb{Z})$, that also vanishes on B_d by composing it with the projection on Z_d, i.e. $\overline{g}_* \to \overline{g}_* \circ p$. Thus, $\overline{g}_* \circ p$ descends to a homomorphism $\mathrm{Hom}(H_d,\mathbb{Z}) \to \ker\Delta_d$. Taking the quotient by $\mathrm{im}\Delta_{d-1}$, we obtain a homomorphism from $\mathrm{Hom}(H_d,\mathbb{Z})$ to H^d. In this way, we proved, that there is a split exact sequence

$$0 \to \ker h \to H^d(\mathfrak{C},\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_d(\mathfrak{C},\mathbb{Z}),\mathbb{Z}) \to 0.$$

In other words, $H^d(\mathfrak{C}, \mathbb{Z}) = \mathrm{Hom}(H_d, \mathbb{Z}) \oplus \ker h$. If H_d is finitely generated, then $\mathrm{Hom}(H_d, \mathbb{Z}) \cong H_d/T_d$. Group $\ker h$ is the torsion part of H^d. It also has the interpretation of the set of isomorphism classes of group extensions of $\mathbb{Z}$ by H_{d-1}, which we denote by $\mathrm{Ext}(H_{d-1}(\mathfrak{C}, \mathbb{Z}), \mathbb{Z})$ (see [58] for a more detailed discussion). A group extension G is defined by an exact sequence

$$0 \to \mathbb{Z} \to G \to H_{n-1} \to 0.$$

In general, group $\mathrm{Ext}(H, G)$ for finitely generated H and abelian G can be computed using the following three properties: i) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$, ii) if H is free, then $\mathrm{Ext}(H, G) = 0$, iii) $\mathrm{Ext}(\mathbb{Z}_p, G) \cong G/pG$. Thus, by taking $H = \mathbb{Z}^k \oplus T$, we see, that $\mathrm{Ext}(H_{d-1}(\mathfrak{C}, \mathbb{Z}), \mathbb{Z}) \cong T_{d-1}$.

Theorem 2.8 can be generalised to describe the change of coefficients from chain complexes over R, which is a principal ideal domain, and their homolgy to cohomology with coefficients in abelian group G by the following split exact sequence.

$$0 \to \mathrm{Ext}_R(H_{d-1}(\mathfrak{C}, R), G) \to H^d(\mathfrak{C}, G) \xrightarrow{h} \mathrm{Hom}(H_d(\mathfrak{C}, R), G) \to 0.$$

In particular, if $R = \mathbb{Q} = G$, $\mathrm{Ext}_R(H_{d-1}(\mathfrak{C}, R), G) = 0$, hence $H^d(\mathfrak{C}, \mathbb{Q}) \cong H_d(\mathfrak{C}, \mathbb{Q})$.

Chapter 3

Vector bundles and their classification

The main motivation for studying (co)homology groups of configuration spaces comes from the fact that they give information about the isomorphism classes of vector bundles over configuration spaces. In the following chapter, we review the main strategies of classifying vector bundles and make the role of homology groups more precise. Throughout, we do not assume that the configuration space is a differentiable manifold, as the configuration spaces of graphs are not differentiable manifolds. We only assume that $C_n(X)$ has the homotopy type of a finite CW-complex. This means that $C_n(X)$ can be deformation retracted to a finite CW-complex. As we explain in chapter 4, configuration spaces of graphs are such spaces. The lack of differentiable structure means, that the flat vector bundles have to be defined without referring the notion of a connection and all the methods that are used have to be purely algebraic. We provide such an algebraic definition of flat bundles in section 3.3.

Let us begin with the definition of a vector bundle. In this thesis, we consider only complex vector bundles, however, throughout this chapter, we provide also examples of real vector bundles. A vector bundle consists of two topological spaces, which we denote by E and B, and a continuous surjection $\pi:\ E \to B$. Space E is the *total space* and B is the *base space*. For each point $p \in B$ the fiber $\pi^{-1}(p)$ is isomorphic to a vector space. In the case of complex vector bundles, we have $\pi^{-1}(p) \cong \mathbb{C}^k$ and in the case of real vector bundles, we have $\pi^{-1}(p) \cong \mathbb{R}^k$. If the dimension of the vector space is the same and equal to k for all $p \in B$, we say that the vector bundle is of rank k. To complete the definition, we have to impose one more condition for map π, which is called the *local triviality* condition and says that for every point $p \in B$, there is a neighbourhood $U \subset B$ of p such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{K}^k$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ respectively. More precisely, there is a homeomorphism

$$\phi_U:\ U \times \mathbb{K}^k \to \pi^{-1}(U),$$

such that for all $p \in U$ i) $(\pi \circ \phi)(p, v) = p$ for all vectors $v \in \mathbb{K}^k$ and ii) map $v \mapsto \phi(p, v)$ is a linear isomorphism between $\mathbb{K}^k$ and $\pi^{-1}(p)$. The set of vector bundles over B will be denoted by

$$Vect_k^{\mathbb{K}}(B).$$

Example 3.1. Möbius band as a nontrivial real line bundle over S^1. There are two isomorphism classes of real line bundles over S^1. The first class is the trivial one $S^1 \times \mathbb{R}$, which is homeomorphic to the cylinder $S^1 \times [0,1]$. The nontrivial line bundle over S^1 is constructed by a band $[0,1] \times [0,1]$ and identifying one pair of its opposite edges by twisting them, i.e. forming the Möbius band $([0,1] \times [0,1])/\sim$, where $(0,t) \sim (1,1-t)$, see Fig. 3.1. Twisting the edges makes the resulting space not homeomorphic to the cylinder, hence the Möbius band and the cylinder are two non-isomorphic vector bundles.

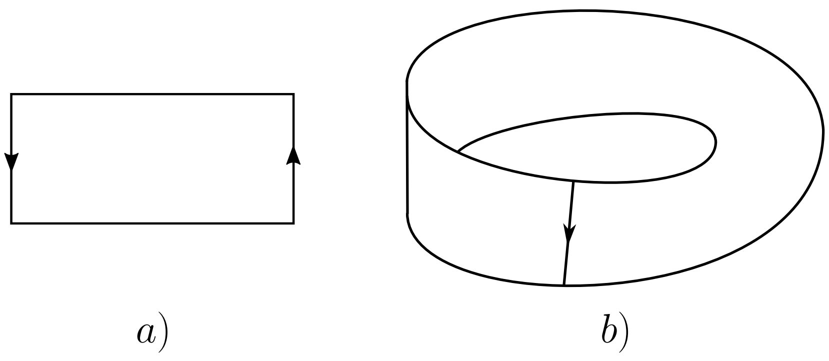


Figure 3.1: Möbius band as a nontrivial real line bundle over S^1.

In quantum mechanics on configuration spaces, we view wave functions as sections of complex vector bundles. A section of a vector bundle is a continuous map $\sigma : B \to E$ such, that $\pi \circ \sigma$ is the identity map on B. While considering wave functions as sections of vector bundles, we would like to have the additional property of their square-integrability. Therefore, technically, when speaking about wave functions, we mean such square-integrable sections.[1] The notion of integrability also requires picking a measure on the base space. As configuration spaces of euclidean spaces are locally diffeomorphic to $\mathbb{R}^n$, we pick the standard measure in $\mathbb{R}^n$. For graph configuration spaces, we do this procedure piecewise with respect to cells of $C_n(\Gamma)$.

There is a natural notion of homomorphism between vector bundles E_1 and E_2, which involves two maps $f: E_1 \to E_2$ and $g: B_1 \to B_2$. Maps f and g are required to commute with projections, i.e. $\pi_2 \circ f = \pi_1 \circ g$, so that diagram (3.1) is commutative. Moreover, for every $p \in B_1$ the map $\pi_1^{-1}(p) \to \pi_2^{-1}(g(p))$ induced by f is a linear map between vector spaces.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array} \tag{3.1}$$

If there exists an invertible homomorphism between two vector bundles, whose inverse is also a homomorphism of vector bundles, we say that the two vector bundles are isomorphic. Going back to example 3.1, there are two isomorphism classes of real line bundles over S^1 - the trivial one and the Möbius band. However if we switch to the complex case, it turns out that there is one isomorphism class of complex line bundles over S^1 - the trivial bundle. Perhaps a more intuitive characterisation of an isomorphism between two vector bundles is the following [68].

Theorem 3.2. *Two vector bundles are isomorphic iff there exists a homeomorphism between their total spaces, which preserves the fibres.*

If two vector bundles belong to different isomorphism classes, there is no continuous function, which transforms the total spaces to each other, while preserving the fibres.

[1]If the base space is a smooth manifold, one can even restrict to considering smooth square-integrable sections with compact support, as they form a dense subspace of square integrable sections [67].

Hence, the wave functions stemming from sections of such bundles must describe particles with different topological properties.

The classification of vector bundles is the task of classifying isomorphism classes of vector bundles. The set of isomorphism classes of vector bundles of rank k will be denoted by

$$\mathcal{E}_k^{\mathbb{K}}(B).$$

Before we proceed to the specific methods of classification of vector bundles, we introduce an equivalent way of describing vector bundles, which involves *principal bundles* (principal G-bundles). A principal G-bundle $\xi : P \to B$ is a generalisation of the concept of vector bundle, where the total space is equipped with a free action of group G [2] and the base space has the structure of the orbit space $B \cong P/G$. Fibre $\pi^{-1}(p)$ is isomorphic to G is the sense that map $\pi : P \to B$ is G-invariant, i.e. $\pi(ge) = \pi(e)$. Moreover, B has a covering by open sets such that homomorphisms

$$\phi_U : U \times G \to \pi^{-1}(U)$$

are G-equivariant, i.e. $\phi_U(p, g'g) = g'\phi_U(p, g)$. Similarly, a morphism of G-bundles is a completely analogous notion to the notion of the vector bundle morphism with an additional requirement for map $f : P_1 \to P_2$ being G-equivariant, i.e. $f(gp) = gf(p)$ for all $g \in G$ and $p \in P_1$. Principal G-bundles have particularly nice properties thanks to the G-equivariance of all maps. In particular, we have the following two facts.

Fact 3.1. *Any morphism between principal G-bundles is an isomorphism.*

Fact 3.2. *A principal G-bundle is trivial iff it admits a section.*

The set of isomorphism classes of principal G-bundles over base space B will be denoted by

$$\mathcal{P}_G(B).$$

The following theorem shows the usefulness of principal bundles in studying vector bundles.

Theorem 3.3. *For a fixed base space B, sets $\mathcal{P}_{GL_k(\mathbb{K})}(B)$ and $\mathcal{E}_k^{\mathbb{K}}(B)$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ are in a bijective correspondence.*

Proof. Let us first describe a map, which assigns a principal $GL_k(\mathbb{K})$-bundle to every complex vector bundle of rank k. To this end, we pass from considering vector spaces $\mathbb{K}^k$ as fibres to considering all sets consisting of k vectors that form a basis of $\mathbb{K}^k$. A collection of vectors $(v_1, \ldots, v_k)$ in $\mathbb{K}^k$ that are linearly independent is called a frame. The set of all possible frames in $\pi^{-1}(b)$ will be denoted by F_b. Taking the sum of all frames for each fibre $\pi^{-1}(b)$, we obtain the frame bundle

$$F(E) := \bigcup_{b \in B} F_b.$$

There is a natural action of $GL_k(\mathbb{K})$ on F_b by complex $k \times k$ matrices with nonzero determinant, which reads $A(v_1, \ldots, v_k) := (Av_1, \ldots, Av_k)$. This action is free, hence it

[2] The action of G on P can be left or right. In this work we pick up the convention of right action. This means, that $g(h(p)) = (gh)(p)$ for $g, h \in G$, $p \in P$. Group action is free iff for all $g \in G$ and $p \in P$, $gp \neq p$.

gives $F(E)$ the structure of a principal $GL_k(\mathbb{K})$-bundle. Conversely, for every principal $GL_k(\mathbb{K})$-bundle P with base space B there is a corresponding complex vector bundle $E(P)$. Bundle $E(P)$ is constructed as the quotient of $P \times \mathbb{K}^k$ with the equivalence relation $(gp, r) \sim (p, g^{-1}r)$ for all $g \in GL_k(\mathbb{K})$. The resulting bundle $E(P) = (P \times \mathbb{K}^k)/\sim$ has base space $P/GL_k(\mathbb{K}) \cong B$ and each fibre is $\mathbb{K}^k$. What remains to show is that both maps $\phi : E \to F(E)$ and $\epsilon : P \to E(P)$ compose to identity. For any E, we have $(\epsilon \circ \phi)(E) = (\phi(E) \times \mathbb{K}^k)/\sim$. The resulting bundle is isomorphic to E by taking $\phi(E) \times \mathbb{K}^k \ni ((v_1, \ldots, v_k), (r_1, \ldots, r_k)) \mapsto r_1v_1 + \cdots + r_kv_k \in E$. To see this, note that this map descends to a map on $(\phi(E) \times \mathbb{K}^k)/\sim$, because expression $r_1v_1 + \cdots + r_kv_k$ is the same when vectors $v_1, \ldots, v_k$ are transformed by $g \in GL_k(\mathbb{K})$ or vector $(r_1, \ldots, r_k)$ is transformed by g^{-1}. The fact that $v_1, \ldots, v_k$ are linearly independent allows one to determine the coefficients $r_1, \ldots, r_k$ from a given point in E. Finally, consider composition $(\phi \circ \epsilon)(P)$, which is the set of all k-frame bundles in $P \times \mathbb{K}^k/\sim$. An element of such a principal bundle is a collection of equivalence classes $([(p, r^1)], \ldots, [(p, r^k)])$, $r^i \in \mathbb{K}^k$. This suggests to define an isomorphism as the map $P \to F(P \times \mathbb{K}^k/\sim)$ that assigns $p \mapsto ([(p, r^1)], \ldots, [(p, r^k)])$. This map is a $GL_k(\mathbb{K})$-equivariant morphism, hence indeed is an isomorphism of principal bundles. □

While interpreting sections of vector bundles as wave functions, we need the notion of a hermitian product on E. This means that we consider hermitian vector bundles, i.e. bundles with hermitian product $\langle \cdot, \cdot \rangle$ on fibres $\pi^{-1}(p)$, $p \in B$, that depends on the base point and varies between the fibres in a continuous way. Then, for a measure ν on B and two sections σ, σ', the overlap of the corresponding wave functions reads

$$\langle \sigma | \sigma' \rangle = \int_B \langle \sigma^*(p), \sigma'(p) \rangle_p d\mu(p).$$

Choosing in the proof of theorem 3.3 sets of unitary frames, we obtain an analogous correspondence between hermitian vector bundles and principal $U(k)$-bundles. If the base space is paracompact, any complex vector bundle can be given a hermitian metric [35]. Using the fact that principal $U(k)$-bundles corresponding to different choices of the hermitian structure are isomorphic [35], we have the following bijection

$$\mathcal{P}_{U(k)}(B) \cong \mathcal{E}_k^{\mathbb{C}}(B).$$

From now on, we will focus only on the problem of classification of principal $U(k)$-bundles.

3.1. Universal bundles and Chern classes

It turns out, that all vector bundles of rank k over a paracompact topological space can be obtained from a vector bundle, which is universal for all base spaces. Let us next outline some main steps of this procedure. The construction of a *pullback bundle* is a key construction used in such a classification of vector bundles. Any continuous map $f : B' \to B$ between base spaces induces a pullback map of vector bundles over B to vector bundles over B'. The pullback bundle is defined as $f^*E = \{(p, e) \in B' \times E : f(p) = \pi(e)\}$. The topology of f^*E is induced by the topology of $B' \times E$ and the projection $\pi' : f^*E \to B'$ is just the projection on the first factor. Projection on the second factor defines map $g: \ f^*E \to E$ so that diagram (3.2) commutes.

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array} \tag{3.2}$$

Similarly, one defines the pullback of principal G-bundles. For a fixed pair of base spaces A and B, we denote the set of homotopy classes of continuous maps from A to B by $[A, B]$. For a fixed principal G-bundle $\xi: \ P \to B$, the pullback map induces a map from $[A, B]$ to the set of isomorphism classes of principal G-bundles over A by $f \mapsto f^*\xi$. A space B for which such a map is bijective regardless the choice of space A, is called a *classifying space* for G and is denoted by BG. If this is the case, bundle ξ is called a *universal bundle*. A fundamental theorem [35] asserts, that if the total space E is weakly contractible (all of its homotopy groups are trivial), then ξ is a universal bundle. Furthermore, for a fixed G, the universal space is unique up to homotopy. For $G = U(k)$ one can construct weakly contractible principal bundles as certain canonical principal bundles over the infinite Grassmannian

$$BU(k) = Gr_k(\mathbb{C}^\infty).$$

Therefore, we have the following bijections classifying isomorphism classes of hermitian vector bundles over a paracompact base space B.

$$\mathcal{E}_k^{\mathbb{C}}(B) \cong \mathcal{P}_{U(k)}(B) \cong [B, Gr_k(\mathbb{C}^\infty)].$$

Let us next briefly describe the construction of the infinite Grassmannian. Recall the standard definition of the finite Grassmannian $Gr_k(\mathbb{C}^{k+n})$ as the set of all subspaces of dimension k in $\mathbb{C}^{k+n}$. This set carries the topology of a certain quotient space. By a choice of basis, a subspace of dimension k can be described as a k-tuple of linearly independent vectors from $\mathbb{C}^{k+n}$, therefore as an element of the k-fold Cartesian product $\mathbb{C}^{k+n} \times \mathbb{C}^{k+n} \times \cdots \times \mathbb{C}^{k+n}$. The set of all such tuples is an open subset of $(\mathbb{C}^{k+n})^{\times k}$ and is called the Stiefel manifold $V_k(\mathbb{C}^{k+n})$. Grassmannian $Gr_k(\mathbb{C}^{k+n})$ is the quotient space $V_k(\mathbb{C}^{k+n})/\sim$ with the equivalence relation that identifies tuples that span the same subspace of $\mathbb{C}^{k+n}$. We say that a subset of $Gr_k(\mathbb{C}^{k+n})$ is open iff its preimage under the quotient map is open as a subset of $(\mathbb{C}^{k+n})^{\times k}$. The infinite Grassmannian $Gr_k(\mathbb{C}^\infty)$ is constructed by taking the limit $n \to \infty$ as the set of all k-dimensional subspaces of $\mathbb{C}^\infty$. In order to define topology of the infinite Grassmannian, consider the natural sequence of subspaces $Gr_k(\mathbb{C}^k) \subset Gr_k(\mathbb{C}^{k+1}) \subset Gr_k(\mathbb{C}^{k+2}) \subset \ldots$ that converges to $Gr_k(\mathbb{C}^\infty)$. We say that a subset of $Gr_k(\mathbb{C}^\infty)$ is open iff its intersection with each $Gr_k(\mathbb{C}^{k+n})$ is an open set. The universal bundle $\gamma_{\mathbb{C}}^k$ is constructed as follows. The total space $E(\gamma_{\mathbb{C}}^k)$ the set of all pairs

$$(k - \text{plane in } \mathbb{C}^\infty, \text{vector in that plane}).$$

The projection map π projects on the first component of the pair. The topology of $E(\gamma_{\mathbb{C}}^k)$ is the topology of a subset of $Gr_k(\mathbb{C}^\infty) \times \mathbb{C}^\infty$. Having introduced the above concepts, we sum up the classification procedure by the following theorem.

Theorem 3.4. *Any principal $U(k)$-bundle over a paracompact Hausdorff space B is the pullback of the universal bundle $\gamma_{\mathbb{C}}^k$ by a continuous map $f: \ B \to Gr_k(\mathbb{C}^\infty)$. The isomorphism class of $f^*(\gamma_{\mathbb{C}}^k)$ is determined uniquely by the homotopy class of f and vice versa.*

Theorem 3.4 gives us an alternative way of classifying principal bundles by homotopy classes of continuous maps from the base space to $BU(k)$. However, the classification of such homotopy classes of maps, as well as differentiating between different classes are difficult tasks. A more computable criterion for comparing isomorphism classes of vector bundles are invariants called characteristic classes. In this work, we apply a specific class of invariants called Chern characteristic classes. Let us next briefly introduce this notion. A characteristic class is a map that assigns to each principal G-bundle $\xi : P \to B$ an element of the cohomology ring of B with some coefficients.

$$c(\xi) \in H^*(B).$$

This assignment is required to satisfy the naturality condition, which states that the pullback of vector bundles induces the pullback of a given characteristic class, i.e.

$$c(f^*(\xi)) = f^*(c(\xi)).$$

From this definition it follows that characteristic classes are invariant under isomorphisms of principal bundles. We are primarily interested in the characteristic classes that describe principal $U(k)$-bundles and have values in $H^*(B, \mathbb{Z})$. Such characteristic classes are called integral Chern classes. The set of all integral Chern classes (i.e. integral Chern classes for principal $U(k)$-bundles over all paracompact base spaces) $Char_{U(k)}(\mathbb{Z})$ has the structure of a ring, which is isomorphic to the cohomology ring of the classifying space $H^*(BU(k), \mathbb{Z})$.

$$Char_{U(k)}(\mathbb{Z}) \cong H^*(BU(k), \mathbb{Z}).$$

This can be seen by evaluating Chern classes on the universal bundle $\gamma^k_{\mathbb{C}}$. Map $c \mapsto c(\gamma^k_{\mathbb{C}})$ is a ring homomorphism. Its inverse is constructed in the following way. Let $a \in H^q(BU(k), \mathbb{Z})$. We assign to this element a characteristic class c_a, which is defined defined by its values on an arbitrary principal bundle $\xi : P \to B$. By the classification theorem, we have $\xi = f^*_\xi(\gamma^k_{\mathbb{C}})$ for some continuous map $f_\xi : B \to BU(k)$. Hence, c_a is evaluated as

$$c_a(\xi) := f^*_\xi(a),$$

where in the above formula $f^*_\xi : H^q(BU(k), \mathbb{Z}) \to H^q(B, \mathbb{Z})$ is the pullback of cohomology rings via map f_ξ. Map f^*_ξ is often called the *characteristic homomorphism.* Because c_a is an element of the cohomology group of degree q, we say that c_a is a Chern class of degree q. It turns out that the only nonzero Chern classes are of even degree.

Theorem 3.5. *The ring of $U(k)$ Chern characteristic classes is a polynomial algebra on k generators*

$$Char_{U(k)}(\mathbb{Z}) \cong H^*(BU(k), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k],$$

where $c_i \in H^{2i}(BU(k), \mathbb{Z})$ is the ith Chern class.

For the proof, we refer the reader to [37, 35]. Another important general property of Chern classes is the *splitting principle*, which states, that each class c_i is completely determined by a polynomial in first Chern classes. This is done as follows. Consider an inclusion of the maximal torus $i : T = U(1)^{\times n} \to U(k)$ (it can be viewed as all diagonal $U(k)$ matrices in some basis). The inclusion map induces map Bi between classifying spaces

$$Bi : BU(1)^{\times n} \to BU(k).$$

The splitting principle asserts, that the corresponding pullback of cohomology groups

$$Bi^* : \ H^*(BU(k), \mathbb{Z}) \to H^*(BU(1)^{\times k}, \mathbb{Z})$$

is injective [53]. Because space $BU(1)^{\times k}$ is isomorphic to the k-fold Cartesian product of $BU(1)$, one can use Künneth theorem, which states that $H^*(BU(1)^{\times k})$ is the polynomial ring on k generators

$$H^*(BU(1)^{\times k}, \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_k].$$

The generators can be regarded as first Chern classes of each copy of $BU(1)$. Moreover, each Chern class c_i is uniquely characterised as the pullback of the ith elementary symmetric polynomial σ_i[3] [53]

$$c_i = Bi^*(\sigma_i(x_1, \dots, x_k)). \tag{3.3}$$

Chern classes are especially useful in classifying line bundles. This is thanks to the fact that the universal space $BU(1) = \mathbb{P}(\mathbb{C}^\infty)$ is an Eilenberg - MacLane space of type $(\mathbb{Z}, 2)$, which means that only its first two homotopy groups are non vanishing and both are isomorphic to $\mathbb{Z}$. This in turn implies that the set of homotopy classes of maps $[B, BU(1)]$ is in a bijective correspondence with $H^2(B, \mathbb{Z})$. For more details, see [36, 37]. Hence, we arrive at the first direct application of the knowledge of cohomology ring of space B, namely

$$\mathcal{E}_1^{\mathbb{C}}(B) \cong H^2(B, \mathbb{Z}).$$

The abelian group structure of $\mathcal{E}_1^{\mathbb{C}}(B)$ is realised here by the fibrewise tensor product of vector bundles [49]. The identity element is the trivial line bundle. More applications of Chern classes and cohomology ring $H^*(B, \mathbb{Z})$ follow in the remaining parts of this section. In particular, they appear in K-theory and while studying characteristic classes of flat vector bundles.

For more details and complete proofs of facts used in this section regarding the structure of characteristic classes, see also [37, 69].

3.2. K-theory

The set of vector bundles over base space B acquires under proper modifications the structure of an abelian group. This group is denoted by $K(B)$ and called the Grothendieck group. Let us next briefly review the construction of the Grothendieck group. As the main objects of interest are complex vector bundles, we immediately specify all constructions to the complex K-theory. We start with the notion of a fibrewise direct sum of vector bundles, which is also called the Whitney sum of vector bundles. Given two vector bundles $\xi : \ E \to B$, $\xi' : \ E \to B$, we denote their fibrewise sum by $\xi \oplus \xi' : \ E \oplus_B E' \to B$, where

$$E \oplus_B E' := \{(v, v') \in E \times E' : \ \pi(v) = \pi(v')\} \subset E \times E'.$$

[3]Elementary symmetric polynomials form a basis of symmetric polynomials. In terms of generators, they are given by formula

$$\sigma_i = \sum_{1 \leq a_1 < a_2 < \dots < a_i \leq k} x_{a_1} \dots x_{a_i}.$$

This operation of Whitney sum is associative and has the unit element - the trivial zero-dimensional vector bundle. Whitney sum descends to the isomorphism classes and gives the set $\mathcal{E}^{\mathbb{C}}(B)$ the structure of an abelian semigroup with unit (there are no inverse elements). We denote the addition of isomorphism classes as

$$[\xi] + [\xi'] =: [\xi \oplus \xi'].$$

This situation is analogous to the addition of natural numbers. In fact, there is a homomorphism between the $\mathcal{E}^{\mathbb{C}}(B)$ and $\mathbb{N}$, which assigns to a vector bundle its rank. As one passes from $\mathbb{N}$ to $\mathbb{Z}$ by considering a certain quotient map, a similar construction for vector bundles yields Grothendieck group $K(B)$.

Definition 3.1. *The Grothendieck group of complex vector bundles over B, $K(B)$, is the set of pairs of isomorphism classes of vector bundles $([\xi_+], [\xi_-])$, subject to the following equivalence relation.*

$$\begin{aligned}(([\xi_+], [\xi_-]) \sim (([\eta_+], [\eta_-])) \iff \exists_{\mu,\nu \in Vect^{\mathbb{C}}(B)}\ ([\xi_+ \oplus \mu], [\xi_- \oplus \mu]) =& \\ = ([\eta_+ \oplus \nu], [\eta_- \oplus \nu]).& \qquad (3.4)\end{aligned}$$

The equivalence class of $([\xi_+], [\xi_-])$ will be denoted by $[([\xi_+], [\xi_-])]$ or simply as $[\xi_+] - [\xi_-] \in K(B)$.

Equivalently, we can formulate relation (3.1) as

$$(([\xi_+], [\xi_-]) \sim (([\eta_+], [\eta_-])) \iff \exists_{\gamma \in Vect^{\mathbb{C}}(B)}\ [\xi_+ \oplus \eta_- \oplus \gamma] = [\eta_+ \oplus \xi_- \oplus \gamma]. \qquad (3.5)$$

The addition of vector bundles gives rise to associative and commutative operation of addition in $K(B)$, which works pairwise

$$[([\xi_+], [\xi_-])] + [([\eta_+], [\eta_-])] = [([\xi_+ \oplus \eta_+], [\xi_- \oplus \eta_-])]\,. \qquad (3.6)$$

The zero element is of the form $[([\xi], [\xi])]$ for any ξ. The inverse of $[([\xi_+], [\xi_-])]$ is $[([\xi_-], [\xi_+])]$. Hence, we have introduced the notion of subtraction between isomorphism classes of vector bundles via an equivalence relation on $\mathcal{E}^{\mathbb{C}}(B) \times \mathcal{E}^{\mathbb{C}}(B)$. Consequently, there is a homomorphism between $K(B)$ and the integers

$$[\xi_+] - [\xi_-] \mapsto \mathrm{rk}(\xi_+) - \mathrm{rk}(\xi_-).$$

If B is compact and Hausdorff, the above definitions simplify due to the following fact [49].

Fact 3.3. *For every vector bundle $\xi : E \to B$ over a compact Hausdorff space B there exists a vector bundle $\tilde{\xi} :\ \tilde{E} \to B$ such that*

$$E \oplus_B \tilde{E} \cong B \times \mathbb{C}^n.$$

Using fact 3.3, every element of $K(B)$ over a compact Hausdorff space can be represented as

$$[\xi] - [\tau_n] \text{ for some } n, \qquad (3.7)$$

where τ_n is the trivial vector bundle of rank n. This is because for any $\xi_+,\ \xi_-$ we have

$$[([\xi_+], [\xi_-])] \sim \left[([\xi_+ \oplus \tilde{\xi}_-], [\xi_- \oplus \tilde{\xi}_-])\right],$$

where $\tilde{\xi}_-$ can be chosen so that $\xi_- \oplus \tilde{\xi}_- \cong \tau_n$. A similar argument shows that equivalence relation (3.5) can be realised just by taking $\gamma = \tau_n$ for some n. Note that representation (3.7) is not unique, because by passing to group $K(B)$ we identify some isomorphism classes of vector bundles. More precisely, let us see when two pairs $([\xi], \tau_n)$ and $([\xi'], \tau_m)$ represent the same element of $K(B)$ in the case, when B is a compact Hausdorff space. By definition (3.5) and fact 3.3, $([\xi], \tau_n) \sim ([\xi'], \tau_m)$ implies that

$$\exists_{k \in \mathbb{Z}} \; [\xi \oplus \tau_m \oplus \tau_k] = [\xi' \oplus \tau_n \oplus \tau_k].$$

This means that bundes $\xi \oplus \tau_m$ and $\xi' \oplus \tau_n$ become isomorphic when summed with a sufficiently large trivial bundle. Such a fact gives rise to the notion of *stable equivalence* of vector bundles.

Definition 3.2. *Vector bundles ξ and ξ' are stably equivalent $\xi \sim_s \xi'$ iff*

$$\exists_{k_1, k_2 \in \mathbb{Z}} \; [\xi \oplus \tau_{k_1}] = [\xi' \oplus \tau_{k_2}].$$

In terms of stable equivalence, the above considerations can be written in a concise way as

$$([\xi] - [\tau_n] = [\xi'] - [\tau_m]) \implies (\xi \sim_s \xi') \,. \tag{3.8}$$

The set of stable equivalence classes of vector bundles over a compact Hausdorff space has the structure of an abelian group stemming from the addition of vector bundles. This group is called the reduced Grothendieck group $\tilde{K}(B)$.

Definition 3.3. *Let B be a compact Hausdorff space. The reduced Grothendirck group $\tilde{K}(B)$ is an abelian group of stable equivalence classes of vector bundles over B, where the addition operation reads*

$$[\xi]_s + [\xi']_s := [\xi \oplus \xi']_s.$$

As a consequence, the inverse element of $[\xi]_s$ is the unique class $[\tilde{\xi}]_s$ such that $\xi \oplus \tilde{\xi} \cong \tau_n$ for some n. The neutral element is the equivalence class of the zero-dimensional vector bundle $[\tau_0]_s$.

Equation (3.8) suggests, that group $K(B)$ is roughly the same as $\tilde{K}(B)$. To be more precise, let us study the following group homomorphism $K(B) \to \tilde{K}(B)$

$$[\xi] - [\tau_n] \mapsto [\xi]_s. \tag{3.9}$$

Homomorphism (3.9) is surjective and its kernel consists of elements $[\xi] - [\tau_n]$, where ξ is stably equivalent to τ_0. This is only possible, when ξ is a trivial vector bundle τ_m. Hence, kernel of (3.9) is the subgroup of $K(B)$ generated by elements of the form $[\tau_m] - [\tau_n]$. This subgroup is isomorphic to $\mathbb{Z}$. Thus, we have the following splitting [50]

$$K(B) = \tilde{K}(B) \oplus \mathbb{Z}.$$

If the base space has the homotopy type of a finite CW-complex, group $\tilde{K}(B)$ fully describes isomorphism classes of vector bundles that have a sufficiently high rank. This claim is based on the following two theorems, whose proofs can be found for instance in [52]. The following theorems concern vector bundles, whose rank is in the *stable range*, i.e. is greater than or equal to

$$k_s := \left\lceil \frac{1}{2} \dim B \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x.

Theorem 3.6. *If ξ and ξ' are two vector bundles of rank k, where $k \geq k_s$ and $\xi \oplus \tau_n \cong \xi \oplus \tau_n$ for some n, then ξ and ξ' are isomorphic.*

Theorem 3.7. *Every complex vector bundle ξ, whose rank k is greater than or equal to k_s, is isomorphic to $\eta \oplus \tau_{k-k_s}$ for some $\eta \in Vect^{\mathbb{C}}_{k_s}(B)$.*

Theorem 3.6 means that for vector bundles of a fixed rank in the stable range the standard notion of isomorphism classes is equivalent to the notion of stable equivalence classes. Theorems 3.7 and 3.6 together assert, that set of stable equivalence classes of $Vect^{\mathbb{C}}(B)$ is equal to $\mathcal{E}^{\mathbb{C}}_{k_s}(B)$. Moreover, by theorem 3.7 $\mathcal{E}^{\mathbb{C}}_{k}(B)$ is the same for all $k \geq k_s$ and equal to $\mathcal{E}^{\mathbb{C}}_{k_s}(B)$. Therefore,

$$\mathcal{E}^{\mathbb{C}}_{k}(B) \cong \tilde{K}(B) \text{ for } k \geq k_s.$$

Relation between K-theory and cohomology

The relation between K-theory and cohomology is phrased via the *Chern character*. Chern character is a ring homomorphism from K-theory to rational cohomology

$$ch: \ K(B) \to H^*(B, \mathbb{Q}).$$

We view $K(B)$ as a commutative ring, where multiplication corresponds to tensor products of vector bundles

$$[([\xi_+], [\xi_-])] \cdot [([\eta_+], [\eta_-])] := [([(\xi_+ \otimes \eta_+) \oplus (\xi_- \otimes \eta_-)], [(\xi_+ \otimes \eta_-) \oplus (\xi_- \otimes \eta_+)])]$$

and addition works as defined in equation (3.6).

Recall the splitting principle, which asserts that the Chern characteristic classes of a complex bundle of rank k can be represented as elementary symmetric polynomials in k variables (equation 3.3). We denote the elementary symmetric polynomials as $\sigma_1, \ldots, \sigma_k$. Such a representation allows one to compute the values of formal symmetric polynomials of Chern classes. Namely, any symmetric polynomial $f(x_1, \ldots, x_k)$ can be expressed as a polynomial in elementary symmetric polynomials $\tilde{f}(\sigma_1, \ldots, \sigma_k)$. In order to determine the value of $\tilde{f}(c_1, \ldots, c_k)$, we replace σ_i with c_i in the formula defining $\tilde{f}$. The Chern character is such a polynomial. Let us next state its precise definition. Consider a family of symmetric polynomials of the form

$$x_1^m + \cdots + x_k^m.$$

The prescription of how to represent such polynomials by the elementary symmetric polynomials is known as the Newton's identities. As a result, we get

$$x_1^m + \cdots + x_k^m = P_m(\sigma_1(x_1, \ldots, x_k), \ldots, \sigma_k(x_1, \ldots, x_k)),$$

where P_m is a polynomial. The Chern character is defined as the sum

$$ch(\xi) = \sum_m \frac{1}{m!} P_m(c_1(\xi), \ldots, c_k(\xi)). \tag{3.10}$$

Note, that for line bundles, $ch(\xi)$ is equal to $\exp(c_1(\xi))$. Moreover the splitting principle implies that

$$ch(\xi \oplus \xi') = ch(\xi) + ch(\xi'), \ ch(\xi \otimes \xi') = ch(\xi)ch(\xi')$$

and ch descends to a homomorphism of rings between $K(B)$ or $\tilde{K}(B)$ and $H^*(B)$. The coefficients of $H^*(B)$ must be rational, so that formula (3.10) makes sense. The key theorem of this section is the following.

Theorem 3.8. *Assume B has the homotopy type of a finite CW-complex. Then,*

$$ch: \ \tilde{K}(B) \otimes \mathbb{Q} \to \bigoplus_{i=1} H^{2i}(B, \mathbb{Q})$$

is an isomorphism.

This theorem follows from a more general result of paper [11], but for finite CW-complexes it can also be proved using less advanced methods via induction with respect to attaching cells of B and cohomological exact sequences, see e.g. [54].

As a consequence of theorem 3.8, the classification of vector bundles in the stable range asserts that

$$\mathcal{E}_k^{\mathbb{C}} \cong \bigoplus_{i=1} H^{2i}(B, \mathbb{Q}), \text{ for } k \geq \frac{1}{2} \dim B,$$

on condition that the even integral cohomology groups of B are torsion-free. In the case, when there is non-trivial torsion in $H^*(B, \mathbb{Z})$, torsion of $\tilde{K}(B)$ is determined by the Atiyah-Hirzebruch spectral sequence [11]. However, the correspondence between torsion of even cohomology and $\tilde{K}(B)$ is not an isomorphism (for examples, see [65]). In particular, torsion in $\tilde{K}(B)$ can vanish, despite the existence of nonzero torsion in $H^{2i}(B, \mathbb{Z})$.

3.3. Flat bundles and quantum statistics

In this section, we describe the structure of the set of flat principal G-bundles over base space B. More precisely, we consider the set of pairs $(\xi, \mathcal{A})$, where ξ is a principal G-bundle, and $\mathcal{A}$ is a connection 1-form on ξ. We divide the set of such pairs into equivalence classes $[\xi, \mathcal{A}]$ that consist of vector bundles isomorphic to ξ and the set of flat connections that are congruent to $\mathcal{A}$ under the action of the gauge group. The quotient space with respect to this equivalence relation is called the *moduli space of flat connections* and is denoted by $\mathcal{M}(B, G)$. The culminating point of this section is to introduce the fundamental relation, which says that $\mathcal{M}(B, G)$ is in a bijective correspondence with the set of conjugacy classes of homomorphisms of the fundamental group of B.

$$\mathcal{M}(B, G) \cong \mathrm{Hom}(\pi_1(B), G)/G. \tag{3.11}$$

We use this relation to explain some key properties of quantum statistics that were sketched in the introduction of this thesis.

Let us begin the description of the moduli space of flat connections in the case, when B is a smooth manifold. We define a principal connection on principal G-bundle as a decomposition of the tangent bundle TP that has certain properties. Firstly, we have $H \oplus \ker \pi^* = TP$, where $\pi^*: \ TP \to TB$ is the push-forward of tangent vectors with respect to the projection map $\pi: P \to B$. Secondly, we require space H to be invariant under the G-action, i.e. the push-forward with respect to any $g \in G$ of all vectors from $H_p, \ p \in P$ gives space H_{gp}[4]. Principal connection H is flat iff the distribution $H \subset TP$ is integrable, i.e. vectors from H are tangent to the leaves of a foliation. Another criterion for deciding flatness of a principal connection comes from the fact that a connection can be uniquely encoded in a 1-form defined on the total

[4]We denote this fact by $R_g^*(H_p) = H_{gp}$, where R_g^* is the push-forward (tangent map) to the right multiplication by elements of G, $R_g: \ P \to P$.

space P that has values in the Lie algebra of G. This is done in the following way. Space $\ker \pi^*$ at point $p \in P$ is the space tangent to the fibre $\pi^{-1}(\pi(p))$ at p. Using the fact that G acts transitively on the fibres, we identify $\ker \pi^*$ at p with the Lie algebra of G denoted by $\mathfrak{g}$.

$$T_p\left(\pi^{-1}(\pi(p))\right) \cong \mathfrak{g} \text{ by } \mathfrak{g} \ni \alpha \mapsto \hat{\alpha}_p := \frac{d}{dt}\Big|_{t=0} e^{\alpha t} p.$$

Using such identifications pointwise over P, we regard a connection H at every $p \in P$ as the kernel of a linear projection map $\mathcal{A}_p : \ T_pP \to \mathfrak{g}$. We require $\mathcal{A}_p$ to vary smoothly with p and to restrict to the identity map on $\ker \pi^*_p$. The G-invariance of H is equivalent to $\mathcal{A}_p$ being G-equivariant in the following sense.

$$\mathcal{A}_{gp}(R_g^* v) = Ad_{g^{-1}} A_p(v), \ v \in T_pP, \ g \in G,$$

where Ad_g is the adjoint action of G on its algebra, which reads $Ad_g(\alpha) = \frac{d}{dt}\big|_{t=0} g e^{\alpha t} g^{-1}$. Such a linear map $\mathcal{A}_p$ can be viewed an element $T^*_pP \otimes \mathfrak{g}$. Collecting such projections pointwise over P we get the following fact.

Fact 3.4. *There is a bijective correspondence between principal connections on P and equivariant one-forms over P with values in $\mathfrak{g}$. In other words, a connection one-form ω is an element of $C^\infty(P, T^*P \otimes \mathfrak{g})$ that satisfies the following conditions.*

- $R_g^*\omega = Ad_{g^{-1}}\omega$,
- $\omega(\hat{\alpha}) = \alpha$, *where $\hat{\alpha}$ is the fundamental vector field of $\alpha \in \mathfrak{g}$.*

Such one-form ω determines H as $H_p = \ker \omega_p$.

A connection is flat iff its one-form satisfies the Maurer-Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Having fixed a principal connection on P, we consider parallel transport of elements of P around loops in B. Parallel transport around loop $\gamma \subset B$ is a morphism of fibres $\Gamma_\gamma : \ \pi^{-1}(b) \to \pi^{-1}(b)$, where b is the base point of loop γ. Map Γ_γ is defined as follows. For a fixed loop $\gamma \in B$, we construct a curve $\tilde{\gamma} \in P$ such that $\pi(\tilde{\gamma}) = \gamma$ and all vectors that are tangent to curve $\tilde{\gamma}$ belong to H. Curve $\tilde{\gamma}$ is called the *horizontal lift* of γ and is described by an ordinary differential equation. Hence, $\tilde{\gamma}$ is fully determined by the choice of its initial point $\tilde{\gamma}(0) \in \pi^{-1}(b)$. The parallel transport assigns the end point of $\tilde{\gamma}$ to its initial point

$$\Gamma_\gamma : \ \tilde{\gamma}(0) \mapsto \tilde{\gamma}(1).$$

Because fibres are homogeneous spaces for the action of G, for every choice of the initial point $p = \tilde{\gamma}(0)$ there is a unique group element $g \in G$ such that $\tilde{\gamma}(1) = gp$. We denote this element by $\mathrm{hol}_p(H, \gamma)$ and call the *holonomy* of connection H around loop γ at point p. Moreover, by the G-equivariance of the connection, we get that

$$\Gamma_\gamma(gp) = g\Gamma_\gamma(p), \ p \in P.$$

This means that $\mathrm{hol}_{gp}(H, \gamma) = g^{-1}\mathrm{hol}_p(H, \gamma)g$. If connection H is flat, the parallel transport depends only on the topology of the base space [66], i.e.

- Γ_γ depends only on the homotopy class of γ,
- parallel transport around a contractible loop is trivial,
- parallel transport around two loops that have the same base point is the composition of parallel transports along the two loops.

$$\Gamma_{\gamma_1\circ\gamma_2}=\Gamma_{\gamma_1}\circ\Gamma_{\gamma_2}.$$

The above facts show that if H is flat, map $\pi_1(B)\ni[\gamma]\mapsto \mathrm{hol}_p(H,\gamma)\in G$ is a homomorphism of groups. Because holonomies at different points from the same fibre differ only by conjugation in G, it is not necessary to specify the choice of the initial point. Instead, we consider map

$$\mathcal{S}_H:\ \pi_1(B)\ni[\gamma]\mapsto \mathrm{Hol}(H,\gamma)\in Conj(G),$$

where $\mathrm{Hol}(H,\gamma)=\{\mathrm{hol}_p(H,\gamma):\ p\in\pi^{-1}(\gamma(0))\}$ is a conjugacy class of group G. If $\pi_1(B)$ is finitely generated, by picking generators $a_1,\dots,a_r$ of the fundamental group and considering set $(\mathrm{hol}_p(H,a_1),\dots,\mathrm{hol}_p(H,a_r))\in G^{\times r}$, one fixes a group homomorphism $\mathrm{Hom}(\pi_1(B),G)$. Taking the quotient by the conjugation in G and disregarding the point p, one can view map $\mathcal{S}_H$ as an element of $\mathrm{Hom}(\pi_1(B),G)/G$. There is one more symmetry of this map that we have not discussed so far, namely the gauge symmetry. A gauge transformation is a map $f:\ P\to G$, which is G-equivariant, i.e. $f(gp)=g^{-1}f(p)g$. A gauge transformation induces an automorphism of P, which acts as $p\to f(p)p$. Consequently, transformation f induces a pullback of connection forms. It can be shown that map $\mathcal{S}_H$ is gauge invariant [48, 66], i.e. depends only on the gauge equivalence class of connection H.

Example 3.9. – Gauge transformations of the $U(1)$ bundle over $\mathcal{S}^1$. Parametrise points from S^1 by angle $\theta\in[0,2\pi[$. In local coordinates, points in P are pairs $(\theta,e^{i\phi})$ and the action of $U(1)$ reads $e^{it}(\theta,e^{i\phi})=(\theta,e^{i(\phi+t)})$. Because $U(1)$ is abelian, the equivariance of gauge map means that $f(gp)=f(p)$, i.e. f is constant on the fibres. In other words, for $p=(\theta,e^{i\phi})$, we have $f(p)=e^{i\mathcal{F}(\theta)}$, where $\mathcal{F}(\theta)$ is a continuous function. This gauge transformation translates to the pullback of a connection one-form as the well-known formula $\mathcal{A}\to\mathcal{A}+d\mathcal{F}$.

An important conclusion regarding flat bundles on spaces that do not have a differential structure comes from the second part of correspondence (3.11). This is the reconstruction of a flat principal bundle from a given homomorphism $\mathrm{Hom}(\pi_1(B),G)$. It turns out that any flat bundle over B can be realised as a particular quotient bundle of the trivial bundle over a space, which is called the *universal cover* of B. In order to formulate the correspondence, we first introduce the notion of a covering space and a universal cover. A covering space of B is a topological space C with a continuous surjective map $p:\ C\to B$ such that for every $b\in B$ there exists an open neighbourhood U of b in B such that $p^{-1}(U)$ is a disjoint union of sets in C that are homeomorphic with U. An universal covering space (universal cover) $\tilde{B}$ is the space that covers any connected cover of space B. The universal cover is unique up to a homeomorphism. For a general topological space B there is an issue of the existence of the universal cover. However, all spaces that are considered in this paper have universal covers. Universal covers of graph configuration spaces have a particularly nice structure, as they have the homotopy type of a $CAT(0)$ cube complex [28], which is contractible. The following theorem will also serve as a definition of a flat principal bundle for spaces that are not differential manifolds.

Theorem 3.10. *Any flat principal G-bundle $P \to B$ can be constructed as the following quotient bundle of the trivial bundle over the universal cover of B.*

$$P = (\tilde{B} \times G)/\pi_1(B).$$

In the above formula, group $\pi_1(B)$ acts on $\tilde{B}$ via deck transformations. Action on G is defined by picking a homomorphism $\rho : \pi_1(B) \to G$. Then the action reads $ag := \rho(a)g$ for $a \in \pi_1(B)$, $g \in G$.

Deck transformations are a notion, which is analogous to the notion of parallel transport. Namely, every loop $\gamma \subset B$ based at point b can be lifted to a path $\tilde{\gamma} \subset \tilde{B}$ uniquely by specifying the initial point of $\tilde{\gamma}$ in fibre $p^{-1}(b)$. The endpoint of $\tilde{\gamma}$ also belongs to $p^{-1}(b)$. We define the action of $[\gamma] \in \pi_1(B)$ in $\tilde{B}$ as map $\tilde{\gamma}(0) \to \tilde{\gamma}(1)$. This definition is independent of the choice of γ in the given homotopy class of loops. Proofs concerning the above lifting properties can be found in [38]. Intuitively, the fundamental group acts on $\tilde{B}$ by permuting points in fibres.

If the base space is a differential manifold, the corresponding flat connection can be constructed as follows. For $\tilde{B} \times G$ we consider tangent vectors in $T(\tilde{B} \times G)$ that have components only in the direction of $\tilde{B}$, i.e. tangent vectors that correspond to curves $\gamma(t) = (b(t), g)$ in local coordinates. The quotient map $q : \tilde{B} \times G \to P$ is a local diffeomorphism, and the push-forward of such tangent vectors by q^* defines the corresponding horizontal flat bundle on P. For such a connection, homomorphism ρ arises as the holonomy.

Summing up, in order to describe the moduli space of flat G-bundles, one has to classify conjugacy classes of homomorphisms $\pi_1(B) \to G$. All spaces that are considered in this thesis have finitely generated fundamental group. This fact makes the classification procedure easier. Namely, one can fix a set of generators $a_1, \ldots, a_r$ of $\pi_1(B)$ and represent them as group elements $g_1, \ldots, g_r$. Matrices $g_1, \ldots, g_r$ realise $\pi_1(B)$ in G in a homomorphic way iff they satisfy the relations between the generators of $\pi_1(B)$. In general, the relations have the form of words in the generators composing to the identity element

$$a_{i_1}^{p_1} a_{i_2}^{p_2} \ldots a_{i_k}^{p_k} = e, \; p_j \in \mathbb{Z}.$$

Assume there are n_R relations. They give rise to n_R equations for chosen elements of group G. In the case, when $G = U(n)$, these are just matrix equations and the classification problem reduces to the problem of solving sets of polynomial equations. This means that the moduli space of flat connections can be given the structure of an algebraic variety. In other words, we consider map

$$\mathcal{Q} : \; G^{\times r} \to G^{\times n_R},$$

which returns the values of words describing the relations between generators of $\pi_1(B)$. Then,

$$\mathcal{M}(B, G) = \mathcal{Q}^{-1}(e, \ldots, e)/G.$$

We view $\mathcal{Q}^{-1}(e, \ldots, e)$ as the zero locus of a set of multivariate polynomials. In general, such a zero locus has many path connected components. This reflects the topological structure of $\mathcal{M}(B, G)$. Namely, one can decompose the moduli space of flat connections into a number of disjoint components, that are enumerated by the isomorphism classes of bundles

$$\mathcal{M}(B, G) = \bigsqcup_{[\xi] \in \mathcal{P}_G(B)} \mathcal{M}_{[\xi]}(B, G).$$

$\mathcal{M}_{[\xi]}(B,G)$ is the space of flat connections on principal bundles from the isomorphism class $[\xi]$ modulo the gauge group. The following fact gives a necessary condition for two flat structures to be non-isomorphic.

Fact 3.5. *Two points in $\mathcal{M}(B,G)$ that correspond to two non-isomorphic flat bundles, belong to different path-connected components of $\mathcal{M}(B,G)$.*

Equivalently, if two flat structures are mapped to the same path-connected component of $\mathcal{M}(B,G)$, they are isomorphic. Path connecting the two points in $\mathcal{M}(B,G)$ gives a homotopy between the corresponding flat structures.

Example 3.11. – The moduli space of flat $U(1)$ bundles over spaces with finitely generated fundamental group. As conjugation in $U(1)$ is trivial, we have

$$\mathcal{M}(B,U(1)) \cong \mathrm{Hom}(\pi_1(B),U(1)).$$

Moreover, $\mathrm{Hom}(\pi_1(B),U(1))$ is the same as the space of homomorphisms from the abelianization of $\pi_1(B)$ to $U(1)$. A standard result from algebraic topology says that

$$\pi_1(B)/[\pi_1(B),\pi_1(B)] \cong H_1(B,\mathbb{Z}).$$

$H_1(B,\mathbb{Z})$ as any finitely generated abelian group decomposes as the sum of a free component and a cyclic (torsion) part

$$H_1(B,\mathbb{Z}) = \mathbb{Z}^p \oplus \bigoplus_{i=1}^{q} \mathbb{Z}_{p_i}.$$

Therefore, we can generate $H_1(B,\mathbb{Z})$ as

$$H_1(B,\mathbb{Z}) = \langle a_1,\ldots,a_p,b_1,\ldots,b_q:\ b_i^{p_i}=e\rangle.$$

We represent a_i as $e^{\iota\phi_i}$, $\phi_i \in [0,2\pi[$ and the cyclic generators as roots of unity $e^{\iota 2k_i\pi/p_i}$, where $k_i = 0,1,2,\ldots,p_i-1$. This way, we get $\prod_{i=1}^q p_i$ connected components in the space of homomorphisms $\mathrm{Hom}(H_1(B,\mathbb{Z}),U(1))$ that are enumerated by different choices of numbers k_i. Each connected component is homeomorphic to a p-torus, whose points correspond to phases ϕ_i. In fact, the connected components are in a one-to-one correspondence with isomorphism classes of flat bundles. To see this, recall the fact that set of $U(1)$-bundles has the structure of a group, which is isomorphic to $H^2(B,\mathbb{Z})$. Moreover, as we explain in remark 3.1, Chern classes of flat bundles are torsion. This means that flat $U(1)$-bundles form a subgroup of the group of all $U(1)$-bundles, which is isomorphic to the torsion of $H^2(B,\mathbb{Z})$. By the universal coefficient theorem, torsion of $H^2(B,\mathbb{Z})$ is the same as torsion of $H_1(B,\mathbb{Z})$. Note that there is exactly the same number of connected components in $\mathrm{Hom}(H_1(B,\mathbb{Z}),U(1))$ as the number of group elements in the torsion component of $H_1(B,\mathbb{Z})$. In this case, fact 3.5 implies that each connected component represents one isomorphism class of flat bundles.

Recall that for particles in $\mathbb{R}^2$ and $\mathbb{R}^3$, we had

$$H_1(C_n(\mathbb{R}^2),\mathbb{Z}) = \mathbb{Z},\ H_1(C_n(\mathbb{R}^3),\mathbb{Z}) = \mathbb{Z}_2.$$

Hence, the moduli spaces read (see also Fig. 3.2)

$$\mathcal{M}(C_n(\mathbb{R}^2),U(1)) \cong \mathrm{Hom}(\mathbb{Z},U(1)) \cong S^1,$$
$$\mathcal{M}(C_n(\mathbb{R}^3),U(1)) \cong \mathrm{Hom}(\mathbb{Z}_2,U(1)) \cong \{*,*'\} \subset T^2.$$

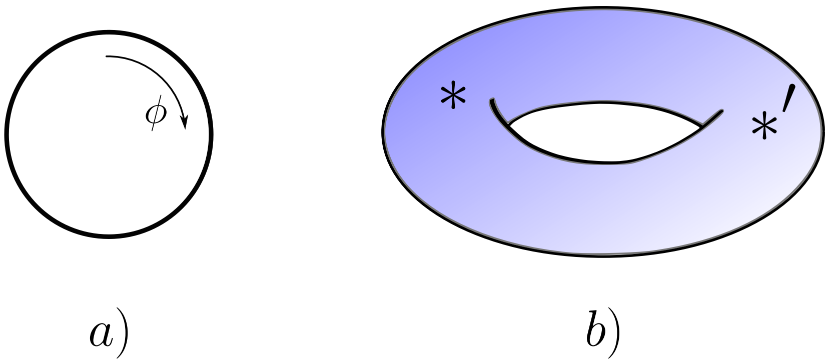


Figure 3.2: The moduli space of flat $U(1)$ bundles a) for n particles on a plane, b) n particles in $\mathbb{R}^3$. Homomorphisms from $\mathbb{Z}$ to $U(1)$ are parametrised by points from S^1 via the map $\phi \mapsto e^{\iota\phi}$. The corresponding homomorphism reads $n \mapsto e^{\iota n\phi}$. There is only one path-connected component in $\mathrm{Hom}(\mathbb{Z}, U(1))$, which reflects the fact that there is only one flat $U(1)$ bundle over $C_n(\mathbb{R}^2)$ (the trivial one) and points form the circle parametrise different flat connections. For particles in $\mathbb{R}^3$, there are two homomorphisms of $\mathbb{Z}_2 = \{1, -1\}$ - the trivial one and $1 \mapsto e^{2\pi\iota}$, $-1 \mapsto e^{\iota\pi}$. They correspond to two isolated points on the torus $T^2 = U(1) \times U(1)$. The trivial homomorphism corresponds to the bosonic bundle, while the other homomorphism corresponds to the fermionic bundle. The fundamental difference between these two types of quantum statistics is that anyons arise as different flat connections on the trivial bundle, whereas bosons and fermions arise as canonical flat connections on two non-isomorphic flat bundles.

In the following example we demonstrate some of the nontrivial problems that can be encountered when studying the moduli space of flat $U(n)$ bundles over configuration spaces of particles for $n > 1$.

Example 3.12. – The moduli space of flat $U(2)$ bundles over $C_3(\mathbb{R}^2)$. For three particles on a plane, the fundamental group is the braid group on three strands

$$\pi_1(C_3(\mathbb{R}^2)) = Br_3 = \langle a, b : \ aba = bab \rangle.$$

Hence, space $\mathrm{Hom}(Br_3, U(2))$ is the space of solutions to the following matrix equation

$$ABAB^{-1}A^{-1}B^{-1} = \mathbb{1}, \ A, B \in U(2). \tag{3.12}$$

One can study regularity properties of map $\mathcal{Q} : \ (A, B) \mapsto ABAB^{-1}A^{-1}B^{-1}$ in a straightforward way by differentiating curves $A(t), B(t) \subset U(2)$. Such methods proved to be useful in studying moduli spaces of flat connections on surfaces [47, 48], where the fundamental group is also described by one relation between generators. One can show, that map $\mathcal{Q}$ is singular, when matrices A and B commute and regular (of full rank) for generic A, B. The solution of (3.12) for commuting A, B is $A = B$. After taking the quotient by $U(2)$, this singular set gives rise to a singular point in $\mathcal{M}(C_3(\mathbb{R}^2), U(2))$. Further study of topological properties of $\mathcal{Q}^{-1}(e)/G$ would be an interesting and nontrivial problem.

Characteristic classes of flat bundles

From this point, we can move away from considering connections and use the wider definition of flat G-bundles, which makes sense for bundles over spaces that have a universal covering space. As stated in theorem 3.10, such flat bundles have the form

$$P = (\tilde{B} \times G)/\pi_1(B),$$

where we implicitly use a group homomorphism $\rho : \ \pi_1(B) \to G$ in the definition of the quotient. Diagram 3.13 combines two perspectives on the classification of flat $U(n)$-bundles.

$$\begin{array}{ccccc} \tilde{B} \times U(n) & \xrightarrow{q} & P & \xrightarrow{g_P} & \gamma^n_{\mathbb{C}} \\ \downarrow & & \pi' \downarrow & & \downarrow \pi \\ \tilde{B} & \xrightarrow{p} & B & \xrightarrow{f_P} & BU(n) \end{array} \tag{3.13}$$

On one hand, P is realised as the image of quotient map q, where the mapping between base spaces is projection p from the universal covering space to B. On the other hand, P is the pullback of the canonical $U(n)$ bundle via classifying map f_P.

$$f_P^*(\gamma^n_{\mathbb{C}}) = P = (\tilde{B} \times U(n))/\pi_1(B).$$

For such flat $U(n)$-bundles over connected CW-complexes we have the following general result about the triviality of rational Chern classes [39].

Theorem 3.13. *Let G be a compact Lie group, B a connected CW-complex and $\xi :$ $P \to B$ a flat G-bundle over B. Then, the characteristic homomorphism*

$$f_\xi^* : \ H^*(BG, \mathbb{Q}) \to H^*(B, \mathbb{Q})$$

is trivial.

Remark 3.1. *Theorem 3.13 in particular means that if B is a finite CW-complex, then by the universal coefficient theorem for cohomology (see e.g. [40]), the image of the characteristic map $f_\xi^* : \ H^*(BG, \mathbb{Z}) \to H^*(B, \mathbb{Z})$ consists only of torsion elements of $H^*(B, \mathbb{Z})$.*

Specifying the above results for $U(n)$-bundles, we get that the lack of nontrivial torsion in $H^{2i}(B, \mathbb{Z})$ has the following implications for the stable equivalence classes of flat vector bundles.

Proposition 3.14. *Let B be a finite CW complex. If the integral homology groups of B are torsion-free, then every flat complex vector bundle over B is stably equivalent to a trivial bundle.*

Proof. If the integral cohomology of B is torsion-free, then by the Chern character we get, that the reduced Grothendieck group is isomorphic to the direct sum of even cohomology of B. Thus, if all Chern classes of a given bundle vanish, this means that this bundle represents the trivial element of the reduced Grothendieck group, i.e. is stably equivalent to a trivial bundle. □

Interestingly, in the following standard examples of configuration spaces, there is torsion in cohomology.

1. Configuration space of n particles on a plane. Space $C_n(\mathbb{R}^2)$ is aspherical, i.e. is an Eilenberg-Maclane space of type $K(\pi_1, 1)$, where the fundamental group is the braid group on n strands Br_n. Cohomology ring $H^*(C_n(\mathbb{R}^2), \mathbb{Z}) = H^*(Br_n, \mathbb{Z})$ is known [41, 42, 43]. Its key properties are i) **finiteness** – $H^i(Br_n, \mathbb{Z})$ are cyclic groups, except $H^0(Br_n, \mathbb{Z}) = H^1(Br_n, \mathbb{Z}) = \mathbb{Z}$, ii) **repetition** – $H^i(Br_{2n+1}, \mathbb{Z}) = H^i(Br_{2n}, \mathbb{Z})$, iii) **stability** – $H^i(Br_n, \mathbb{Z}) = H^i(Br_{2i-2})$ for $n \geq 2i-2$. Description of nontrivial flat $U(n)$ bundles over $C_n(\mathbb{R}^2)$ for $n > 2$ is an open problem.

2. Configuration space of n particles in $\mathbb{R}^3$. Much less is known about $H^*(C_n(\mathbb{R}^3))$. Some computational techniques are presented in [44, 46] (see also [51] for an up-to-date review), but no explicit results are given. Ring $H^*(C_3(\mathbb{R}^3)$ is equal to $\mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_3$ [45] and $H^q(C_3(\mathbb{R}^3)) = 0$ for $q > 4$. However, it has been shown that there are no nontrivial flat $SU(n)$ bundles over $C_3(\mathbb{R}^3)$.

3. Configuration space of n particles on a graph (a 1-dimensional CW-complex Γ). Spaces $C_n(\Gamma)$ are Eilenberg-Maclane spaces of type $K(\pi_1, 1)$. The calculation of their homology groups is a subject of this thesis. Group $H_1(C_n(\Gamma), \mathbb{Z})$ is known [20, 22] for an arbitrary graph. We review the structure of $H_1(C_n(\Gamma))$ in section 4.4. By the universal coefficient theorem, the torsion of $H^2(C_n(\Gamma))$ is equal to the torsion of $H_1(C_n(\Gamma))$, which is known to be equal to a number of copies of $\mathbb{Z}_2$, depending on the structure of Γ. We interpret this result as the existence of different bosonic or fermionic statistics in different parts of Γ. The existence of torsion in higher (co)homology groups of $C_n(\Gamma)$, which is different than $\mathbb{Z}_2$, is an open problem. In this thesis, we compute homology groups for certain canonical families of graphs. However, the computed homology groups are either torsion-free, or have $\mathbb{Z}_2$-torsion.

Concluding remark and an open problem As we have seen while studying the example of anyons, the parametrisation of different path-connected components of the moduli space of flat bundles corresponds physically to changing some fields. On the other hand, while studying the example of particles in $\mathbb{R}^3$, we learned that on each path-connected component of $\mathcal{M}(B, G)$ there may exist points that correspond to nontrivial action of the holonomy without the requirement of introducing any additional fields in the physical model. Such points are for example the isolated points of $\mathcal{M}(B, G)$. It is worthwhile to pursue the search of such canonical points in $\mathcal{M}(B, G)$, as they may lead to some new spontaneously occurring quantum statistical phenomena. The existence of torsion in higher cohomology groups suggests, that such phenomena may be possible for higher dimensional bundles over $C_n(\mathbb{R}^2)$. However, by inspection of the table of cohomology of $C_n(\mathbb{R}^2)$ [41], they are expected to appear in systems that consist of at least 10 particles, which makes their rigorous study difficult. Perhaps a more tractable discrete model can be found for graph configuration spaces. In view of proposition 3.14, the first step in this direction would be construction of non-trivial flat vector bundles for graphs, whose configuration spaces have torsion-free cohomology ring. In chapter 5 we provide examples of graphs that have such a property.

Chapter 4

Configuration spaces of graphs

The general structure of configuration spaces of graphs has been introduced in section 1.2. For computational purposes, we use discrete models of graph configuration spaces. By a discrete model we understand a CW-complex, which is a deformation retract of $C_n(\Gamma)$. The existence of discrete models for graph configuration spaces enables us to use standard tools from algebraic topology to compute homology groups of graph configuration spaces. In particular, we use different kinds of homological exact sequences. There are two discrete models that we use.

1. Abram's discrete configuration space [28]. The Abram's deformation retract of $C_n(\Gamma)$ is denoted by $D_n(\Gamma)$. We use Abram's discrete model mainly in the first part of this thesis, where we apply discrete Morse theory to the computation of homology groups of some small canonical graphs (section 5.2) and compute homology groups of configuration spaces of certain graphs of connectivity 1 (section 5.3).

2. The discrete model by Świątkowski [14], that we denote by $S_n(\Gamma)$. We use this model in sections 5.4-5.6 to compute homology groups of configuration spaces of wheel graphs and some families of complete bipartite graphs.

Świątkowski model has an advantage over Abram's model in the sense that its dimension agrees with the homological dimension of $C_n(\Gamma)$, and as such, stabilises for sufficiently large n. The dimension of Abram's model is equal to n for sufficiently large n. Hence, the Świątkowski model is more suitable for rigorous calculations. However, sometimes it is more convenient to use Abram's model with the help of discrete Morse theory. The computational complexity of numerically calculating the homology groups of $C_n(\Gamma)$ for a generic graph is comparable in both approaches.

4.1. Elements of graph theory

In this section we define the notions from graph theory, that we use in the following part of the thesis. An *undirected graph* Γ is a 1-dimensional CW-complex. In other words, this is a collection of vertices, that are connected by edges. The set of edges is denoted by $E(\Gamma)$, and the set of vertices is denoted by $V(\Gamma)$. So far, we have not made any assumptions on the structure of connections, i.e. we allow multiple edges and self-loops to occur in the graph (see figure 4.1a). We distinguish two types of edges according to the way they are connected to the vertices of Γ. Edges, whose endpoints are connected with distinct vertices are called *regular*, while edges, whose endpoints are connected to the same vertex are called the *self-loops*.

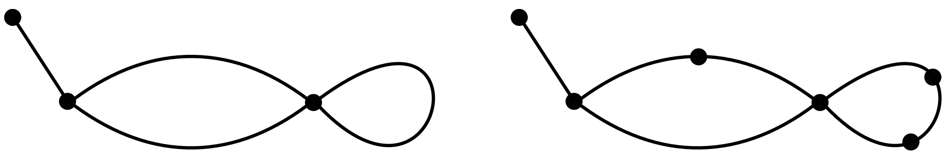


Figure 4.1: a) Example of a graph with a double edge and a self-loop. b) Topologically isomorphic graph, which has been subdivided, so that it is now a simple graph.

Definition 4.1. *The degree of vertex $v \in V(\Gamma)$ denoted by $d(v)$ is equal to the number of regular edges adjacent to v plus twice the number of self-loops at v.*

Vertices, whose degree is greater than or equal to 3, are called *essential*. Often we need to assume certain regularity properties by requiring the graphs to be *simple*.

Definition 4.2. *A simple graph is a graph, where every pair of vertices is connected by at most one edge.*

This is often a technical condition, as any non-simple graph can be made simple by suitably subdividing its edges, i.e. adding a number of vertices of degree 2 (see figure 4.1b). Clearly, a subdivision of edges does not affect the topological properties of $C_n(\Gamma)$. If we want to emphasise the fact, that a graph is not subdivided, we call it a *smooth graph*.

Definition 4.3. Γ *is a smooth graph iff all its vertices have degree different than* 2.

Any graph can be made smooth by removing the vertices of degree 2 and gluing the endpoints of edges, that were incident to those vertices.

By assigning arbitrary orientations to the edges of Γ, we make the considered graph a *directed graph*. Every edge of the directed graph has its initial vertex and terminal vertex, denoted by $\iota(e)$ and $\tau(e)$ respectively. After assigning $\pm$ orientations to the vertices of Γ, the graph becomes a chain complex, that consists of 0-cells and 1-cells. The Betti numbers read

$$\beta_0(\Gamma) = N,\ \beta_1(\Gamma) = E(\Gamma) - V(\Gamma) + N,$$

where N is the number of connected components of Γ. The first Betti number gives the number of linearly independent cycles in directed Γ. A cycle is an embedding of S^1 in Γ. Using the language of CW complexes, we define cycles via the notion of *paths*.

Definition 4.4. *A path $P \subset \Gamma$ is a subset $P = e_1 \cup e_2 \cup \cdots \cup e_l$, where $e_i \in E(\Gamma)$ for $i = 1, \ldots, l$ and $e_i \cap e_{i+1} \subset V(\Gamma)$ for $i = 1, \ldots, l-1$. Number l is the length of the path.*

A path is simple iff it is a simple graph as a subgraph of Γ, and closed iff $e_l \cap e_1 \neq \emptyset$. Two paths are called independent iff their sets of interior vertices are disjoint.

Definition 4.5. *A cycle $O \subset \Gamma$ is a closed, simple path.*

Remark 4.1. *To every path $P = e_1 \cup e_2 \cup \dots e_l$ we can assign a chain*

$$c_P = \sum_{e \in P} \pm e.$$

Chain c_P can be made unique up to an overall sign by choosing the signs, so that the common vertices $e_i \cap e_{i+1}$ for $i = 1, \dots, l-1$ have coefficients 0 in expression $\pm e_i \pm e_{i+1}$.

A tree graph is a graph, which has no cycles. For every simple graph Γ one defines a *spanning tree* as a connected tree graph $T \subset \Gamma$, which contains all vertices of Γ, $V(T) = V(\Gamma)$. Sometimes, given a tree graph T, we fix the *root* of T, which is a vertex of T of degree 1.

As a final paragraph of this section, we introduce different notions of graph connectivity. We say, that a given graph is connected, if any two vertices of Γ can be connected by a path. Let us next define further connectivity properties. To this end, we need the notion of vertex deletion.

Definition 4.6. *A vertex deletion of $v \in V(\Gamma)$ is the operation of removing v from Γ together with edges incident to v. The resulting graph is denoted by $\Gamma_{/v}$. Its edges and vertices are*

$$V(\Gamma_{/v}) = V(\Gamma) - \{v\}, \quad E(\Gamma_{/v}) = \{e \in E(\Gamma) : \ e \cap v = \emptyset\}.$$

If one deletes a set of vertices $W \subset V(\Gamma)$, the resulting graph is denoted by $\Gamma_{/W}$.

Definition 4.7. *Graph Γ is k-connected, if Γ contains at least $k+1$ vertices and the minimal size of set $W \subset V(\Gamma)$ such, that $\Gamma_{/W}$ is disconnected is larger than $k-1$.*

Menger's theorem [71] gives an equivalent characterisation of k-connectivity.

Theorem 4.1. *Let u and v be two nonadjacent vertices of Γ. The minimal number of consecutive vertex deletions disconnecting u and v is equal to the number of pairwise independent paths from u to v.*

We denote the number of independent paths between u and v by $\kappa(u, v)$. By Menger's theorem, one can compute the largest k such, that Γ is k-connected as

$$\kappa = \min_{u,v \in V(\Gamma)} k(u, v).$$

Definition 4.8. *We say, that Γ has connectivity $\kappa \in \mathbb{N}$ iff κ is the largest k such, that Γ is k-connected.*

Note, that the property of being k-connected is not well-defined for complete graphs, i.e. graphs, whose every pair of vertices is connected by a single edge. This is because there is no set of vertices, whose deletion disconnects the graph. For complete graphs on n vertices (denoted by K_n) $k(u, v)$ is equal to $n-1$, regardless of the choice of u and v. Therefore, K_n has connectivity $(n-1)$. If Γ is not a complete graph, then the above definitions work well.

4.2. Abrams discrete model

Let us next describe in detail the discrete configuration spaces $D_n(\Gamma)$ by Abrams. For the deformation retraction from $C_n(\Gamma)$ to $D_n(\Gamma)$ to be valid, the graph must be simple and *sufficiently subdivided*, which means that

- each path between distinct vertices of degree not equal to 2 passes through at least $n-1$ edges,
- each nontrivial loop passes through at least $n+1$ edges.

The discrete configuration space $D_n(\Gamma)$ is a cubic complex. The n-dimensional cells in $D_n(\Gamma)$ are of the following form.

$$\Sigma^n(D_n(\Gamma)) = \{\{e_1,\ldots,e_n\}:\ e_i \in E(\Gamma),\ e_i \cap e_j = \emptyset \ \text{ for }\ i\neq j\}.$$

We denote cells of $D_n(\Gamma)$ by the set notation using curly brackets. Lower dimensional cells are described by sets of edges and vertices from Γ, that are mutually disjoint. A d-dimensional cell consists of d edges and $n-d$ vertices. In other words, cells from $\Sigma^d(D_n(\Gamma))$ are of the form

$$\Sigma^d(D_n(\Gamma)) = \{\sigma \subset E(\Gamma)\cup V(\Gamma):\ |\sigma| = n,\ |\sigma\cap E(\Gamma)| = d,\ \epsilon\cap\epsilon' = \emptyset\ \forall_{\epsilon,\epsilon'\in\sigma}\}.$$

In particular, when there are not enough pairwise disjoint edges in the sufficiently subdivided Γ, the dimension of the discrete configuration space can be smaller than n.

The corresponding chain complex is formed by the basis composed of cells of $D_n(\Gamma)$, i.e.

$$\mathfrak{C}(D_n(\Gamma), R) = \bigoplus_{\sigma\in\Sigma(D_n(\Gamma))} R, \quad R = \mathbb{Z} \text{ or } R = \mathbb{Q}.$$

In order to define the boundary map, we introduce a suitable order on vertices of Γ, following [24, 22]. To this end, we choose a spanning tree $T \subset \Gamma$ and fix its planar embedding. We also fix the root $*$ of T by picking a vertex of degree 1 in T. For every $v \in V(\Gamma)$ there is the unique path in T that joins v and $*$, called the geodesic $g_{v,*}$. For every vertex with $d(v) \geq 2$ we enumerate the edges adjacent to v with numbers $0, 1, \ldots, d(v)-1$. The edge contained in $g_{v,*}$ has label 0. The remaining edges are labelled increasingly, according to their clockwise order starting from edge 0. The enumeration procedure for vertices goes in an inductive manner. The root has number 1. If vertex v has label k and $d(v) = 2$, the vertex adjacent to v is given label $k+1$. Otherwise, if $d(v) \geq 2$, the vertex adjacent to v in the lowest direction with vertices that have not been yet labelled is given label $k_{max}+1$, where k_{max} is the maximal label among all of the already labelled vertices. If $d(v) = 1$, we look for essential vertices in $g_{v,*}$ and go back to the closest essential vertex that contains a direction with unlabelled vertices. In other words, the vertices are labelled in the clockwise direction. This way every edge is given an initial and terminal vertex that we denote by $\iota(e)$ and $\tau(e)$ respectively. The terminal vertex is the vertex with the lower index, i.e. $\tau(e) < \iota(e)$. We can unambiguously specify an edge by calling its initial and terminal vertices, hence we denote the edges by e^{ι}_{τ}. Given a cell from $D_n(\Gamma)$

$$\sigma = \{e_1,\ldots,e_d,v_1,\ldots,v_{n-d}\},$$

we order the edges from σ according to their terminal vertices, i.e. $\tau(e_1) < \tau(e_2) < \cdots < \tau(e_d)$. The ith pair of faces from the boundary of σ reads

$$(\partial^{\iota}\sigma)_i := \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d, v_1, \ldots, v_{n-d}, \iota(e_i)\},$$
$$(\partial^{\tau}\sigma)_i := \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d, v_1, \ldots, v_{n-d}, \tau(e_i)\}.$$

The full boundary of σ is given by the following alternating sum of faces.

$$\partial\sigma = \sum_{i=1}^{k} (-1)^i \left((\partial^{\iota}\sigma)_i - (\partial^{\tau}\sigma)_i \right). \tag{4.1}$$

For examples, see section 4.4 and chapter 5.

4.3. Świątkowski discrete model

Świątkowski complex is denoted by $S_n(\Gamma)$. In order to define it, we regard graph Γ as a set of edges E, vertices V and half-edges H. A half-edge of $e \in E(\Gamma)$ assigned to vertex v, $h(v) \subset e$, is the part e, which is an open neighbourhood of vertex v. Intuitively, the half-edges are places, where the particles are allowed to 'slide'. By $e(h)$ we will denote the unique edge, for which $e \cap h \neq \emptyset$. Similarly, we have vertex $v(h)$ as the vertex, for which h is a neighbourhood. By $H(v)$ we will denote all half edges that are incident to vertex v. Chain complex $S(\Gamma) = \bigoplus_n S_n(\Gamma)$ reads

$$S(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S_v,$$

where $S_v = \mathbb{Z}\langle v, h \in H(v), \emptyset \rangle$. This is a bigraded module with respect to the multiplication by $E(\Gamma)$ (a bigraded $\mathbb{Z}[E]$ module). The degrees of the components are

$$|v| = (0,1),\ |e| = (0,1),\ |h| = (1,1).$$

The boundary map reads

$$\partial v = \partial e = 0,\ \partial h = e(h) - v(h).$$

The boundary map for elements of a higher degree is determined by the Eilenberg-Zilber theorem:

$$\partial(\chi \otimes \eta) = (\partial\chi) \otimes \eta + (-1)^d \chi \partial\eta$$

for d-chain χ. There is a canonical basis for $S(\Gamma)$, whose elements of degree (d,n) are of the form

$$h_1 \ldots h_d v_1 \ldots v_k e_1^{n_1} \ldots e_l^{n_l},\ \{v_1 \ldots v_k\} \cap \{h_1, \ldots h_d\} = \emptyset, \qquad d + k + n_1 + \cdots + n_l = n. \tag{4.2}$$

The basis elements form a cube complex. In calculations we use the notion of support of a given cell or a chain.

Definition 4.9. *The support of d-cell $c = h_1 \ldots h_d v_1 \ldots v_k e_1^{n_1} \ldots e_l^{n_l} \in S_n(\Gamma)$ is the set of the corresponding edges and vertices of Γ*

$$\mathrm{Supp}(c) := \left(\bigcup_{i=1}^{d} \{e(h_i), v(h_i)\} \right) \cup \{v_1, \ldots, v_k\} \cup \{e_1, \ldots, e_l\} \subset E(\Gamma) \cup V(\Gamma).$$

The support of a chain $b = \sum_i p_i c_i$, $p_i \in \mathbb{Z}$ is given by

$$\mathrm{Supp}(b) := \bigcup_{i=1}^{d} \mathrm{Supp}(c_i).$$

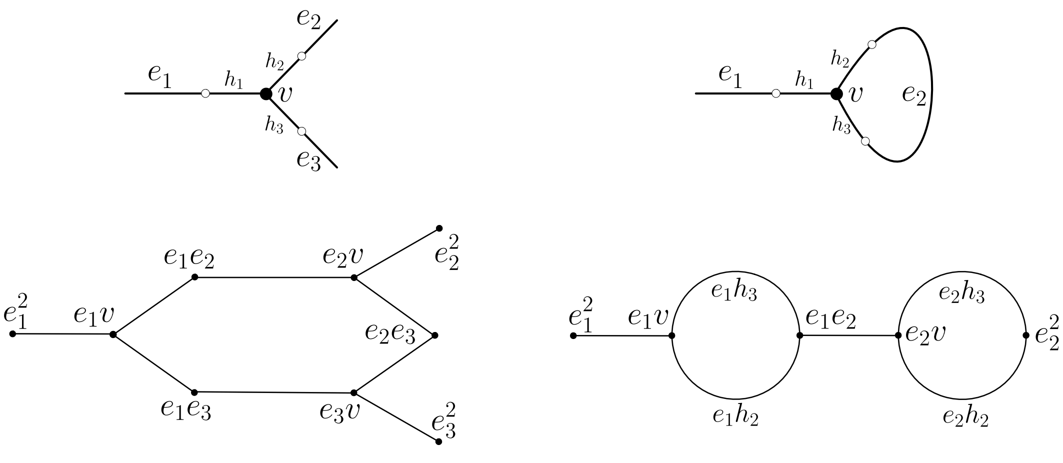
In this thesis we will also use a variation of $S(\Gamma)$, which we will call the reduced Świątkowski complex with respect to a subset of vertices $U \subset V(\Gamma)$ and denote by $\tilde{S}^U(\Gamma)$. In most cases, the reduced complexes lack a canonical basis, however they have a smaller number of generators than $S(\Gamma)$. The reduction is done by changing the generators at vertex v to differences of half edges $h_{ij} := h_i - h_j,\ h_i, h_j \in H(v)$, $\tilde{S}_v := \mathbb{Z}\langle \emptyset, h_{ij} \rangle$.

$$\tilde{S}^U(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V \setminus U} S_v \otimes \bigotimes_{v \in U} \tilde{S}_v.$$

Intuitively, this means, that effectively, the particles always slide from one half-edge to another without staying at the central vertex. Both reduced and the non-reduced Świątkowski complexes have the same homology groups [61]. From now on, the default complex we will work with is the complex, which is reduced with respect to all vertices of degree one. Intuitively, this means that we do not consider redundant cells, where particles move from an edge to some vertex of valency one. Such complexes have the canonical basis, which corresponds to cells of a cube complex of the form (4.2). By a slight abusion of notation, we will denote such a default reduced complex by $S(\Gamma)$. In other words, from now on

$$S(\Gamma) := \mathbb{Z}[E] \otimes \bigotimes_{v \in V:\ d(v) > 1} S_v.$$

For examples, see figure 4.2. As a direct consequence of the dimension of $S_n(\Gamma)$, we



a) b)

Figure 4.2: Świątkowski complex of the Y-graph and of the lasso graph, where vertices of degree 1 have been reduced. a) Świątkowski complex of $C_2(Y)$. Only vertices of $S_2(\Gamma)$ are captioned. The Y-cycle reads $e_1(h_2 - h_3) + e_2(h_3 - h_1) + e_3(h_1 - h_2)$. b) Świątkowski complex of $C_2(\Gamma)$ for the lasso graph. Vertices and some chosen edges of $S_2(\Gamma)$ are captioned. The O-cycles are $e_1(h_2 - h_3)$ and $e_2(h_2 - h_3)$. The Y-cycle is their sum, hence can be written as $(e_1 - e_2)(h_2 - h_3)$.

get the following fact.

Fact 4.1. *Let Γ be a graph. Then, the following homology groups of $C_n(\Gamma)$ vanish.*

$$H_d(C_n(\Gamma)) = 0 \text{ if } d < n \text{ or } d > N_\Gamma,$$

where $N_\Gamma = |\{v \in V(\Gamma) : \ d(v) \geq 3\}|$.

Vertex blowup

In the following, we will explore relations on homology groups that stem from blowing up a vertex of Γ: $\Gamma \to \Gamma_v$ (Fig. 4.3).

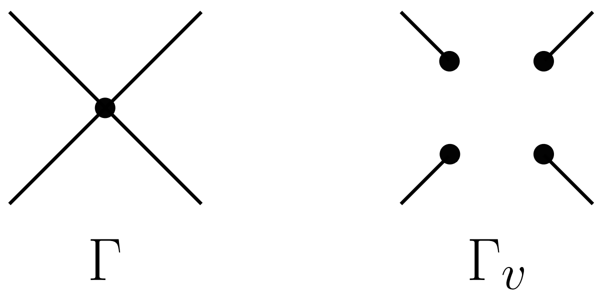


Figure 4.3: Vertex blow up at vertex v in Γ.

We borrow this nomenclature and the methodology of this subsection from [61]. We start with the reduced complex with respect to vertex v, $\tilde{S}^v(\Gamma)$. Any chain $b \in \tilde{S}^v(\Gamma)$ can be decomposed in a unique way by extracting the part that involves generators from $\tilde{S}_v$. In order to do it, we fix a half-edge $h_0 \in H(v)$ and write b as

$$b = b_0 + \sum_{h \in H(v) \setminus h_0} (h_0 - h) b_h.$$

Note that chains b_0 and b_h belong to $S(\Gamma_v)$. We associate two chain maps to the above decomposition. The first map ϕ is the embedding of any chain b_0 from $S(\Gamma_v)$ to $\tilde{S}^v(\Gamma)$. Clearly, this map is injective and commutes with the boundary operator.

$$\phi_n : \ S_n(\Gamma_v) \to \tilde{S}^v_n(\Gamma), \phi(b_0) = b_0 \in \tilde{S}^v(\Gamma),$$

The other map ψ is the projection of $b \in \tilde{S}^v(\Gamma)$ to its h-components. It assigns a number of $n-1$-particle $d-1$-chains to a n-particle d-chain in the following way

$$\psi_n : \tilde{S}^v_n(\Gamma) \to \bigoplus_{h \in H(v) \setminus h_0} S_{n-1}(\Gamma_v), \ \psi(b) = \bigoplus_{h \in H(v) \setminus h_0} b_h.$$

Map ψ is surjective, because any chain $b' \in S_{n-1}(\Gamma_v)$ can be obtained by ψ for exmaple from chain $(h_0 - h)b' \in \tilde{S}^v_n(\Gamma)$. In order to see that ψ is a chain map, consider a cycle $c \in \tilde{S}^v_n(\Gamma)$. We have

$$0 = \partial c = \partial c_0 + \sum_{h \in H(v) \setminus h_0} \left((e(h_0) - e(h)) c_h - (h_0 - h) \partial c_h \right).$$

Grouping the summands that entirely belong to $S_{n-1}(\Gamma_v)$, we get

$$\partial c_0 + \sum_{h\in H(v)\backslash h_0} (e(h_0) - e(h))c_h = 0,$$
$$\sum_{h\in H(v)\backslash h_0} (h_0 - h)\partial c_h = 0.$$

By the same argument, the second equation implies that $\partial c_h = 0$ for all $h \in H(v)\backslash h_0$. We can write down the two maps as a short exact sequence

$$0 \to S_n(\Gamma_v) \xrightarrow{\phi_n} \tilde{S}^v_n(\Gamma) \xrightarrow{\psi_n} \bigoplus_{h\in H(v)\backslash h_0} S_{n-1}(\Gamma_v) \to 0. \tag{4.3}$$

Short exact sequence (4.3) of chain maps implies the long exact sequence of homology groups

$$\ldots \xrightarrow{\Psi_{n,d+1}} \bigoplus_{h\in H(v)\backslash h_0} H_d\left(S_{n-1}(\Gamma_v)\right) \xrightarrow{\delta_{n,d}} H_d\left(S_n(\Gamma_v)\right) \xrightarrow{\Phi_{n,d}} H_d\left(\tilde{S}^v_n(\Gamma)\right) \xrightarrow{\Psi_{n,d}} \tag{4.4}$$
$$\xrightarrow{\Psi_{n,d}} \bigoplus_{h\in H(v)\backslash h_0} H_{d-1}\left(S_{n-1}(\Gamma_v)\right) \xrightarrow{\delta_{n,d-1}} H_{d-1}\left(S_n(\Gamma_v)\right) \xrightarrow{\Phi_{n,d-1}} \ldots,$$

where the connecting homomorphism reads

$$\delta[b_h] = [\partial\left((h_0 - h)b_h\right)] = e(h_0)[b_h] - e(h)[b_h].$$

Long exact sequence (4.4) implies a collection of short exact sequences

$$0 \to \operatorname{coker}\left(\delta_{n,d}\right) \to H_d\left(\tilde{S}^v_n(\Gamma)\right) \to \ker\left(\delta_{n,d-1}\right) \to 0.$$

Intuitively, the $\operatorname{coker}(\delta_{n,d})$ identifies different distributions of free particles in $S_n(\Gamma_v)$ on the two sides of the junction $h_0 - h$ and $\ker(\delta_{n,d-1})$ is responsible for creating new cycles at vertex v (for example, the c_Y cycles).

4.4. Review of the structure of the first homology group

With a discrete model at hand, the computation of homology groups for a given graph boils down to studying the kernel and the image of the boundary map. Denote by ∂_d the restriction of the boundary map to d-chains

$$\partial_d : \ \mathfrak{C}_d \to \mathfrak{C}_{d-1}, \ \partial_d := \partial|_{\mathfrak{C}_d}.$$

There are some particular types of cycles that play an important role in this work. These are O-cycles and Y-cycles. We specify them for the Abram's model. The construction for $S_n(\Gamma)$ is fully analogous.

Definition 4.10. *Let $O \subset \Gamma$ be a simple cycle (an embedding of S^1 in Γ). Choose sign coefficients $s_e \in \{-1, 1\}$, $e \in O$ such that $\partial \sum_{e\in O} s_e e = 0$ in $D_1(\Gamma)$. An O-cycle in $D_n(\Gamma)$ is a 1-chain of the form*

$$c_O := \sum_{e\in O} s_e\{e, v_1, \ldots, v_{n-1}\},$$

where $\{v_1,\ldots,v_{n-1}\}\cap O=\emptyset$ *is some choice of vertices. In order to define an* O*-cycle in* $S_n(\Gamma)$*, note that for all* $v\in V(\Gamma)\cap O$*, set* $H(v)\cap O$ *contains exactly two half-edges. We denote these half-edges by* h_v, h'_v*, where the labels are such, that* $\partial\sum_{v\in V(\Gamma)\cap O}(h'_v-h_v)=0$*. Then,*

$$c_O=\left(\sum_{v\in V(\Gamma)\cap O}(h'_v-h_v)\right)\otimes\left(\bigotimes_{w\in W}w\right)\otimes\left(\bigotimes_{e\in E(\Gamma)}e^{n_e}\right),$$
$$W\subset(V(\Gamma)-V(\Gamma)\cap O),\ \#W+\sum_{e\in E(\Gamma)}n_e=n-1.$$

Definition 4.11. *Let* $Y\subset\Gamma$ *be a* Y*-subgraph of* Γ *spanned on vertices* u_0,u_h,u_1,u_2 *such that* $u_0,\ u_1,\ u_2$ *are adjacent to* u_h *and* $u_0<u_h<u_1<u_2$*. The* Y*-cycle in* $D_2(\Gamma)$ *associated to subgraph* Y *is of the following form*

$$c_Y:=\{e^{u_1}_{u_h},u_0\}+\{e^{u_h}_{u_0},u_1\}+\{e^{u_2}_{u_h},u_1\}-\{e^{u_1}_{u_h},u_2\}-\{e^{u_h}_{u_0},u_2\}-\{e^{u_2}_{u_h},u_0\}.$$

A Y*-cycle in* $D_n(\Gamma)$ *is formed by distributing the free particles outside of subgraph* Y*, i.e.*

$$c^{(n)}_Y:=\sum_{\sigma\in c_Y}s_\sigma\left(\sigma\cup\{v_1,\ldots,v_{n-2}\}\right),$$

where $\{v_1,\ldots,v_{n-2}\}\cap Y=\emptyset$ *and* s_σ *is the sign of cell* σ *in cycle* c_Y*. In order to define the* Y*-cycle in* $S_n(\Gamma)$*, denote the half edges of subgraph* Y *as* $\{h_i\}_{i=0}^2$*, where* $h_i\in H(u_h)$ *are such, that* $e(h_0)=e^{u_h}_{u_0}$*,* $e(h_1)=e^{u_1}_{u_h}$*,* $e(h_2)=e^{u_2}_{u_h}$*. Then,*

$$c_Y=e^{u_h}_{u_0}(h_2-h_3)+e^{u_1}_{u_h}(h_3-h_1)+e^{u_2}_{u_h}(h_1-h_2).$$

Cycle $c^{(n)}_Y\in S_n(\Gamma)$ *is formed by multiplying* c_Y *by a suitable polynomial in* $V(\Gamma)$ *and* $E(\Gamma)$*.*

$$c^{(n)}_Y=c_Y\otimes\left(\bigotimes_{w\in W}w\right)\otimes\left(\bigotimes_{e\in E(\Gamma)}e^{n_e}\right),\ W\subset(V(\Gamma)-\{u_h\}),\ \#W+\sum_{e\in E(\Gamma)}n_e=n-2.$$

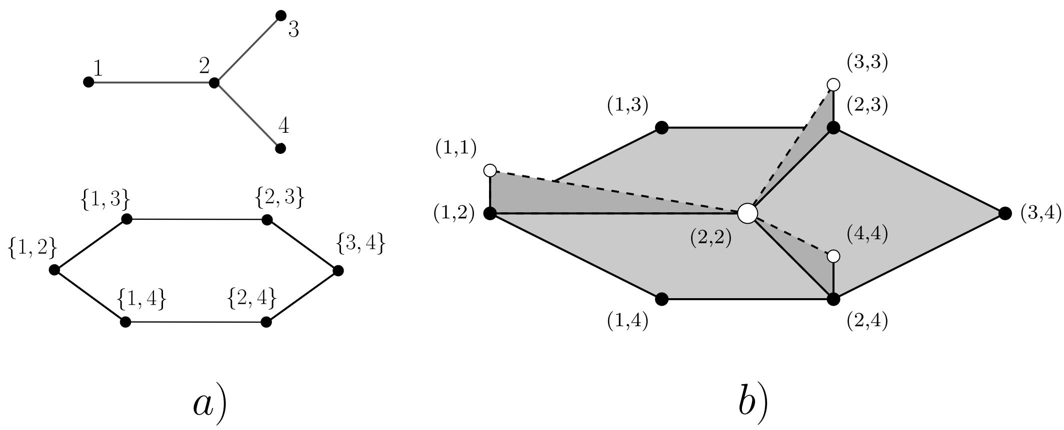


Figure 4.4: A Y-graph, its configuration space (b) and its discrete configuration space $D_2(\Gamma)$ (a).

It has been shown in [20] that subject to certain relations, cycles c_O and $c_Y^{(n)}$ generate $H_1(D_n(\Gamma))$ (see also [61] for the proof of an analogous fact for $H_1(S_n(\Gamma))$). The fundamental relation between Y-cycles is shown on Fig. 4.5 and Fig. 4.6.

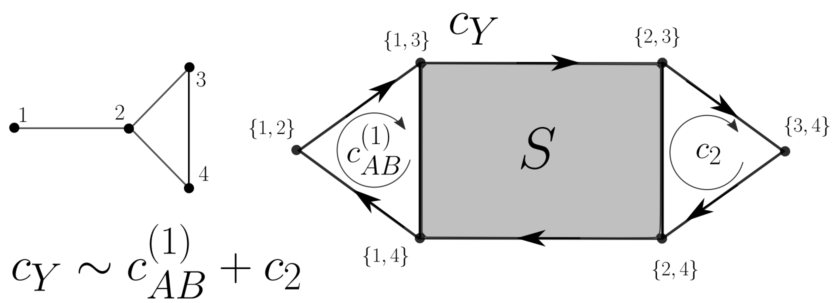


Figure 4.5: The fundamental relation between the two-particle cycle on a Y-graph and the AB-cycle and a two-particle cycle c_2 in the lasso graph.

Cycle $c_{AB}^{(1)}$ is the cycle, where one particle goes around the cycle in the lasso graph and the other particle occupies vertex 1.

$$c_{AB}^{(1)} = c_O \times \{1\} = \{e_2^3, 1\} + \{e_3^4, 1\} - \{e_2^4, 1\}.$$

Cycle c_2 is the cycle, where two particles go around the cycle in lasso.

$$c_2 = \{e_2^4, 3\} - \{e_2^3, 4\} - \{e_3^4, 2\}.$$

It is straightforward to check that

$$c_{AB}^{(1)} + c_2 - c_Y = \partial S, \tag{4.5}$$

where $S = \{e_1^2, e_3^4\}$. Consider next a situation, where two disjoint Y-graphs share one cycle c_O and their free ends are connected by a path p_{v_1,v_2}, which is disjoint with c_O (Fig. 4.6). In other words, consider an embedding of a graph, which is isomorphic to the Θ- graph[1].

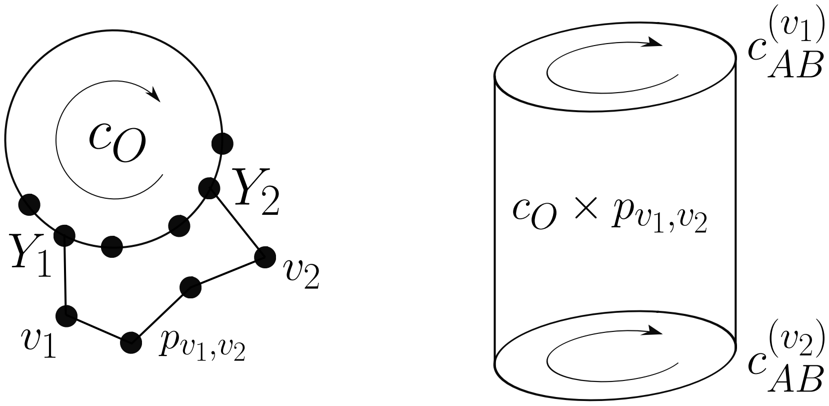


Figure 4.6: Cycles c_{Y_1} and c_{Y_2} are homologically equivalent.

Then,

$$c_{AB}^{(v_1)} + c_2 - c_{Y_1} = \partial S_1,$$
$$c_{AB}^{(v_2)} + c_2 - c_{Y_2} = \partial S_2.$$

[1]The Θ graph consists of two vertices, which are connected by three edges. It can be also viewed as complete bipartite graph $K_{2,3}$.

Subtracting both equations, we get

$$c_{Y_1} - c_{Y_2} = \partial(S_2 - S_1) + c_{AB}^{(v_1)} - c_{AB}^{(v_2)}. \tag{4.6}$$

But the existence of p_{v_1,v_2} gives us that $c_{AB}^{(v_1)} - c_{AB}^{(v_2)} = \partial\left(c_O \times p_{v_1,v_2}\right)$. This in turn means that c_{Y_1} and c_{Y_2} are homologically equivalent. Relation

$$c_{Y_1} - c_{Y_2} = \partial\left(S_2 - S_1 + c_O \times p_{v_1,v_2}\right) \tag{4.7}$$

will be called a Θ-relation. It turns out that considering all Θ-relations stemming from different Θ-subgraphs and relations (4.6) that express different distributions of particles in the O-cycles as differences of Y-cycles, one can compute the first homology group of $D_n(\Gamma)$. Let us next summarise the results concerning the structure of the first homology group of graph configuration spaces. We formulate the results assuming, that the considered graphs are simple. The general form of the first homology group reads

$$H_1(D_n(\Gamma), \mathbb{Z}) = (\mathbb{Z})^N \oplus (\mathbb{Z}_2)^L,$$

where N and L are the numbers of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ respectively. Numbers N and L depend on the connectivity and some combinatorial properties of the given graph. The simplest case is, when Γ is 3-connected. Then, if Γ is planar, we have $L = 0$ and $N = \beta_1(\Gamma) + 1$. Otherwise, if Γ is non-planar, we have $L = 1$ and $N = \beta_1(\Gamma)$. This is because for 3 connected graphs, all c_Y-cycles are equal up to a sign in $H_1(D_n(\Gamma))$. For planar graphs they generate a $\mathbb{Z}$-component and for non-planar graphs they generate the $\mathbb{Z}_2$-component in $H_1(D_n(\Gamma))$. The remaining $\mathbb{Z}^{\beta_1(\Gamma)}$-component is generated by the AB-cycles. The generation of torsion can be proved using Kuratowski theorem [62], which asserts, that every non-planar graph contains a graph isomorphic to K_5 or $K_{3,3}$ (called Kuratowski graphs) as a subgraph. The 2-particle configuration space of a Kuratowski graph is a non-orientable closed surface of dimension 2 [28]. Such surfaces embedded in $D_n(\Gamma)$ of a non-planar graph give relations, that are responsible for torsion in $H_1(D_n(\Gamma))$.

If the given graph is 2-connected, the first homology group is stable with respect to the number of particles in the sense, that $H_1(D_n(\Gamma)) = H_2(D_n(\Gamma))$ for $n \geq 2$. For a general connected graph the rank of $H_1(D_n(\Gamma))$ grows polynomially with n for $n \geq 2$. In order to fully describe $H_1(D_n(\Gamma))$ for any connected graph, we need to decompose Γ by performing a sequence of vertex cuts, that results with a decomposition of Γ into a number of 3-connected components [20, 22, 73]. Let us next briefly describe the decomposition procedure.

Definition 4.12. *A cut of graph Γ is a set of vertices $W \subset V(\Gamma)$, whose removal from Γ results with a decomposition of Γ into disconnected components. In other words, $\Gamma - W$ is disconnected as a topological space.*

A cut is closely related to the vertex deletion from definition 4.6. Namely, the number of connected components of $\Gamma_{/W}$ and $\Gamma - W$ is the same. Therefore, if Γ has connectivity k, it has a k-vertex cut. We denote the components by G_i, so that

$$\Gamma - W = \bigsqcup_{i=1}^{\mu} G_i.$$

By giving the connected components of $\Gamma - W$ the topology of a subspace of Γ, we consider their closures in Γ. We refer to them as the *closed components* of $\Gamma - W$.

Each closed component is a subgraph of Γ and contains a copy of vertices from W. By denoting the closed components by Γ_i, we have

$$V(\Gamma_i) = W \cup (V(\Gamma) \cap G_i), \quad E(\Gamma_i) = \{e \in E(\Gamma) : \; e \cap G_i \neq \emptyset\}.$$

The relevant types of cuts are the one- and two-vertex cuts (see figure 4.7). If the

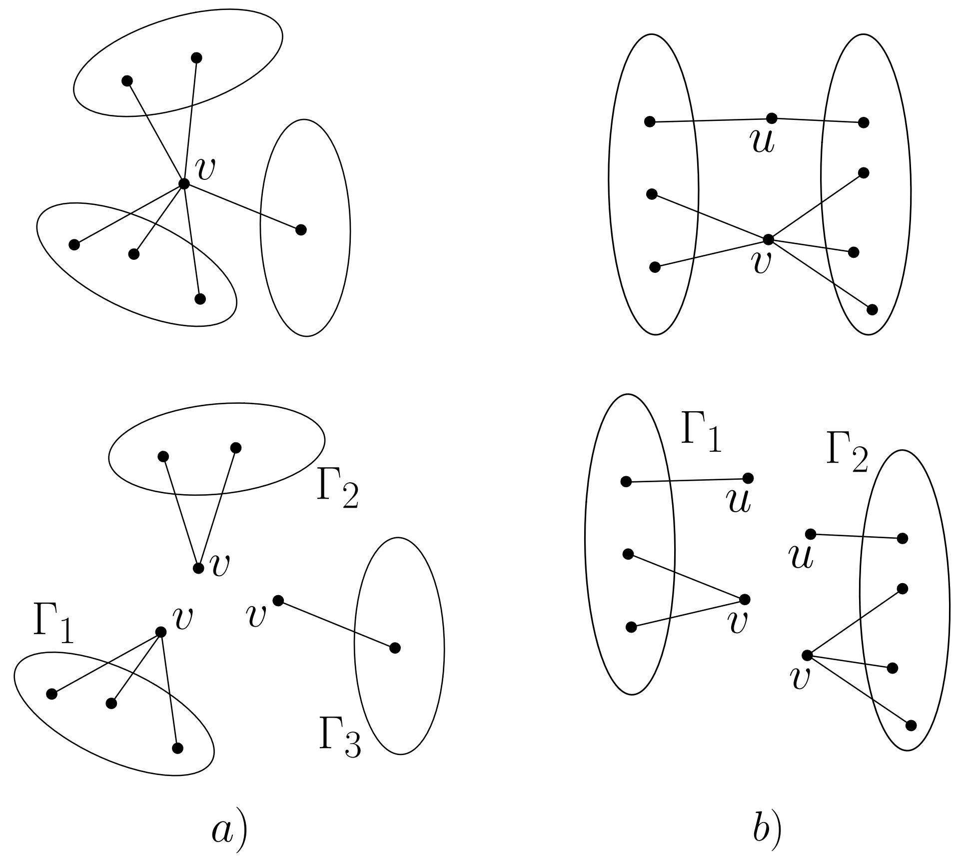


Figure 4.7: Examples of one- and two-vertex cuts. a) A one-vertex cut of a graph of connectivity one with three components. b) A two-vertex cut of a graph of connectivity two with two components.

given graph has connectivity one, we perform a series of one-vertex cuts, until the components are either graphs of connectivity 2, or are isomorphic to line segments (single edges). A one-vertex cut affects the first homology group as follows.

Lemma 4.2. *Let $v \in V(\Gamma)$ be a one-vertex cut, which decomposes Γ into μ components. Then,*

$$H_1(D_n(\Gamma)) \cong \left(\bigoplus_{i=1}^{\mu} H_1(D_n(\Gamma_i)) \right) \oplus \mathbb{Z}^{N_1(n,\mu,v)},$$

where

$$N_1(n, \mu, v) = (d(v) - 2)\binom{n + d(v) - 2}{n - 1} - \binom{n + d(v) - 2}{n} - (d(v) - \mu - 1).$$

Denote by $W_1 \subset V(\Gamma)$ the set of one-vertex cuts. We collect the overall change of $H_1(D_n(\Gamma))$ to $N_1(n)$.

$$N_1(n, \Gamma) := \sum_{v \in W_1} N_1(n, \mu(v), v).$$

Then, for the components of connectivity 2, we perform further two-vertex cuts (Fig. 4.7b) and form the *marked components*. The marked components are formed from the standard components as follows. Let u, v be a two-vertex cut of Γ. Then, the marked components are of the form $\hat{\Gamma}_i := \Gamma_i \cup e_u^v$, where e_u^v is an additional edge that joins u and v in Γ_i. Group $H_1(D_n(\Gamma))$ is expressed by the first homology of the marked components as follows.

Lemma 4.3. *Let u, v be a two-vertex cut of graph Γ, that decomposes Γ into μ marked components denoted by $\{\hat{\Gamma}_i\}_{i=1}^{\mu}$. Then,*

$$H_1(D_n(\Gamma)) \oplus \mathbb{Z} \cong \left(\bigoplus_{i=1}^{\mu} H_1(D_n(\hat{\Gamma}_i)) \right) \oplus \mathbb{Z}^{(\mu-1)(\mu-2)/2}.$$

By iterating the two-vertex cuts, we obtain components, that are either cycles or 3-connected graphs [73]. Denote by W_2 the set of pairs of vertices of Γ, at which the two-vertex cuts are done. We collect the overall impact on $H_1(D_n(\Gamma))$ in $N_2(n)$.

$$N_2(\Gamma) := \sum_{\{u,v\} \in W_2} \frac{1}{2}(\mu(\{u,v\}) - 1)(\mu(\{u,v\}) - 2).$$

Moreover, denote by $N_3(\Gamma)$ the number of resulting 3-connected planar components and by $N_3'(\Gamma)$ the number of 3-connected non-planar components. We summarise the above review by a theorem.

Theorem 4.4. *Let Γ be a simple graph. Then, for $n \geq 2$*

$$H_1(D_n(\Gamma)) = \mathbb{Z}^{N_1(n)+N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}.$$

Chapter 5

Calculation of homology groups of graph configuration spaces

This chapter contains the techniques that we use for computing homology groups of graph configuration spaces. We tackle this problem from the 'numerical' and the 'analytical' perspective. The numerical approach means using a computer code for creating the boundary matrices and then employing the standard numerical libraries for computing the kernel and the elementary divisors of given matrices. The procedures for calculating the boundary matrices of $D_n(\Gamma)$, $S_n(\Gamma)$ and the Morse complex (see section 5.2) were written by the author of this thesis, based on papers [24, 22]. The analytical approach means computing the homology groups for certain families of graphs by suitably decomposing a given graph into simpler components and using various homological exact sequences. Recently in the mathematical community, there has been a growing interest in computing the homology groups of graph configuration spaces. A significant part of the recent work has been devoted to explaining certain regularity properties of the homology groups of $C_n(\Gamma)$ [30, 31, 32, 34, 33]. In this thesis, by a direct study of some families of graphs, we give new evidence supporting the regularity of Betti numbers of graph configuration spaces. By the regularity property we understand the following conjecture, which has been stated in a similar form (without giving explicit bound on n) in [30].

Conjecture 5.1. *For $n \geq 2d$, the behaviour of $\beta_d(C_n(\Gamma))$ becomes regular, i.e. $\beta_d(C_n(\Gamma))$ grows polynomially with n.*

In view of conjecture 5.1, we will call $H_d(C_{2d}(\Gamma))$ the first regular homology group of order d.

5.1. Product cycles

Considering simultaneous exchanges of pairs of particles on disjoint Y-subgraphs of Γ and the O-type cycles with the remaining particles distributed on the free vertices of Γ, one can construct some generators of $H_*(D_n(\Gamma))$ or $H_*(S_n(\Gamma))$. Such cycles are products of 1-cycles, hence are isomorphic to tori embedded in the discrete configuration space. To construct a product d cycle in $D_n(\Gamma)$, we choose Y-subgraphs of Γ $\{Y_i\}_{i=1}^{d_Y}$ and cycles in Γ (O-subgraphs of Γ) $\{O_i\}_{i=1}^{d_O}$, where $d_Y + d_O = d$. All the chosen subgraphs must be mutually disjoint.

$$Y_i \cap Y_j = O_i \cap O_j = \emptyset \text{ for } i \neq j,\ Y_i \cap O_j = \emptyset \text{ for all } i, j.$$

Moreover, we choose vertices $\{v_1, \ldots, v_{n-2d_Y-d_O}\} \subset V(\Gamma)$, so that $v_i \cap O_j = v_i \cap Y_j = \emptyset$ for all i, j. Product cycle on $Y_1 \times \cdots \times Y_{d_Y} \times O_1 \times \cdots \times O_{d_O}$ with the free particles distributed on $\{v_1, \ldots, v_{n-2d_Y-d_O}\}$ is the following chain.

$$c_{Y_1} \otimes \cdots \otimes c_{Y_{d_Y}} \otimes c_{O_1} \otimes \cdots \otimes c_{O_{d_O}} \otimes \{v_1, \ldots, v_{n-2d_Y-d_O}\}.$$

In an analogous way, we form product cycles in $S_n(\Gamma)$.

We study such product cycles for configuration spaces of different graphs and describe relations between them. We show that in some cases such a cycles generate the entire homology groups. These cases are

- configuration spaces of tree graphs (section 5.3),
- configuration spaces of wheel graphs (section 5.4),
- all homology groups of the configuration space of graph $K_{3,3}$, except the third homology group (section 5.5),

In sections 5.5 and 5.6 we also discuss examples of cycles that are different than tori. In particular, we compute all homology groups of configuration spaces of complete bipartite graphs $K_{2,p}$, that are often pointed out in the literature as an unsolved example, where the simple use of product cycles is not sufficient to generate the homology groups. We show, that some of the generators of $H_*(S_n(K_{2,p}))$ are cycles of a new type, that have the homotopy type of triple tori.

5.2. Discrete Morse theory for Abrams model

In this section, we review a version of Forman's discrete Morse theory [23] for Abram's discrete model that was formulated in [24]. As a final part of this chapter, we formulate a pseudocode that we use for computing homology groups of graph configuration spaces with small numbers of partices for some canonical graphs. The results are listed in tables 5.1 and 5.2.

The central notion of discrete Morse theory is that of a discrete vector field. A discrete vector field W on $D_n(\Gamma)$ is a collection of maps $W_d : \Sigma^d(D_n(\Gamma)) \to \Sigma^{d+1}(D_n(\Gamma))$ that satisfies the following conditions.

1. Each W_d is injective.
2. If $W_d(\sigma) = \tau$, then σ is a face of τ.
3. The image of W_d is disjoint from the domain of W_{d+1}.

For a given discrete vector field, we distinguish three types of d-cells.

- Redundant cells, i.e. cells that belong to the domain of W_d.
- Collapsible cells, i.e. cells that lie in the image of W_{d-1}.
- Critical cells, i.e. cells that lie neither in the image of W_{d-1} nor in the domain of W_d.

We are interested in a particular class of discrete vector fields. These are called discrete gradient vector fields and they are distingushed by the property that they have no closed, non-stationary paths. A path is a sequence of d-cells $\sigma_1 \to \sigma_1 \to \cdots \to \sigma_r$ such that i) if σ_i does not belong to the domain of W_d, then $\sigma_{i+1} = \sigma_i$, ii) if σ_i belongs to the domain of W_d, then $\sigma_{i+1} \neq \sigma_i$ and $\sigma_{i+1} \in \partial W_d(\sigma_i)$. A path is non-stationary if $\sigma_2 \neq \sigma_1$ and closed if $\sigma_1 = \sigma_r$.

From a discrete gradient vector field W we construct the discrete flow $F: \mathfrak{C}(D_n(\Gamma)) \to \mathfrak{C}(D_n(\Gamma))$. Map F is linear and has the property that

$$\forall_{c\in\mathfrak{C}(D_n(\Gamma))} \exists_r \; F^r(c) = F^{r+1}(c).$$

In other words, the action of F on a given chain becomes trivial, when applied sufficiently many times. The Morse complex is the chain complex of chains invariant under F.

$$\mathcal{M}(D_n(\Gamma)) := \{c \in \mathfrak{C}(D_n(\Gamma)) : \; F(c) = c\}.$$

The chain map form $\mathfrak{C}(D_n(\Gamma))$ to $\mathcal{M}(D_n(\Gamma))$ will be denoted by F^∞. It is realised by iterating the action of F on c so many times that the chain becomes invariant under F. In terms of the discrete gradient vector field, map F is defined on $\Sigma^d(D_n(\Gamma))$ as follows. If σ is a critical cell, then $F(\sigma) = \sigma$. If σ is collapsible, then $F(\sigma) = 0$. If σ is redundant

$$F(\sigma) = \pm\partial W(\sigma) + \sigma,$$

where the sign is chosen such that the coefficient of σ in $\pm\partial W(\sigma)$ is -1.

Let us next state the precise form of gradient vector field W on $D_n(\Gamma)$ following [24]. Fix a spanning tree $T \subset \Gamma$ and assume that the vertices of Γ are ordered as described in chapter 4. The idea is to prescribe a motion of particles on Γ that attracts the particles to the root of T. More precisely, for every $v \in V(\Gamma)$ there is unique $e \in E(T)$ such that $v = \iota(e)$. We denote this edge by $e(v)$ - this is the possible direction of movement of particle that occupies vertex v. A d-cell $\sigma = \{e_1, \ldots, e_d, v_1, \ldots, v_{n-d}\}$ can be viewed as a movement of d particles along the edges $e_1, \ldots, e_d$ and $n-d$ particles occupying vertices $v_1, \ldots, v_{n-d}$. Cell σ belongs to the domain of W_d (is redundant) iff it does not belong to the image of W_{d-1} (which we are about to characterise) and there exists $i \in \{1, \ldots, n-d\}$ so that for all $j \in \{1, \ldots, d\}$ we have $e(v_i) \cap e_j = \emptyset$. We call such a vertex v_i an *unblocked vertex*. Cell $W(\sigma)$ corresponds to the movement of the unblocked vertex of lowest index among the vertices from σ, i.e. it substitutes such a vertex in σ with the edge, which is its direction of movement. In other words, if v_j is the unblocked vertex of lowest index, then

$$\begin{aligned} &W(\{e_1, \ldots, e_d, v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_{n-d}\}) = \\ &= \{e_1, \ldots, e_d, e(v_j), v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n-d}\}. \end{aligned}$$

This rule means that on the junctions in T, the particle with lowest index has the priority to move. In order to identify the critical d-cells, we have to determine the conditions, under which a cell whose all vertices are blocked, does not lie in the image of W_{d-1}. This happens, whenever every $e \in \sigma$ breaks the priority rule or $e \notin T$. Edge e breaks the priority rule iff for all $v \in \sigma$ the fact that $e(v) \cap e = \tau(e)$ implies $v > \iota(e)$.

The boundary matrices of the Morse complex $\tilde{\partial}_d : \; \mathcal{M}_d(D_n(\Gamma)) \to \mathcal{M}_{d-1}(D_n(\Gamma))$ are calculated in bases, which consist of critical d-cells and critical $(d-1)$-cells. The explicit formula reads

$$\tilde{\partial}_d(\sigma) = F^\infty(\partial_d(\sigma)).$$

Example 5.2. – Morse complex of the two-particle configuration space on the lasso graph.

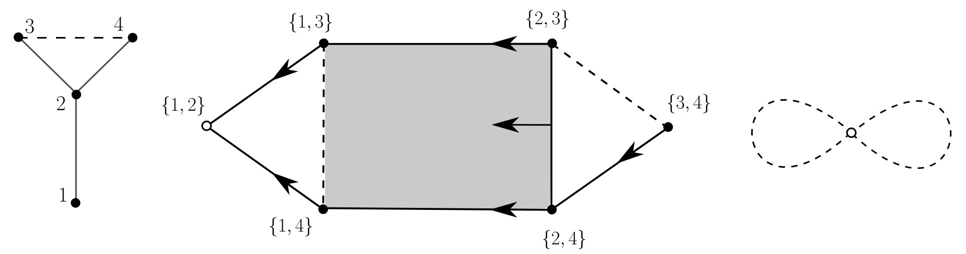


Figure 5.1: A discrete Morse theory for the lasso graph. We have $H_1(C_2(\Gamma)) = \mathbb{Z} \oplus \mathbb{Z}$. Arrows denote the gradient vector field, whose flow contracts the configuration space to a bow-tie. The critical 1-cells are marked by dashed lines. There is only one critical 0-cell – $\{1, 2\}$.

The spanning tree of Γ is denoted on figure 5.1 by solid lines. There are two critical cells - one, that contains the edge, which is not contained in the spanning tree and the unique vertex disjoint with this edge, $\sigma_1 = \{e_3^4, 1\}$. The second critical cell has the priority-breaking edge e_2^4, which blocks particle at vertex 3, $\sigma_2 = \{e_2^4, 3\}$. Let us compute the boundary of σ_2 in the Morse complex. From formula 4.1, we have $\partial\left(\{e_\tau^\iota, v\}\right) = -\left(\{\iota, v\} - \{\tau, v\}\right)$. Hence, $\partial\left(\{e_2^4, 3\}\right) = \{2, 4\} - \{3, 4\}$. Both cells are redundant, as in general there cannot be any collapsible 0-cells. The discrete gradient vector field acts as follows

$$W\left(\{2, 4\}\right) = \{e_1^2, 4\}, \quad W\left(\{3, 4\}\right) = \{e_2^3, 4\}.$$

That said, we have $F(\{2, 4\}) = \{1, 4\}$, $F(\{3, 4\}) = \{2, 4\}$. By iterating the procedure, we get $F^\infty(\{2, 4\}) = \{1, 2\} = F^\infty(\{3, 4\})$. Hence, by linearity of F^∞, we have $\tilde{\partial}(\sigma_2) = 0$. Similarly, $\tilde{\partial}(\sigma_1) = 0$. As there are no critical 2-cells, we conclude, that $H_1(D_2(\Gamma)) = \mathbb{Z} \oplus \mathbb{Z}$.

Some chosen homology groups over integers for configuration spaces of exemplary graphs have been calculated using the above formulation of discrete Morse theory. The results are collected in tables 5.1 and 5.2. One can readily see the regularity of second Betti numbers in table 5.1. Namely, for $n \geq 4$, the second Betti numbers grow quadratically, as the differences between $\beta_2(C_n(\Gamma))$ and $\beta_2(C_{n+1}(\Gamma))$ become constant. As we explain in the following sections, higher Betti numbers also become regular, with polynomial behaviour of a higher degree.

Table 5.2 presents the results for the second and third homology groups for graphs from the Petersen family (fig. 5.2). These graphs serve as examples, where torsion in higher homology groups appears. Interestingly, the torsion subgroups are always equal to a number of copies of $\mathbb{Z}_2$. This phenomenon can be explained by embedding a nonplanar graph in Γ and considering suitable product cycles, as discussed in subsection 5.3.5. The question about the existence of torsion different than $\mathbb{Z}_2$ in higher homologies remains open.

Algorithm 1 Main steps of the algorithm for computing $H_d(D_n(\Gamma))$ via discrete Morse theory

```
Input: Sufficiently subdivided graph Γ, number of particles n.
Output: β_d(D_n(Γ)), T_d(D_n(Γ))
F ← flow of the discrete gradient vector field
∂ ← the boundary map in 𝕮(D_n(Γ))
procedure MorseBoundaryMap(d)
    critcellsd ← list of critical d-cells
    critcellsdminus ← list of critical d − 1-cells
    D_M ← integer matrix of size Length(critcellsd)×Length(critcellsdminus)
    for i = 0 to Length(critcellsd) do
        b ← ∂(critcellsd[i])
        repeat
            b ← F(b)
        until F(b) == b
        for σ' in b do
            D_M[i][Index(σ', critcellsdminus)] ← Coefficent(σ', b)
    return D_M
D_d ← MorseBoundaryMap(d)
dimker ← Length(D_d[0]) − MatrixRank(MorseBoundaryMap(d))
D_{d+1} ← MorseBoundaryMap(d + 1)
divisors ← ElementaryDivisors(D_{d+1})
nonzerodivisors ← number of nonzero elements of divisors
torsion ← list of elements of divisors that are greater than 1
return (dimker − nonzerodivisors), torsion
```

Γ	n	$\beta_2(C_n(\Gamma))$	$\beta_3(C_n(\Gamma))$	$\beta_4(C_n(\Gamma))$
K_4	3	3	0	-
	4	9	0	0
	5	15	0	0
	6	21	4	0
	7	27	16	0
	8	33	40	1
	9	39	80	6
$K_{3,3}$	2	0	-	-
	3	8	0	-
	4	19	1	0
	5	28	10	0
	6	37	39	0
	7	46	88	0
	8	55	157	15
K_5	2	0	-	-
	3	30	0	-
	4	76	1	0
	5	116	77	0
	6	156	381	0
	7	196	961	0

Table 5.1: Betti numbers for chosen graphs computed using the discrete Morse theory [24]. The calculated groups were torsion-free.

	K_6	P_7	$K_{3,3,1}$	$K_{4,4}$	P_8	P_9	P_{10}
$\beta_2(C_4(\Gamma))$	264	177	172	144	114	70	40
$T_2(C_4(\Gamma))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\beta_3(C_6(\Gamma))$	4137	2058	1919	1460	986	452	191
$T_3(C_6(\Gamma))$	0	0	0	$(\mathbb{Z}_2)^{73}$	0	0	0

Table 5.2: The first regular homology groups of order 2 and 3 for the Petersen family.

5.3. Graphs of connectivity one

This section is taken almost verbatim from paper [10]. It concerns graphs of connectivity one, by which we understand connected graphs, that have a vertex, which when removed together with its adjacent edges, results with a decomposition of the graph into a number of disjoint components. The existence of such a vertex has implications for the structure of the discrete configuration space. Namely, $D_n(\Gamma)$ has a natural decomposition into simpler components, where Mayer-Vietoris sequences can be conveniently used. This allows us to give recursive formulae for computing the homology groups of $D_n(\Gamma)$. In particular, we show first computation of the homology groups for particles on tree graphs, that has been repeated later in the work [30] using different methods. We argue, that for particles on tree graphs the homology groups are free, hence in view of proposition 3.14 tree graphs provide a simple playground for studying flat complex vector bundles over configuration spaces. In comparison to

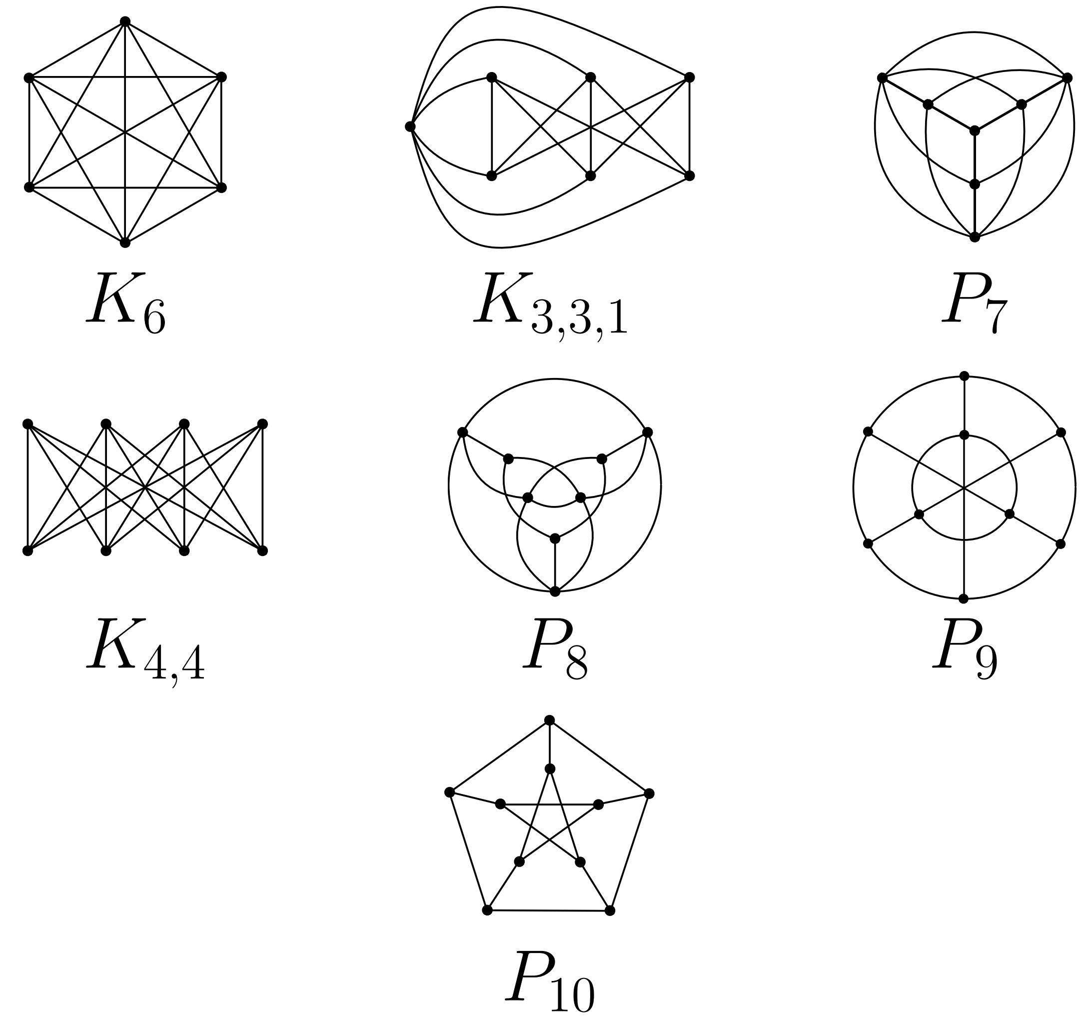


Figure 5.2: Graphs that form the Petersen family.

paper [10], this section contains some additional comments about the generating set of $H_2(D_2(\Gamma))$ (subsection 5.3.2) and about the structure of homology groups of configuration spaces of two graphs connected by a single edge (subsection 5.3.4). We also provide an extensive discussion concerning torsion in homology groups for particles on graphs of connectivity one (subsection 5.3.5).

Let us begin with an overview of the general methodology used in this section. We will regard a tree graph as a loopless lattice of star graphs [1]. Namely, for every tree graph one can construct the underlying tree, whose vertices denote the central vertices of the star graphs and the edges symbolise the connections between the star graphs, see Fig. 5.3. The homology groups for star graphs are well-known. In particular, $D_n(S)$ is homotopy equivalent to a wedge sum of circles [64, 28]. Recall that a wedge sum of topological spaces X and Y is a space, which is created by identifying a point in X with a point in Y. In other words, this is the space $(X \sqcup Y)/\sim$, where $\sim$ is the quotient map that identifies the two distinguished points. Hence, $H_k(D_n(S)) = 0$ for $k \geq 2$. Moreover, the dimension of $H_1(D_n(S))$ is given by [20, 22]

$$\beta_1^{(n)}(S) = \binom{n+E-2}{E-1}(E-2) - \binom{n+E-2}{E-2} + 1, \tag{5.1}$$

[1] A star graph is a connected graph with a single vertex of degree larger than 2 (called the hub, or the central vertex) and a number of edges attached to the vertex.

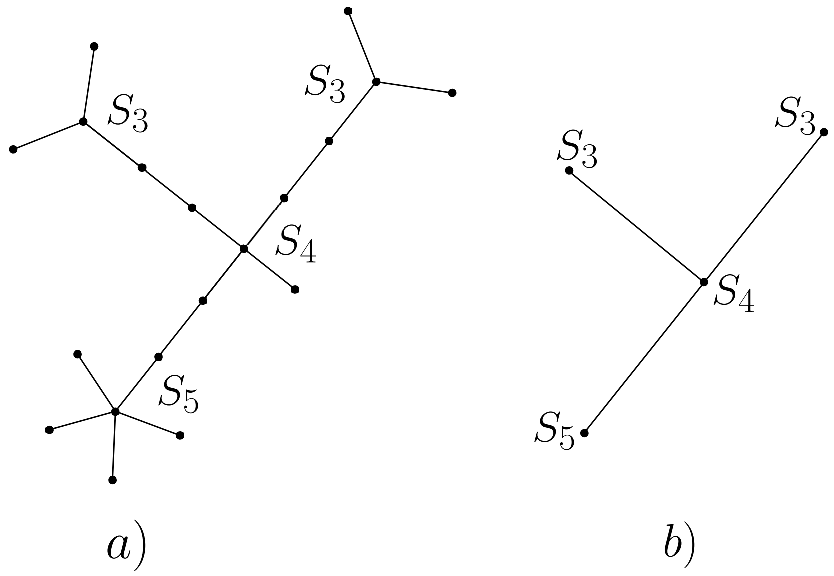


Figure 5.3: a) A tree graph regarded as a lattice of star graphs. b) The underlying scheme of connections.

where E is the number of edges adjacent to the central vertex of S.

Homology groups of $D_n(T)$ have been studied from the Morse-theoretic point of view by Farley and Sabalka in [26]. The authors show that $H_k(D_n(T))$ are free with rank equal to the number of k-dimensional critical cells of the discrete vector field. However, as they point out, it is a difficult task to give a simple formula for the number of critical cells. In this paper, we construct an over-complete basis of $\ker \partial_k$ for $D_n(T)$ using the knowledge of the critical cells of the discrete vector field. The idea is to construct a k-cycle for a given critical k-cell, which contains the critical cell and which is carried by the vector field's flow to the corresponding cell in the Morse complex. Critical cells of the discrete vector field for trees are known [22, 26]. A critical k-cell, like any k-cell, contains k edges and $n-k$ vertices, all disjoint. Each edge from the considered critical k-cell is incident to a vertex of degree ≥ 3. Let us call such a vertex the *hub* of a star graph. The vertices from the critical cell are *blocked*, i.e. are stacked behind the hubs or stacked behind the tree's root. For a full description of critical cells of the discrete vector field, see section 5.2. See Fig. 5.4 for an example of a critical 2-cell.

The corresponding cycle, that is carried by the vector field's flow to a proper critical cell is of the form

$$c_{Y_1} \otimes c_{Y_2} \otimes \cdots \otimes c_{Y_k} \otimes \{v_1, \ldots, v_{n-2k}\},$$

where each of the Y-subgraphs consists of one edge from the critical cell, one edge, which contains the hub and a free vertex in the star graph, and one edge, which contains the hub and a vertex from the critical cell (Fig. 5.4). Clearly, such a cycle contains a single critical cell, which is the desired one. Moreover, one can check that the remaining cells from such a cycle are collapsible or redundant, i.e. are collapsed by the vector field's flow. The over-complete basis for tree graphs that we consider, consists of all such cycles, however we do not require the vertices $\{v_1, \ldots, v_{n-2k}\}$ to be blocked. They can be arbitrary vertices of T. Hence, our over-complete basis is much larger than the number of critical cells. Subsection 5.3.3 uses a proper topological machinery to handle the relations between the cycles from the over-complete basis. These relations come from the $(k+1)$-cells from $D_n(T)$ and from linear dependence

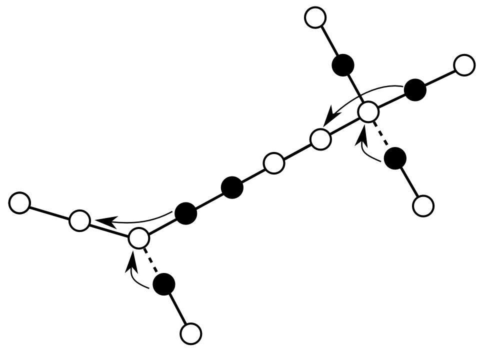


Figure 5.4: The correspondence between the critical cells of the discrete gradient vector field and cycles in the configuration space. The edges from the critical cell are marked with dashed lines. Arrows mark the Y-subgraphs, where the pairs of particles exchange. The occupied vertices (black dots) denote the free particles, which are stacked behind the hubs.

within $\ker \partial_p$.

The fact that the homology groups are free, can be also proved using the above correspondence between the critical cells and cycles in $D_n(T)$. Namely, there are no relations between the k-cycles of the Morse complex that stem from the boundaries of $(k+1)$-cells. This is because every cell of the Morse complex has no boundary, as it is the image under the chain map F^∞ of a cycle in the configuration space.

Let us next describe the intuition standing behind the proof of the formulae for the ranks of the homology groups. Particles on a tree graph can exchange only on Y-subgraphs. Using the above arguments from Morse theory, it is enough to consider exchanges of pairs of particles that involve separate Y-subgraphs. There are two kinds of relations between the cycles corresponding to such exchanges

- relations between the exchanges on Y-subgraphs from the same star subgraph,
- relations between the exchanges on distinct star subgraphs, that stem from the connections between the subgraphs.

The relations of the first kind can be handled by choosing the 1-cycles, which are the representatives of the basis of the first homology group for the proper star subgraphs. The number of independent 1-cycles for particles on a star graph is given by formula (5.1). For example, consider a tree, which consists of exactly two star graphs (Fig. 5.10), S and S'. The representatives of the second homology group for four particles on such a tree are the 2-cycles, which are products of two-particle 1-cycles from S and two-particle 1-cycles from S'. Hence, the rank of $H_2(D_4(S,S'))$ reads

$$\beta_2^{(4)}(S,S') = \beta_1^{(2)}(S)\beta_1^{(2)}(S').$$

When the number of particles on T is larger than 4, the relations of the second kind come into play. There are 2-cycles, that come from all possible distributions of particles between S and S'. In the case of 5 particles, the number of such cycles is $\beta_1^{(2)}(S)\beta_1^{(3)}(S') + \beta_1^{(3)}(S)\beta_1^{(2)}(S')$. However, each such 2-cycle has one free particle, that does not take part in the exchange. Hence, the 2-cycles, where the free particle is sitting on the path connecting S and S', were counted twice. To obtain the rank of $H_2(D_5(S,S'))$, we have to subtract the double-counted cycles, whose number is $\beta_1^{(2)}(S)\beta_1^{(2)}(S')$. In the case of n particles, we have to subtract the cycles, where at least one particle is sitting on the connecting path, which is exactly the number of (over-complete) cycles for $n-1$ particles. The formula reads

$$\beta_2^{(n)}(S,S') = \sum_{l=2}^{n-2} \left(\beta_1^{(l)}(S) - \beta_1^{(l-1)}(S)\right) \beta_1^{(n-l)}(S').$$

A similar result holds for a situation, where star graph S is connected with a single edge to a tree graph T', i.e. S' can be replaced in the above formula by T'. Recall that $\beta_1^{(n-l)}(T')$ is the sum of $\beta_1^{(n-l)}(S')$ for all $S' \subset T'$. Hence, we have $\beta_2^{(n)}(T) = \sum_{(S,S')\subset T} \beta_2^{(n)}(S,S')$ for any tree. The same reasoning can be used to compute the rank of the dth homology group. The conditions for H_d to be nonzero are: i) the tree contains at least d star subgraphs, ii) the number of particles is at least $2d$. The simplest case is when $n=2d$ and the tree contains exactly d star subgraphs. Then, it is enough to multiply the two-particle 1-cycles from the distinct star subgraphs, i.e.

$$\beta_d^{(2d)}(T) = \prod_{S\subset T} \beta_1^{(2)}(S) \text{ for } \#T = d.$$

For a larger number of particles, handling the multiply-counted cycles is a more difficult task than in the case of H_2, because there are more connecting paths, where the free particles can be distributed. However, this problem can be tackled recursively. Consider a star graph S connected by an edge with a tree, which consists of $m-1$ star subgraphs. Every d-cycle in $H_d(D_n(S,T'))$, $n > 2d$, is a product of a 1-cycle from $D_l(S)$ and a $(d-1)$-cycle from $D_{n-l}(T')$. Multiplying the $(d-1)$-cycles with the 1-cycles and subtracting the multiply-counted cycles, we get

$$\beta_d^{(n)}(S,T') = \sum_{l=2}^{n-2} \left(\beta_1^{(l)}(S) - \beta_1^{(l-1)}(S)\right) \beta_{d-1}^{(n-l)}(T').$$

Considering tree T' as a star S' connected by an edge with tree T'', we get a similar relation for $\beta_{d-1}^{(n-l)}(T')$. Proceeding in this way, we end up with a formula, which expresses $\beta_d(n)(T)$ by the first Betti numbers of the star subgraphs of T for different distributions of particles. The final expression is given in equation (5.17) in subsection 5.3.3.

The above reasoning is just a sketch of the main ideas standing behind the rigorous proof, which is given in the following subsections of this section.

5.3.1. Configuration space for graphs of connectivity one

In this subsection, we describe the structure of $D_n(\Gamma)$ for graphs of connectivity one. The results of this subsection play a key role in the method of computing homology groups by Mayer-Vietoris sequences. As a preliminary exercise, consider the case of n particles on a disjoint sum of two graphs, $\Gamma = \Gamma_1 \sqcup \Gamma_2$. One can distinguish different parts of $D_n(\Gamma)$, which correspond to distributing k particles on Γ_1 and l particles on Γ_2, $k+l=n$. Because the two graphs are disjoint, such a component is isomorphic to the Cartesian product of the corresponding configuration spaces for Γ_1 and Γ_2, i.e. $D_k(\Gamma_1) \times D_l(\Gamma_2)$. The following lemma shows, that there are no connections between different components.

Lemma 5.3. *The n-particle configuration space for a disjoint pair of graphs is a disjoint union of the following connected components.*

$$D_n(\Gamma_1 \sqcup \Gamma_2) = \bigsqcup_{k+l=n} D_k(\Gamma_1) \times D_l(\Gamma_2).$$

Proof. Because Γ_1 and Γ_2 are disjoint, there is no possibility for the particles to move from one graph to another. Such a possibility is essential for the existence of connections between different components of $D_n(\Gamma_1 \sqcup \Gamma_2)$. More formally, for any two cells

$$c \in D_k(\Gamma_1) \times D_l(\Gamma_2) \text{ and } c' \in D_{k'}(\Gamma_1) \times D_{l'}(\Gamma_2),$$

where $k \neq k'$ or $l \neq l'$, there is no path in $D_n(\Gamma_1 \sqcup \Gamma_2)$ joining c with c'. To see this, recall that the n-particle configuration space is a cubic complex. The existence of a path joining two vertices of a cubic complex, is equivalent to the existence of a 1-chain in the complex that joins the two vertices. Such a 1-chain necessarily contains a 1-cell, whose endpoints belong to $D_k(\Gamma_1) \times D_l(\Gamma_2)$ and $D_{k'}(\Gamma_1) \times D_{l'}(\Gamma_2)$ respectively. We will next show that such a 1-cell does not exist. The endpoints of such a cell are the 0-cells of the form

$$c^{(0)} = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_k^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_l^{(2)}\} \in D_k(\Gamma_1) \times D_l(\Gamma_2),$$
$$c'^{(0)} = \{v'^{(1)}_1, v'^{(1)}_2, \ldots, v'^{(1)}_{k'}, v'^{(2)}_1, v'^{(2)}_2, \ldots, v'^{(2)}_{l'}\} \in D_{k'}(\Gamma_1) \times D_{l'}(\Gamma_2),$$

where $v_i^{(1)}, v'^{(1)}_i \in \Gamma_1,\ v_i^{(2)}, v'^{(2)}_i \in \Gamma_2$. For $c'^{(0)}$ and $c^{(0)}$ to be the endpoints of a 1-cell, there must exist a pair of vertices $(v_i^{(1)}, v'^{(1)}_j)$ or $(v_i^{(2)}, v'^{(2)}_j)$, who are adjacent in Γ_1 or Γ_2 respectively. Without loss of generality, we can assume that $(v_1^{(1)}, v'^{(1)}_1)$ is such a pair. Then, any 1-cell that contains $c^{(0)}$ is of the form

$$c^{(1)} = \{e, v_2^{(1)}, \ldots, v_k^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_l^{(2)}\},$$

where $\partial_1(e) = \pm(v_1^{(1)} - v'^{(1)}_1)$. Therefore, both endpoints of any 1-cell containing $c^{(0)}$ belong to $D_k(\Gamma_1) \times D_l(\Gamma_2)$, which is a contradiction. □

Because the configuration space is a disjoint sum, we have

$$H_d(D_n(\Gamma_1 \sqcup \Gamma_2)) = \bigoplus_{k+l=n} H_d(D_k(\Gamma_1) \times D_l(\Gamma_2)).$$

Furthermore, by Künneth theorem (2.6)

$$H_d(D_k(\Gamma_1) \times D_l(\Gamma_2)) = \left(\bigoplus_{i+j=d} H_i(D_k(\Gamma_1)) \otimes H_j(D_l(\Gamma_2)) \right) \oplus$$
$$\oplus \bigoplus_{i+j=d-1} T(H_i(D_k(\Gamma_1))) \otimes T(H_j(D_l(\Gamma_2)))$$

Hence, for a disjoint sum of graphs, the knowledge of the homology groups of the configuration spaces of the components is sufficient to compute the homology groups of $D_n(\Gamma)$. Let us next move to the case of graphs of connectivity one. Recall, that graph of connectivity one is a graph, which becomes disconnected after removing a particular vertex together with the adjacent edges. In other words, any graph of connectivity one

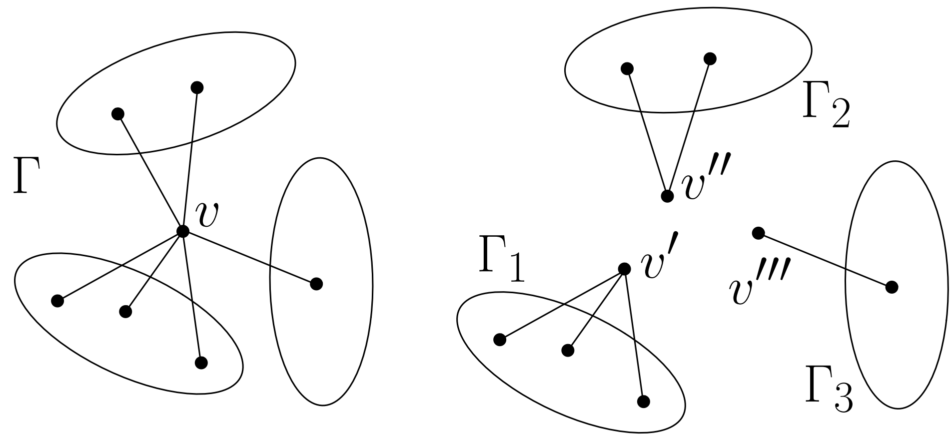


Figure 5.5: A graph of connectivity one Γ as a wedge sum of three components. $\Gamma = (\sqcup_i \Gamma_i)/\sim$, where $v' \sim v''$, $v'' \sim v'''$.

can be viewed as a wedge sum of graphs, which we call components. Consider first a simpler case, where Γ has two components (see Fig. 5.5). Our goal is to describe the connections in $D_n(\Gamma)$ between the components $D_k(\Gamma_1) \times D_l(\Gamma_2)$ that are induced by the gluing map. In fact, we have to consider some disjoint subgraphs of Γ, hence sometimes we have to remove vertex v from each component. To this end, we do an extra subdivision of edges that connect Γ_i with v and remove the last segment of each such edge. The component after such an operation will be denoted by $\tilde{\Gamma}_i$ (see Fig. 5.6). Let us next show how the different components of Γ come into play in $D_2(\Gamma)$. Cells of

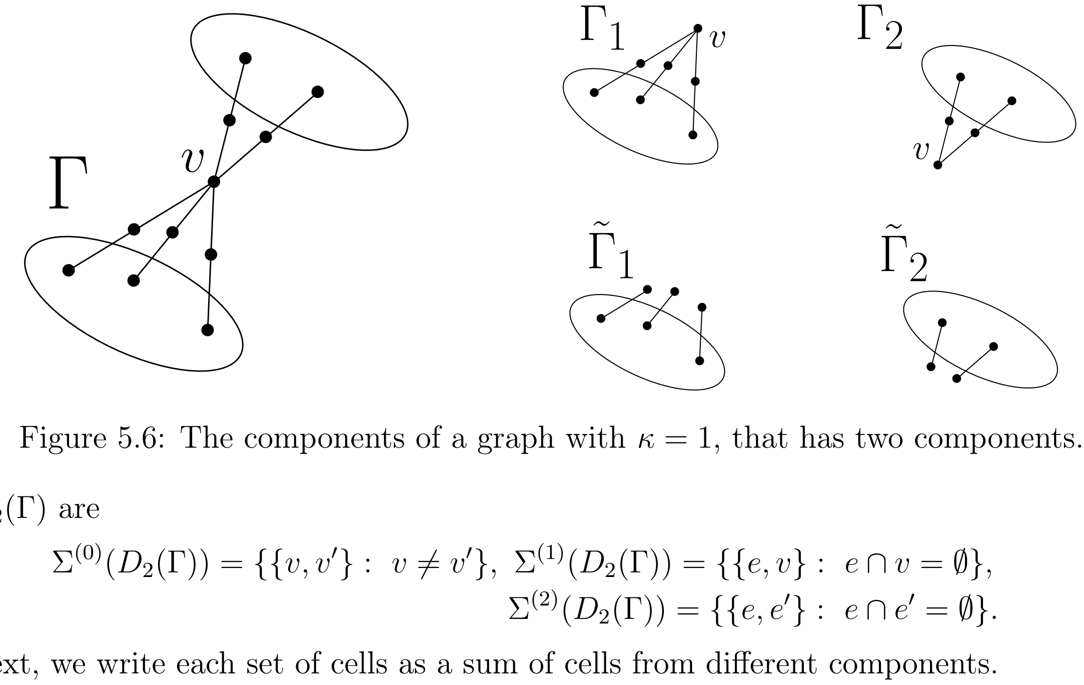


Figure 5.6: The components of a graph with $\kappa = 1$, that has two components.

$D_2(\Gamma)$ are

$$\Sigma^{(0)}(D_2(\Gamma)) = \{\{v, v'\} : \; v \neq v'\}, \; \Sigma^{(1)}(D_2(\Gamma)) = \{\{e, v\} : \; e \cap v = \emptyset\},$$
$$\Sigma^{(2)}(D_2(\Gamma)) = \{\{e, e'\} : \; e \cap e' = \emptyset\}.$$

Next, we write each set of cells as a sum of cells from different components.

$$\Sigma^{(0)}(D_2(\Gamma)) = \{\{v, v'\} : \; v \neq v' \text{ and } v, v' \in V(\Gamma_1)\} \cup$$
$$\{\{v, v'\} : \; v \neq v' \text{ and } v, v' \in V(\Gamma_2)\} \cup \cup \{\{v, v'\} : \; v \in V(\Gamma_1) \text{ and } v' \in V(\tilde{\Gamma}_2)\} \cup$$
$$\cup \{\{v, v'\} : \; v \in V(\tilde{\Gamma}_1) \text{ and } v' \in V(\Gamma_2)\}.$$

The sets of cells of a higher dimension can be written in an analogous way. In other words,

$$\Sigma^{(i)}(D_2(\Gamma)) = \Sigma^{(i)}(D_2(\Gamma_1)) \cup \Sigma^{(i)}(D_2(\Gamma_2)) \cup \Sigma^{(i)}(\Gamma_1 \times \tilde{\Gamma}_2) \cup \Sigma^{(i)}(\tilde{\Gamma}_1 \times \Gamma_2).$$

Some of the above summands are not disjoint, i.e.

$$\begin{aligned}
\Sigma^{(i)}(D_2(\Gamma_1)) \cap \Sigma^{(i)}(\tilde{\Gamma}_1 \times \Gamma_2) &= \Sigma^{(i)}(\tilde{\Gamma}_1 \times v), \\
\Sigma^{(i)}(\tilde{\Gamma}_1 \times \Gamma_2) \cap \Sigma^{(i)}(\Gamma_1 \times \tilde{\Gamma}_2) &= \Sigma^{(i)}(\tilde{\Gamma}_1 \times \tilde{\Gamma}_2), \\
\Sigma^{(i)}(D_2(\Gamma_2)) \cap \Sigma^{(i)}(\Gamma_1 \times \tilde{\Gamma}_2) &= \Sigma^{(i)}(\tilde{\Gamma}_2 \times v).
\end{aligned}$$

The structure of $D_2(\Gamma)$ is shown on Fig. 5.7. Configuration space from Fig. 5.7 can be

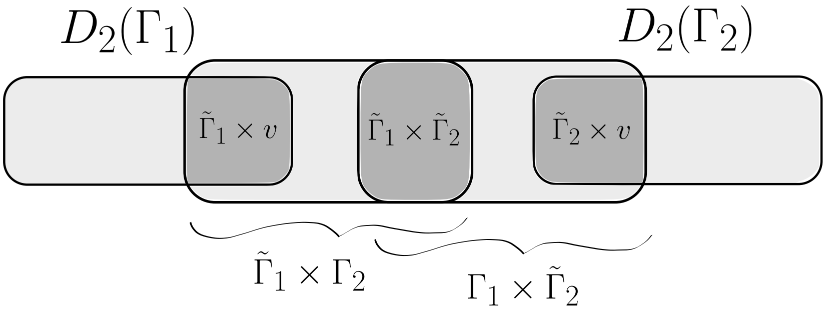


Figure 5.7: A scheme of $D_2(\Gamma)$ for Γ with two components.

represented as a diagram, Fig. 5.8. A node represents a subcomplex of $D_2(\Gamma)$, while the edges describe the common parts of neighbouring subcomplexes. For $n > 2$, one

$$D_2(\Gamma_1) \xrightarrow{\tilde{\Gamma}_1 \times v} \tilde{\Gamma}_1 \times \Gamma_2 \xrightarrow{\tilde{\Gamma}_1 \times \tilde{\Gamma}_2} \Gamma_1 \times \tilde{\Gamma}_2 \xrightarrow{\tilde{\Gamma}_2 \times v} D_2(\Gamma_2)$$

Figure 5.8: Configuration space diagram of $D_2(\Gamma)$ for Γ from Fig. 5.6.

has many possibilities of distributing the particles among the components. However, the structure of $D_n(\Gamma)$ is still linear, see Fig. 5.9. There are two kinds of connections.

$$\begin{array}{ccccccc}
D_n(\Gamma_1) & \xrightarrow{D_{n-1}(\tilde{\Gamma}_1) \times v} & D_{n-1}(\tilde{\Gamma}_1) \times \Gamma_2 & \xrightarrow{D_{n-1}(\tilde{\Gamma}_1) \times \tilde{\Gamma}_2} & D_{n-1}(\Gamma_1) \times \tilde{\Gamma}_2 & \xrightarrow{D_{n-2}(\tilde{\Gamma}_1) \times \tilde{\Gamma}_2 \times v} & D_{n-2}(\tilde{\Gamma}_1) \times D_2(\Gamma_2) \\
 & & & & & & \Big| \; D_{n-2}(\tilde{\Gamma}_1) \times D_2(\tilde{\Gamma}_2) \\
D_n(\Gamma_2) & \xrightarrow{D_{n-1}(\tilde{\Gamma}_2) \times v} & \Gamma_1 \times D_{n-1}(\tilde{\Gamma}_2) & \xrightarrow{\tilde{\Gamma}_1 \times D_{n-1}(\tilde{\Gamma}_2)} & \cdots\cdots & \xrightarrow{D_{n-3}(\tilde{\Gamma}_1) \times D_2(\tilde{\Gamma}_2) \times v} & D_{n-2}(\Gamma_1) \times D_2(\tilde{\Gamma}_2)
\end{array}$$

Figure 5.9: Configuration space diagram of $D_n(\Gamma)$ for Γ from Fig. 5.6.

Namely, the connections, where the number of particles between the components is the same, and the connections, where one particle moves from Γ_1 to Γ_2. Connections of the first kind exist between $D_k(\tilde{\Gamma}_1) \times D_l(\Gamma_2)$ and $D_k(\Gamma_1) \times D_l(\tilde{\Gamma}_2)$, where the common part is $D_k(\tilde{\Gamma}_1) \times D_l(\tilde{\Gamma}_2)$. Connections of the second kind describe a change in the number of particles, hence they exist between $D_k(\Gamma_1) \times D_l(\tilde{\Gamma}_2)$ and $D_{k-1}(\tilde{\Gamma}_1) \times D_{l+1}(\Gamma_2)$, where the common part is $D_{k-1}(\tilde{\Gamma}_1) \times D_l(\tilde{\Gamma}_2) \times v$.

As a final remark, note that for a graph, which is a wedge sum of a larger number of components, the presented results for two components can be applied inductively. Choose Γ_1 to be one of the components and Γ'_2 to be the wedge sum of the remaining components, $\{\Gamma_i : \; i \geq 2\}$. Subgraph $\tilde{\Gamma}'_2$ is a disjoint sum of graphs. Therefore, $D_k(\Gamma_1) \times D_l(\tilde{\Gamma}'_2) = \sqcup_i D_k(\Gamma_1) \times D_l(\tilde{\Gamma}_i)$ and the configuration space diagram splits in such a node. Detaching inductively the remaining components of Γ'_2, we obtain the full configuration space diagram.

5.3.2. Second homology group of $D_2(\Gamma)$

In this subsection, we continue the considerations regarding the example of two particles on graphs of connectivity one with two (Fig. 5.6), or more components. Using this example, we introduce tools that we finally apply for $D_n(\Gamma)$, where Γ is a tree graph. In the end of this subsection we also give a formula for the second Betti number of $D_2(\Gamma)$, which is a generalisation of the formula by Farber [29] for two graphs connected by a single edge.

Consider graph Γ, which has two components. By the construction of $D_2(\Gamma)$, there are no 3-cells in the complex, hence $H_2(D_2(\Gamma))$ is free. To compute the second homology, we use Mayer-Vietoris sequence for different components of the configuration space diagram. Let X be any subcomplex of $D_2(\Gamma)$ and let A and B be subcomplexes of X such that $A \cup B = X$. The Mayer-Vietoris sequence for X reads [58]

$$0 \to H_2(A \cap B) \xrightarrow{\Phi} H_2(A) \oplus H_2(B) \xrightarrow{\Psi} H_2(X) \xrightarrow{\delta} H_1(A \cap B) \xrightarrow{\Phi} \dots \quad (5.2)$$

Long exact sequence (5.2) implies the short exact sequence

$$0 \to \mathrm{coker}(\Phi) \to H_2(X) \to \mathrm{coker}(\Psi) \to 0.$$

If the homology groups in the Mayer-Vietoris sequence are free (in fact, it is enough to require coker(Ψ) to be free abelian), which is the case for $H_2(D_2(\Gamma))$, the sequence splits, i.e.

$$H_2(X) = \mathrm{coker}(\Phi) \oplus \mathrm{coker}(\Psi) = \mathrm{coker}(\Phi) \oplus \mathrm{im}\delta.$$

Often we will consider elements of $\mathrm{im}\delta$ as elements of $H_2(X)$, i.e. use the isomorphism $\mathrm{im}\delta \cong \mathrm{coim}\delta = H_2(X)/\ker\delta$. Then,

$$H_2(X) = \mathrm{coker}(\Phi) \oplus \mathrm{coim}\delta. \quad (5.3)$$

Theorem 5.4. *Let Γ be a graph of connectivity one with two components and let $\Gamma_1, \tilde{\Gamma}_1, \Gamma_2, \tilde{\Gamma}_2$ be the components of Γ, as on Fig. 5.6. Then,*

$$\beta_2\left(D_2(\Gamma)\right) = \beta_2(D_2(\Gamma_1)) + \beta_2(D_2(\Gamma_2)) + \beta_1(\tilde{\Gamma}_1)\beta_1(\Gamma_2) + \beta_1(\Gamma_1)\beta_1(\tilde{\Gamma}_2) - \beta_1(\tilde{\Gamma}_1)\beta_1(\tilde{\Gamma}_2). \quad (5.4)$$

Proof. Decompose the configuration space part-by-part, as follows

$$X_0 = D_2(\Gamma),\ A_0 = D_2(\Gamma_1),\ B_0 = (\tilde{\Gamma}_1 \times \Gamma_2) \cup (\Gamma_1 \times \tilde{\Gamma}_2) \cup D_2(\Gamma_2), A_0 \cap B_0 = \tilde{\Gamma}_1 \times v,$$
$$X_1 = B_0,\ A_1 = \tilde{\Gamma}_1 \times \Gamma_2,\ B_1 = (\Gamma_1 \times \tilde{\Gamma}_2) \cup D_2(\Gamma_2), A_1 \cap B_1 = \tilde{\Gamma}_1 \times \tilde{\Gamma}_2,$$
$$X_2 = B_1,\ A_2 = \Gamma_1 \times \tilde{\Gamma}_2,\ B_2 = D_2(\Gamma_2),\ A_2 \cap B_2 = \tilde{\Gamma}_2 \times v.$$

The ansatz is to write Mayer-Vietoris sequence for each $X_i = A_i \cup B_i$ and proceed inductively, beginning with X_2. Namely,

$$\mathrm{coker}\Phi_2 = (H_2(A_2) \oplus H_2(B_2))/\mathrm{im}\Phi_2 = (H_1(\Gamma_1) \otimes H_1(\tilde{\Gamma}_2)) \oplus H_2(D_2(\Gamma_2)), \quad (5.5)$$

where in $H_2(A_2)$ we used the Künneth theorem. The image of Φ_2 is trivial, because $H_2(A_2 \cap B_2) = H_2(\tilde{\Gamma}_2 \times v) = 0$, therefore $\mathrm{coim}\Phi_2 = 0$. Next, we give a characterisation of elements of $\mathrm{coim}\delta$ for $i = 2$. Recall that

$$\delta_2 :\ H_2(X_2) \to H_1(A_2 \cap B_2) \cong H_1(\tilde{\Gamma}_2).$$

Denote by z a representative of $H_2(X_2)$. Two-cycle z can be decomposed as a sum of 2-chains from A_2 and B_2 respectively

$$z = x + y,\ \ x \in \mathfrak{C}_2(A_2),\ \ y \in \mathfrak{C}_2(B_2). \tag{5.6}$$

The above decomposition is in this case unique, because there are no 2-cells in the subcomplex $A_2 \cap B_2$. The boundary map assigns to $[z]$ the 1-cycle class $[-\delta_2 y]$. Then it follows from (5.3) and (5.5) that

$$H_2(B_1) = H_2(X_2) = H_2(D_2(\Gamma_2)) \oplus \mathrm{coim}\delta_2 \oplus (H_1(\Gamma_1) \otimes H_1(\tilde{\Gamma}_2)). \tag{5.7}$$

Let us proceed with $i = 1$. We have $\mathrm{coim}\Phi_1 = H_1(\tilde{\Gamma}_1) \otimes H_1(\tilde{\Gamma}_2)$, hence

$$\mathrm{coker}\Phi_1 = \left(H_2(B_2) \oplus (H_1(\tilde{\Gamma}_1) \otimes H_1(\Gamma_2)\right) / \left(H_1(\tilde{\Gamma}_1) \otimes H_1(\tilde{\Gamma}_2)\right).$$

Note that the quotient does not affect $H_2(D_2(\Gamma_2))$ and $\mathrm{coim}\delta_2$ from equation (5.7), hence the quotient can be taken with respect to the summand $\left(H_1(\tilde{\Gamma}_1) \otimes H_1(\Gamma_2)\right) \oplus \left(H_1(\Gamma_1) \otimes H_1(\tilde{\Gamma}_2)\right)$ only. This is because every 2-cycle from $\mathrm{coim}\Phi_1$ is of the form

$$z = c \otimes c',\ [c] \in H_1(\tilde{\Gamma}_1),\ [c'] \in H_1(\tilde{\Gamma}_2). \tag{5.8}$$

Cycle z is a 2-cycle from $D_2(\tilde{\Gamma}_1 \times \tilde{\Gamma}_2)$. Therefore, decomposition (5.6) for such a 2-cycle yields $y = 0$, i.e. $z \in \ker \delta_2$. Moreover, none of the representatives of a homology class from $H_2(D_2(\Gamma_2))$ is of the form (5.8). Therefore,

$$H_2(B_1) = H_2(D_2(\Gamma_2)) \oplus \mathrm{coim}\delta_2 \oplus \mathrm{coim}\delta_1 \oplus \frac{(H_1(\Gamma_1) \otimes H_1(\tilde{\Gamma}_2)) \oplus (H_1(\tilde{\Gamma}_1) \otimes H_1(\Gamma_2))}{H_1(\tilde{\Gamma}_1) \otimes H_1(\tilde{\Gamma}_2)}.$$

Finally, for $i = 0$, by an analogous reasoning as in the case of $i = 2$, we have $\mathrm{im}\Phi_0 = 0$, hence

$$H_2(D_2(\Gamma)) = H_2(D_2(\Gamma_1)) \oplus H_2(D_2(\Gamma_2)) \oplus \mathrm{coim}\delta \oplus \frac{(H_1(\Gamma_1) \otimes H_1(\tilde{\Gamma}_2)) \oplus (H_1(\tilde{\Gamma}_1) \otimes H_1(\Gamma_2))}{H_1(\tilde{\Gamma}_1) \otimes H_1(\tilde{\Gamma}_2)},$$

where $\mathrm{coim}\delta = \oplus_{i=0}^{2}\mathrm{coim}\delta_i$. Using a theorem by Farber [29], we prove in lemma 5.6 that $\mathrm{coim}\delta = 0$, which completes the proof. □

The last thing to show is the fact that $\mathrm{im}\delta \cong \mathrm{coim}\delta = 0$. To this end, we consider a specific over-complete basis of the second homology group. First, we briefly review the known facts about the second homology group of the two-particle configuration spaces.

Theorem 5.5 (Farber [29]). *For a planar graph Γ, there exists a basis of $H_2(D_2(\Gamma))$, where the representatives are of the form*

$$z = c \otimes c', [c], [c'] \in H_1(\Gamma).$$

Moreover, cycles c and c' are necessarily disjoint, i.e. for

$$c = \sum_i a_i e_i,\ \ c' = \sum_j b_j e'_j,$$

we have $e_i \cap e'_j = \emptyset$ for all i, j. Then, $z = \sum_{i,j} a_i b_j e_i \times e'_j$.

Hence, for planar graphs the generating set of $H_2(D_2(\Gamma))$ consists of all possible pairs of disjoint cycles in Γ. The construction of an over-complete basis for two particles on non-planar graphs carries an additional subtlety. Recall a theorem by Kuratowski [62], which states that every non-planar graph contains a subgraph that is isomorphic to graph $K_{3,3}$ or K_5 (which we call the Kuratowski graphs). It was shown by Abrams [28] that graphs $K_{3,3}$ and K_5 are the only possible graphs, whose two-particle discrete configuration spaces are closed surfaces. However, the question whether all closed surfaces in the two-particle configuration space of a graph Γ can be obtained by embedding a pair of disjoint cycles or a Kuratowski graph in Γ remained open until recently [63]. Paper [63] shows, that there is one more type of surfaces, stemming from certain subgraphs of Γ, called *quad subgraphs*.

Definition 5.1. *A quad of graph G is subgraph $K = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup R_1 \cup R_2 \cup R_3$ such, that*

- *K consists of four distinct vertices a, b, c, d,*
- *paths P_1, P_2, P_3 connect a and b, are independent (mutually internally disjoint) and contain at least one vertex,*
- *paths R_1, R_2, R_3 connect c and d, are independent and contain at least one vertex,*
- *paths Q_1, Q_2, Q_3 are independent, each Q_i has ends u_i, v_i such, that $u_i \cap (P_i - \{a,b\}) = \emptyset$, $u_i \cap (R_i - \{c,d\}) = \emptyset$, and $(Q_i - \{u_i, v_i\}) \cap (P_1 \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup R_3) = \emptyset$,*
- *for every $i \in \{1,2,3\}$, the vertices of $P_i \cap (R_1 \cup R_2 \cup R_3)$ are contained in the set of vertices of Q_i.*

Hence, in order to get the generating set of $H_2(D_2(\Gamma))$ for non-planar graphs, it is enough to consider all 2-cycles, that are products of disjoint cycles in Γ and the two-particle configuration spaces of all Kuratowski and quad subgraphs of Γ.

Lemma 5.6. *Let Γ be a graph of connectivity one with two components and $\{(A_i, B_i)\}_{i=0}^2$ be the subcomplexes of $D_2(\Gamma)$ defined in the proof of theorem 5.4. Moreover, let δ_i be the boundary map from the Mayer-Vietoris sequence*

$$\delta_i : \ H_2(A_i \cup B_i) \to H_1(A_i \cap B_i).$$

Then, $\mathrm{im}\delta_i = 0$ *for all i.*

Proof. Because $H_2(D_2(\Gamma))$ contains $\mathrm{coim}\delta$ as an independent contribution, theorem 5.5 applies also to 2-cycles representing $\mathrm{coim}\delta$. The strategy for the proof in the case of planar graphs is to show that all possible products of disjoint 1-cycles in $X_i = A_i \cup B_i$ are in $\ker \delta_i$.

For $\delta_2 : \ H_2((\tilde{\Gamma}_1 \times \Gamma_2) \cup D_2(\Gamma_2)) \to H_1(\tilde{\Gamma}_2 \times v)$, a 2-cycle, which does not belong to $\ker \delta_2$, has a nonzero part in both $\mathfrak{C}_2(\tilde{\Gamma}_1 \times \Gamma_2)$ and $\mathfrak{C}_2(D_2(\Gamma_2))$. Let z denote such a product 2-cycle, i.e. $z = c \otimes c'$. Note, that every 1-cycle in Γ can be written as a sum of cycles that are wholly contained in Γ_1 or Γ_2. Therefore, the only possibility for the choice of z to contain cells from both A_2 and B_2 is to take $c \in \mathfrak{C}_1(\Gamma_2)$ that contains an edge adjacent to v. However, c and c' are disjoint, hence c' must be contained in $\tilde{\Gamma}_1$ or $\tilde{\Gamma}_2$. This is a contradiction, because then $z \in \mathfrak{C}_2(\tilde{\Gamma}_1 \times \Gamma_2)$ or $z \in \mathfrak{C}_2(D_2(\Gamma_2))$

respectively, which implies, that $z \in \ker \delta_2$. Analogous reasoning for A_0 and B_0 leads to the conclusion that $\mathrm{im}\delta_0 = 0$. Finally, consider $\delta_1: \ H_2((\Gamma_1 \times \tilde{\Gamma}_2) \cup (\tilde{\Gamma}_1 \times \Gamma_2)) \to H_1(\tilde{\Gamma}_1 \times \tilde{\Gamma}_2)$. The desired 2-cycle can be a product of

$$c \in \mathfrak{C}_1(\Gamma_1),\ c \cap v \neq \emptyset,\ c' \in \mathfrak{C}_1(\tilde{\Gamma}_2),\ \text{or}\ c \in \mathfrak{C}_1(\tilde{\Gamma}_1),\ c' \in \mathfrak{C}_1(\Gamma_2),\ c' \cap v \neq \emptyset.$$

It is straightforward to see that in both cases $z \in \mathfrak{C}_2(\Gamma_1 \times \tilde{\Gamma}_2)$ or $z \in \mathfrak{C}_2(\tilde{\Gamma}_1 \times \Gamma_2)$ respectively. Therefore, $z \in \ker \delta_1$.

For nonplanar graphs, the argumentation is as follows. Consider the additional elements of the over-complete basis stemming from all quad subgraphs of Γ and subgraphs of Γ that are isomorphic to $K_{3,3}$ or K_5. From the definition of Kuratowski graphs and quad subgraphs, one can see, that it is not possible to embed these graphs in Γ such, that $K \cap \tilde{\Gamma}_1 \neq \emptyset$ and $K \cap \tilde{\Gamma}_2 \neq \emptyset$. Hence, such a subgraph K must be necessarily contained either in Γ_1 or in Γ_2. Therefore, the 2-cycles that are isomorphic to $D_2(K)$ are always contained in the two-particle configuration spaces of Γ_1 or Γ_2 respectively, i.e. are mapped by δ to zero. □

Note that the derived formula for $\beta_2(D_2(\Gamma))$ can be written in a simpler form using $\mu_i := \beta_1(\Gamma_i) - \beta_1(\tilde{\Gamma}_i)$. Then,

$$\beta_2(D_2(\Gamma)) = \beta_2(D_2(\Gamma_1)) + \beta_2(D_2(\Gamma_2)) + \beta_1(\Gamma_1)\beta_1(\Gamma_2) - \mu_1\mu_2. \tag{5.9}$$

Numbers μ_i are the numbers of cycles lost after detaching vertex v. If Γ_i is connected with v with E_i edges, then $\mu_i = E_i - 1$.

The above formulae can be easily extended to a graph with more than two components. Assume that Γ has three components, $\{\Gamma_i\}_{i=1}^3$. Then, Γ can be viewed as a two-component graph, where the first component is Γ_1 and the second component is the wedge sum of Γ_2 and Γ_3, which we denote by Γ_{23}. Moreover, $\tilde{\Gamma}_{23} = \Gamma_2 \sqcup \Gamma_3$. Next, we apply formula (5.4), using the fact that $\beta_1(\tilde{\Gamma}_{23}) = \beta_1(\tilde{\Gamma}_2) + \beta_1(\tilde{\Gamma}_3)$ and $\mu_{23} = \mu_2 + \mu_3$.

$$\beta_2(D_2(\Gamma)) = \beta_2(D_2(\Gamma_1)) + \beta_2(D_2(\Gamma_{23})) + \beta_1(\Gamma_1)\left(\beta_1(\Gamma_2) + \beta_1(\Gamma_3)\right) - \mu_1(\mu_2 + \mu_3).$$

Finally, we put $\beta_2(D_2(\Gamma_{23})) = \beta_2(D_2(\Gamma_2)) + \beta_2(D_2(\Gamma_3)) + \beta_1(\Gamma_2)\beta_1(\Gamma_3) - \mu_2\mu_3$. Then,

$$\beta_2(D_2(\Gamma)) = \sum_i \beta_2(D_2(\Gamma_i)) + \sum_{i<j}(\beta_1(\Gamma_i)\beta_1(\Gamma_j) - \mu_i\mu_j).$$

5.3.3. Tree graphs

Let us first compute the second homology group for n particles on a tree, which consists of two star graphs, S and S', connected by an edge. We will assume that both S and S' are sufficiently subdivided for n particles, see Fig. 5.10. The general procedure for computing $H_2(D_n(T))$ will be to decompose the n-particle configuration space part-by-part from the chain shown on figure 5.9. After the decomposition we obtain a family of complexes $\{X_i\}_{i=0}^{n-1}$, which are related by $X_0 \supset X_1' \supset X_1 \supset X_2' \supset \cdots \supset X_{n-1}$. Complex X_0 is the whole configuration space of T. Detaching the leftmost part of the chain we decompose X_0 as the sum of $A_0 = D_n(S)$ and B_0 being the remaining part of the chain. In the next step B_0 plays the role of complex X_1' and we decompose X_1' in an analogous way, namely $X_1' = A_1' \cup B_1'$, where $A_1' = D_{n-1}(\tilde{S}) \times S'$ is the leftmost part of chain B_0. Next, from $B_1' = X_1$ we detach $A_1 = D_{n-1}(S) \times \tilde{S}'$, and so on. We denote complexes from the second step by primed superscripts, because the third and

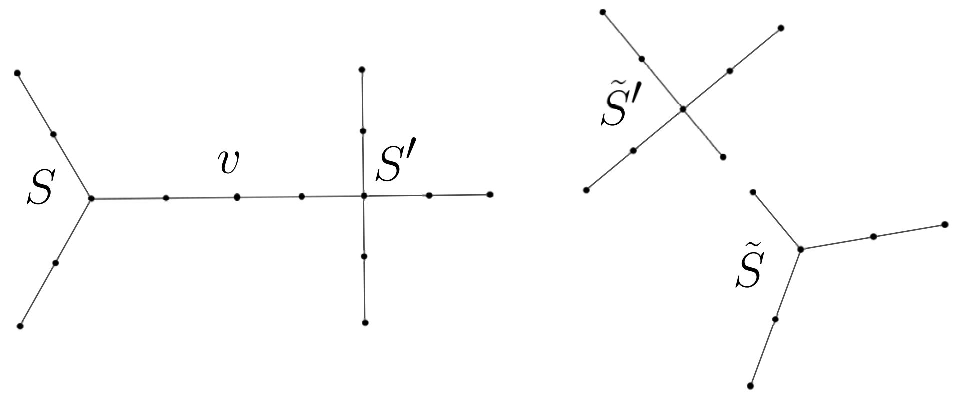


Figure 5.10: A tree graph, which is a wedge of two star graphs, and its components for $n = 3$. Compare with Fig. 5.6.

the second step together describe the transport of one particle from graph S to graph S'. After completing the decomposition we obtain the following list of complexes.

$$X_0 = D_n(T),\ A_0 = D_n(S),\ B_0 = (D_{n-1}(\tilde{S}) \times S') \cup (D_{n-1}(S) \times \tilde{S}') \cup \dots \cup D_n(S'),$$
$$X_1' = B_0,\ A_1' = D_{n-1}(\tilde{S}) \times S',\ B_1' = (D_{n-1}(S) \times \tilde{S}') \cup \dots \cup D_n(S'),$$
$$X_1 = B_1',\ A_1 = D_{n-1}(S) \times \tilde{S}',\ B_1 = (D_{n-2}(\tilde{S}) \times S') \cup \dots \cup D_n(S'),$$
$$\dots$$
$$X_{n-1} = B_{n-1}',\ A_{n-1} = S \times D_{n-1}(\tilde{S}'),\ B_{n-1} = D_n(S').$$

We distinguish two kinds of subcomplexes with respect to the type of their intersections. A pair (A_k', B_k') of the first kind, describes subcomplexes that have the same number of particles on S and S', i.e.

$$A_k' = D_{n-k}(\tilde{S}) \times D_k(S'),\ B_k' = (D_{n-k}(S) \times D_k(\tilde{S}')) \cup \dots \cup D_n(S'),$$
$$A_k' \cap B_k' = D_{n-k}(\tilde{S}) \times D_k(\tilde{S}'),\ X_k' = A_k' \cup B_k',\ k \in \{1, 2, \dots, n-1\}.$$

The second kind of subcomplexes describes pairs, where the numbers of particles on S and S' are different.

$$A_k = D_{n-k}(S) \times D_k(\tilde{S}'),\ B_k = (D_{n-k-1}(\tilde{S}) \times D_{k+1}(S')) \cup \dots \cup D_n(S'),$$
$$A_k \cap B_k = D_{n-k-1}(\tilde{S}) \times D_k(\tilde{S}') \times v,\ X_k = A_k \cup B_k,\ k \in \{0, 1, \dots, n-1\}.$$

Next, we write the Mayer-Vietoris sequence for each pair of subcomplexes, as in (5.2).

$$0 \to H_2(A_k \cap B_k) \xrightarrow{\Phi_k} H_2(A_k) \oplus H_2(B_k) \xrightarrow{\Psi_k} H_2(X_k) \xrightarrow{\delta_k} H_1(A_k \cap B_k) \to \dots \quad (5.10)$$

We use the fact that $H_3(X_k) = 0$, since there are just two connected star graphs, which give no 3-cells in the Morse complex. Because homology groups for tree graphs are free, the Mayer-Vietoris sequence splits and we have

$$H_2(X_k) = \mathrm{coker}(\Phi_k) \oplus \mathrm{coim}\delta_k,\ H_2(X_k') = \mathrm{coker}(\Phi_k') \oplus \mathrm{coim}\delta_k' \quad (5.11)$$

as in (5.3). There are a few differences between the case of tree graphs and a general graph of connectivity one with two components, which allow to compute the second homology for any number of particles. The first simplification comes from the fact that

$D_k(S)$ is homotopy equivalent to $D_k(\tilde{S})$ for $k < n$, and $D_k(S)$ is homotopy equivalent to a wedge of circles. The same holds for distinguishable particles. For $n = 2$, formula (5.9) yields $\beta_2(D_2(T)) = 0$. For $n = 3$, note that all subcomplexes in the configuration space diagram have trivial homology groups in the corresponding Mayer-Vietoris sequences. Therefore, $\beta_2(D_3(T)) = 0$ and the first nontrivial case is $n = 4$.

Theorem 5.7. *Let T be a tree graph with two components S and S' (Fig. 5.10). The rank of the second homology group for n indistinguishable particles on T is*

$$\beta_2^{(n)}(S, S') = \sum_{l=2}^{n-2} \left(\beta_1^{(l)}(S) - \beta_1^{(l-1)}(S)\right) \beta_1^{(n-l)}(S'), \tag{5.12}$$

where $\beta_1^{(k)}(S)$ is the rank of the first homology group for k particles on star graph S, given in equation (5.1).

Proof. Consider two Mayer-Vietoris sequences for two consecutive subcomplexes, X_k and X'_k. We will obtain a recurrence relation for $H_2(B_k)$. Maps from the sequence for X_k are

$$\Phi_k : \; H_2(D_{n-k-1}(\tilde{S}) \times D_k(\tilde{S}') \times v) \to H_2(D_{n-k}(S) \times D_k(\tilde{S}')) \oplus H_2(B_k),$$
$$\delta_k : \; H_2(X_k) \to H_1(D_{n-k-1}(\tilde{S}) \times D_k(\tilde{S}') \times v).$$

Because $H_3(X_k) = 0$, the Mayer-Vietoris sequence implies that map Φ_k is injective. Hence, $\mathrm{im}\Phi_k \cong H_2(A_k \cap B_k)$. Moreover, $\mathrm{im}\delta_k = 0$. This is because each element of the over-complete basis of 2-cycles is a chain, which is properly contained in $D_{n-k}(\tilde{S}) \times D_k(S')$ or $D_{n-k}(S) \times D_k(\tilde{S}')$ for some k. Therefore, we have

$$H_2(B'_k) = \mathrm{coker}\Phi_k \cong \left(\left(H_1(D_{n-k}(S)) \otimes H_1(D_k(\tilde{S}'))\right) \oplus H_2(B_k)\right) / \mathrm{im}\Phi_k.$$

The quotient can be realised as follows. Any element of $\mathrm{coim}\Phi_k$ can be written as a tensor product of chains of the following form

$$[c \otimes c'] \times v : \; [c] \in H_1(D_{n-k-1}(\tilde{S})), \; [c'] \in H_1(D_k(\tilde{S}')).$$

Furthermore, each such 2-cycle can be written as 2-cycle $(c \times v) \otimes c'$, which belongs to $\mathfrak{C}_2(A_k)$, or 2-cycle $c \otimes (c' \times v)$, which belongs to $\mathfrak{C}_2(B_k)$. Map Φ_k acts on the homology classes as

$$\Phi_k([c \otimes c'] \times v) = ([(c \times v) \otimes c'], [-c \otimes (c' \times v)]).$$

On the other hand, every element of $H_2(A_k)$ can be decomposed in the basis of the tensor product

$$[\tilde{c} \otimes \tilde{c}'] : \; [\tilde{c}] \in H_1(D_{n-k}(S)), \; [\tilde{c}'] \in H_1(D_k(\tilde{S}')).$$

By the injectivity of Φ_k, cycles $(c_1 \times v) \otimes c'_1$ and $(c_2 \times v) \otimes c'_2$ represent different classes in $H_2(A_k)$ if $[c_1] \neq [c_2]$ or $[c'_1] \neq [c'_2]$. Therefore, from every element $[a] \in H_2(A_k)$ we can extract in a unique way the part, which belongs to $H_2(A_k \cap B_k)$, i.e.

$$[a] = \sum_{[c],[c']} [(c \times v) \otimes c'] + [\tilde{a}]. \tag{5.13}$$

Therefore, for a pair $([a], [b]) \in H_2(A_k) \oplus H_2(B_k)$, where $[a]$ is decomposed, as in (5.13), we have

$$([a], [b]) \sim \left([\tilde{a}], [b] + \sum_{[c],[c']} [(c \times v) \otimes c']\right)$$

under the quotient by $\mathrm{im}\Phi_k$. Moreover, pairs, where $[a] = [\tilde{a}]$, yield different equivalence classes for different a, b. This means that the quotient by $\mathrm{im}\Phi_k$ can be realised by taking the quotient by $\mathrm{coim}\Phi_k$ only on $H_2(A_k)$. In other words,

$$H_2(B_{k'}) \cong \frac{H_1(D_{n-k}(S)) \otimes H_1(D_k(\tilde{S}'))}{H_1(D_{n-k-1}(\tilde{S})) \otimes H_1(D_k(\tilde{S}'))} \oplus H_2(B_k).$$

A similar result holds for pair (A'_k, B'_k), i.e. Φ'_k is injective and $\mathrm{im}\delta'_k = 0$. The quotient by Φ'_k reads

$$H_2(B_{k-1}) \cong \frac{H_1(D_{n-k}(\tilde{S})) \otimes H_1(D_k(S'))}{H_1(D_{n-k}(\tilde{S})) \otimes H_1(D_k(\tilde{S}'))} \oplus H_2(B'_k) \cong H_2(B'_k).$$

Subtracting ranks in the equation for $H_2(B'_k)$, we obtain a recurrence relation for ranks of $H_2(B_k)$ and $H_2(B_{k-1})$

$$\beta_2(B_{k-1}) = \left(\beta_1^{(k)}(S) - \beta_1^{(k-1)}(S)\right) \beta_1^{(n-k)}(S') + \beta_2(B_k). \tag{5.14}$$

The initial condition is $\beta_2(B_{n-1}) = 0$. □

The following theorem shows how to compute the second homology group of any tree.

Theorem 5.8. *Let T be a tree graph and let a pair (S, S') denote the subgraph of T, which consists of two star graphs S and S' and the unique path in T that connects the essential vertices of S and S' (see Fig. 5.11a). Then,*

$$H_2(D_n(T)) \cong \bigoplus_{(S,S')\subset T} H_2(D_n((S, S'))).$$

Proof. The strategy for the proof is to show that every cycle from the over-complete basis of $H_2(D_n(T))$ is homologically equivalent to a 2-cycle from $D_n(S, S')$ for a pair of star subgraphs of T. Assume first that every star subgraph of T is sufficiently subdivided for n particles. This in particular means that the edges that connect essential vertices (vertices of degree greater than 2) are subdivided twice as much as the condition of sufficient subdivision requires. Every 2-cycle from the over-complete basis of $H_2(D_n(T))$ is isomorphic to a tensor product of two chains, each describing exchange of particles a Y-subgraph of T, and the remaining $n - 4$ particles distributed on free vertices of T. Let S and S' be the star-subgraphs of T that contain the two Y-subgraphs, where the particles exchange (Fig. 5.11a). The remaining particles are distributed on the remaining vertices of T. Some of them may occupy free vertices of S or S'. Assume that $k - 2$ out of free particles occupy star graph S and $l - 2$ free particles occupy S'. The remaining $n - (k + l)$ particles are distributed outside S and S'. The element of the over-complete basis of $H_2(D_n(T))$ that corresponds to such a situation is of the form

$$\sigma = (c \otimes c') \times \{v_1, \ldots, v_{n-k-l}\},\ [c] \in H_1(D_k(S)),\ [c'] \in H_1(D_l(S')),\ \{v_1, \ldots, v_{n-k-l}\} \notin S, S'.$$

We will next give a construction of a path in $D_n(T)$ that connects point $\{v_1, \ldots, v_{n-k-l}\}$ with a point, where all the particles are distributed on star graphs S and S'. To this end, remove from T subgraphs S and S' by removing star subgraphs that are sufficiently subdivided for k and l particles respectively. After removing the star subgraphs, graph

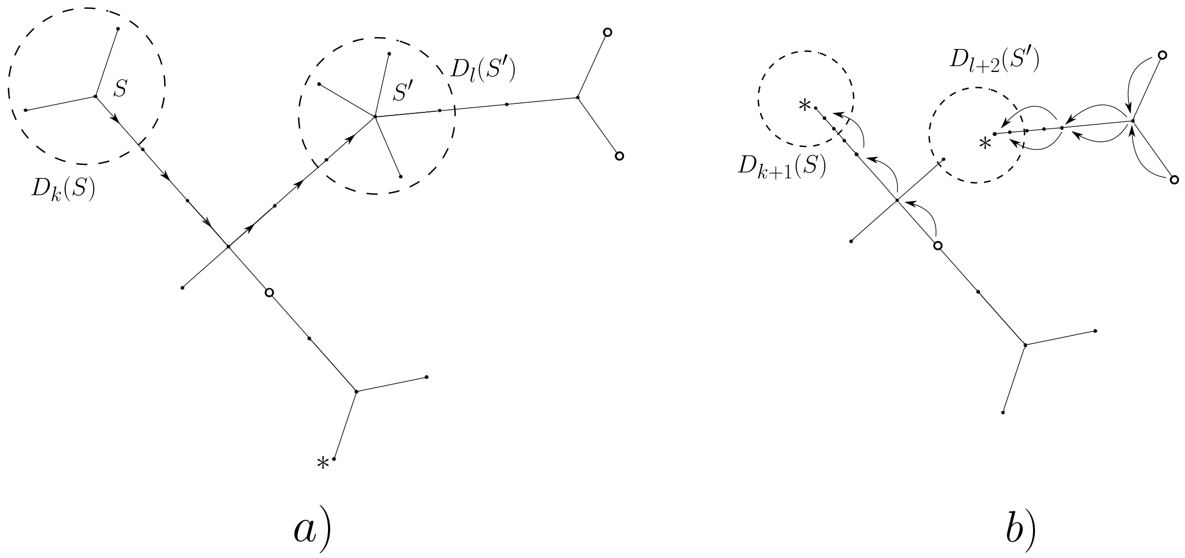


Figure 5.11: Illustration for the proof of Theorem 5.8. White vertices denote vertices that are occupied by particles from the outside of $D_k(S)$ and $D_l(S')$. Starred points denote the roots of the trees. Figure b) shows the construction of the path that brings each particle to a configuration space of one of the star subgraphs, where the particles exchange.

T decomposes into a number of connected components (see Fig. 5.11b). Each component has a number of vertices of degree one, where the star graphs were attached. Order these vertices according to their distance from the root in T. For each component, choose the new root to be the vertex, which was the closest one to the original root in T (Fig. 5.11b). Finally, move all particles in the components to the roots. The resulting configuration is a configuration, where all particles are distributed on star graphs S and S'. □

As a consequence, the rank of the second homology group for n particles reads

$$\beta_2^{(n)}(T) = \sum_{S,S' \subset T} \beta_2^{(n)}(S, S'). \tag{5.15}$$

An analogous result holds for all the higher homology groups. For the dth homology one has to take the sum over all tree subgraphs of T that contain exactly d star subgraphs, i.e.

$$H_d(D_n(T)) \cong \bigoplus_{T' \subset T:\ \#T'=d} H_d(D_n(T')).$$

The proof is the same as the proof of theorem 5.8 – for every d-cycle from the overcomplete basis decompose T by removing the star graphs that belong to T' and move the remaining particles within the components.

Hence, the problem of computing $H_d(D_n(T))$ for any tree boils down to the problem of computing the dth homology for a tree containing d essential vertices. To this end, we consider a bipartition (S, T') of such a tree, where S is one of the star graphs of degree 1 in the sense of the scheme of connections in the tree (see Fig. 5.3b), and T' is the tree with graph S removed.

Theorem 5.9. *Let T' be a tree graph with $d-1$ essential vertices. Construct a tree graph T with d essential vertices as a wedge sum of T' and a star graph S, i.e. $T =$*

$(T' \sqcup S)/\sim$, *where gluing map* $\sim$ *identifies two vertices of degree* 1 *in* T' *and* S. *The rank of the dth homology group for n indistinguishable particles on* T *is*

$$\beta_d^{(n)}(S,T') = \sum_{l=2}^{n-2} \left(\beta_1^{(l)}(S) - \beta_1^{(l-1)}(S)\right) \beta_{d-1}^{(n-l)}(T'). \tag{5.16}$$

Proof. Note first that for such a tree graph, we have $H_{d+1}(D_n(T)) = 0$, because the dimension of the corresponding Morse complex is d. Consider T as a graph of connectivity one with two components, where the components are S and T'. Vertex v connecting the components has degree 2. Next, construct the sequence of subcomplexes

$$D_n(T) = X_0 \supset X_1' \supset X_1 \supset \dots \supset X_{n-1} = \left(S \times D_{n-1}(\tilde{T}')\right) \cup D_n(T'),$$

as in the case of two star graphs. The Mayer-Vietoris sequence for each subcomplex reads

$$0 \to H_d(A_k \cap B_k) \xrightarrow{\Phi_k} H_d(A_k) \oplus H_d(B_k) \xrightarrow{\Psi_k} H_d(X_k) \xrightarrow{\delta_k} H_{d-1}(A_k \cap B_k) \to \dots$$

The above sequence splits and we have

$$H_d(X_k) = \mathrm{coker}(\Phi_k) \oplus \mathrm{coim}\delta_k,\ \ H_d(X_k') = \mathrm{coker}(\Phi_k') \oplus \mathrm{coim}\delta_k'.$$

Consider two Mayer-Vietoris sequences for two consecutive subcomplexes, X_k and X_k'. We will obtain a recurrence relation for $H_d(B_k)$, as in the proof of theorem 5.7. Maps from the sequence for X_k are

$$\Phi_k: \ H_d(D_{n-k-1}(\tilde{S}) \times D_k(\tilde{T}') \times v) \to H_d(D_{n-k}(S) \times D_k(\tilde{T}')) \oplus H_d(B_k),$$
$$\delta_k: \ H_d(X_k) \to H_{d-1}(D_{n-k-1}(\tilde{S}) \times D_k(\tilde{T}') \times v).$$

The corresponding maps for X_k' read

$$\Phi_k': \ H_d(D_{n-k}(\tilde{S}) \times D_k(\tilde{T}')) \to H_d(D_{n-k}(\tilde{S}) \times D_k(T')) \oplus H_d(B_k'),$$
$$\delta_k': \ H_d(X_k') \to H_{d-1}(D_{n-k}(\tilde{S}) \times D_k(\tilde{T}')).$$

Again, from the construction of the over-complete basis, every d-cycle from the basis is contained in A_k or B_k, hence $\mathrm{im}\delta_k = 0$ and $\mathrm{im}\delta_k' = 0$ for all k. Hence, the homology groups of the subcomplexes are

$$H_d(B_k') \cong \left((H_1(D_{n-k}(S)) \otimes H_{d-1}(D_k(\tilde{T}'))) \oplus H_d(B_k)\right)/\mathrm{im}\Phi_k,$$
$$H_d(B_{k-1}) \cong \left((H_1(D_{n-k}(\tilde{S})) \otimes H_{d-1}(D_k(T'))) \oplus H_d(B_k')\right)/\mathrm{im}\Phi_k'.$$

As in the proof of theorem 5.7, the above quotients can be realised by taking the quotient by $\mathrm{coim}\Phi_k$ and $\mathrm{coim}\Phi_k'$ on $H_d(A_k)$ and $H_d(A_k')$ respectively. By doing so, we get $H_d(B_{k-1}) \cong H_d(B_k')$ and

$$H_d(B_k') \cong H_d(B_k') \oplus \frac{(H_1(D_{n-k}(S)) \otimes H_{d-1}(D_k(\tilde{T}'))}{H_1(D_{n-k-1}(\tilde{S})) \otimes H_{d-1}(D_k(\tilde{T}'))}.$$

This gives us the following recursive equation for $\beta_m(B_k)$

$$\beta_d(B_{k-1}) = \beta_d(B_k) + \left(\beta_1^{(n-k)}(S) - \beta_1^{(n-k-1)}(S)\right) \beta_{d-1}^{(k)}(T')$$

with the initial condition $\beta_d(B_{n-1}) = 0$. The solution is equation (5.16). □

Note that equation (5.16) allows one to express $\beta_d^{(n)}(T)$ for a tree with d essential vertices by the ranks of the first homology groups for different numbers of particles on the star subgraphs contained in T. To this end, one has to apply equation (5.16) recursively, until all star subgraphs of T are removed. One can check by a straightforward calculation that the solution to such a recursion with the initial condition $\beta_1^{(n)}(T) = \beta_1^{(n)}(S)$ is

$$\beta_d^{(n)}(T) = \sum_{i=0}^{d-1}(-1)^i \binom{d-1}{i} \sum_{l_1+\dots+l_d=n-i, l_j\geq 2} \beta_1^{(l_1)}(S^{(1)})\beta_1^{(l_2)}(S^{(2)})\dots\beta_1^{(l_d)}(S^{(d)}), \quad (5.17)$$

where $\{S^{(j)}\}_{j=1}^d$ is the set of all star subgraphs from T.

5.3.4. Two graphs connected by a single edge

In this subsection we continue the line of thought from the previous subsections and show how to compute the homology groups for configurations of n particles on a graph, which consists of two arbitrary graphs, that are connected by a single edge. In other words, vertex v on figure 5.6 has degree 2. The results of this subsection can be viewed as another generalisation of the formula for the second homology group for two particles on such a graph from [29]. As in previous subsections, we consider the Mayer-Vietoris sequence for pairs (A_k, B_k), (A'_k, B'_k) from the configuration space diagram, where

$$D_n(\Gamma_1) = X_0 \supset X'_1 \supset X_1 \supset \dots \supset X_{n-1} = \left(\Gamma_1 \times D_{n-1}(\tilde{\Gamma}_2)\right) \cup D_n(\Gamma_2). \quad (5.18)$$

The sequences for decomposition $X_k = A_k \cup B_k$ are of the form

$$\dots \to H_{d+1}(X_k) \xrightarrow{\delta_k} H_d(A_k \cap B_k) \xrightarrow{\Phi_k} H_d(A_k) \oplus H_d(B_k) \xrightarrow{\Psi_k} H_d(X_k) \xrightarrow{\delta_k} \quad (5.19)$$
$$\xrightarrow{\delta_k} H_{d-1}(A_k \cap B_k) \to \dots$$

In paper [10] we conjectured, that in all such sequences, we have $\mathrm{im}\delta = 0$. This fact has been proven independently by considering homology groups of the corresponding Świątkowski complex [61]. Namely, in the Świątkowski complex all cycles are products of cycles, that are entirely contained in $S_k(\Gamma_1)$ and $S_l(\Gamma_2)$. This implies, that cycles in $D_n(\Gamma)$ behave in an analogous way. Hence, we have the following lemma.

Lemma 5.10. *Consider $D_n(\Gamma)$ for Γ consisting of two arbitrary graphs connected by a single edge. Let X_k be any subcomplex in the decomposition (5.18) of $D_n(\Gamma)$. The boundary maps from the Mayer-Vietoris sequence for X_k*

$$\delta: \; H_d(X_k) \to H_{d-1}(A_k \cap B_k), \; d \geq 2.$$

have a trivial image.

By the above lemma, all homology groups of $D_n(\Gamma)$ are determined by the homology groups of the configuration spaces of the components by a collection of short exact sequences.

$$0 \to H_d(A_k \cap B_k) \xrightarrow{\Phi_k} H_d(A_k) \oplus H_d(B_k) \xrightarrow{\Psi_k} H_d(X_k) \to 0, \quad (5.20)$$
$$0 \to H_d(A'_k \cap B'_k) \xrightarrow{\Phi'_k} H_d(A'_k) \oplus H_d(B'_k) \xrightarrow{\Psi'_k} H_d(X'_k) \to 0.$$

The above short exact sequences immediately give

$$H_d(X_k) \cong \mathrm{coker}\Phi_k, \quad H_d(X'_k) \cong \mathrm{coker}\Phi'_k.$$

Theorem 5.11. *Let Γ be a wedge sum of Γ_1 and Γ_2, where gluing map identifies two vertices of degree 1 in Γ_1 and Γ_2. Then, the Betti numbers for n indistinguishable particles on Γ can be expressed as*

$$\beta_d^{(n)}(\Gamma) = \beta_d^{(n)}(\Gamma_2) + \beta_d^{(n)}(\Gamma_1) + \sum_{k=1}^{n} \sum_{p+r=d} \left(\beta_p^{(k)}(\Gamma_1) - \beta_p^{(k-1)}(\Gamma_1)\right) \beta_r^{(n-k)}(\Gamma_2). \quad (5.21)$$

Proof. In order to find the desired relation for Betti numbers, it is easier to work with the homology groups with coefficients in $\mathbb{Q}$. Using the fact, that $D_k(\tilde{\Gamma}_i) \cong D_k(\tilde{\Gamma}_i)$ for $i = 1, 2$ and for all k, we have, that $H_d(B_{k-1}) \cong H_d(B_k')$ and

$$H_d(X_k') = H_d(B_k') \cong H_d(B_k) \oplus \frac{\bigoplus_{p+r=d} H_p(D_{n-k}(\Gamma_1)) \otimes H_r(D_k(\Gamma_2))}{\bigoplus_{p+r=d} H_p(D_{n-k-1}(\Gamma_1)) \otimes H_r(D_k(\Gamma_2))}.$$

The sequence for X_k yields $H_2(B_{k-1}) \cong H_2(B_k')$. From these equations we obtain the following recurrence relation for the rank of $H_d(B_{k-1})$.

$$\beta_d(B_{k-1}) = \beta_d(B_k) + \sum_{p+r=d} \left(\beta_p^{(n-k)}(\Gamma_1) - \beta_p^{(n-k-1)}(\Gamma_1)\right) \beta_r^{(k)}(\Gamma_2)$$

for $k = 0, 1, \ldots, n-1$. The initial condition is $B_{n-1} = D_n(\Gamma_2)$. Solving the recurrence and using the fact that $\sum_{k=0}^{n-1} \left(\beta_d^{(n-k)}(\Gamma_1) - \beta_d^{(n-k-1)}(\Gamma_1)\right) = \beta_d(D_n(\Gamma_1))$ we obtain equation (5.21). □

5.3.5. Inheritance of torsion

As one can see from the Künneth formula, there are two mechanisms of inheritance of torsion in the homology of $D_n(\Gamma_1 \vee \Gamma_2)$ from the homology of configuration spaces of Γ_1 and Γ_2. Consider first the situation from subsection 5.3.4, where graphs Γ_1 and Γ_2 are connected by a single edge. From the decomposition (5.18) and short exact sequences (5.20), we obtain the recursive formula

$$H_d(B_k') \cong H_d(B_k) \oplus \frac{H_d(D_{n-k}(\Gamma_1) \times D_k(\Gamma_2))}{H_d(D_{n-k-1}(\Gamma_1) \times D_k(\Gamma_2))}, \quad k = 0, 1, \ldots, n-1. \quad (5.22)$$

with

$$B_{n-1} = D_n(\Gamma_2) \text{ and } H_d(D_n(\Gamma)) \cong \frac{H_d(D_n(\Gamma_1))}{H_d(D_{n-1}(\Gamma_1))} \oplus H_d(B_0).$$

The two types of torsion are denoted by T_1 and T_2.

$$T(H_d(B_k')) \cong T(H_d(B_k)) \oplus T_1^{(k)} \oplus T_2^{(k)}.$$

The first type of torsion is the torsion stemming from the tensor product of the homology groups of components that multiply to order d.

$$T_1^{(k)} = \frac{\bigoplus_{p+r=d} T\left(H_p(D_{n-k}(\Gamma_1)) \otimes H_r(D_k(\Gamma_2))\right)}{\bigoplus_{p+r=d} T\left(H_p(D_{n-k-1}(\Gamma_1)) \otimes H_r(D_k(\Gamma_2))\right)}.$$

Torsion T_1 can be interpreted as torsion stemming from embedding in $D_n(\Gamma)$ a subcomplex, which has torsion in its homology. The particular realisation is embedding of a graph Γ' in Γ_1 or Γ_2, which has torsion in $H_p(D_k(\Gamma'))$ for some p and k. By considering tensor products of p-cycles from $D_k(\Gamma')$ with some disjoint $(d-p)$-cycles in $D_{n-k}(\Gamma-\Gamma')$, we get a homomorphism $T(H_p(D_k(\Gamma'))) \to T(H_d(D_n(\Gamma)))$. The second type of torsion is a less obvious contribution inherited from $(d-1)$-homology of $D_n(\Gamma)$

$$T_2^{(k)} = \frac{\bigoplus_{p+r=d-1} T\left(H_p(D_{n-k}(\Gamma_1))\right) \otimes T\left(H_r(D_k(\Gamma_2))\right)}{\bigoplus_{p+r=d-1} T\left(H_p(D_{n-k-1}(\Gamma_1))\right) \otimes T\left(H_r(D_k(\Gamma_2))\right)}.$$

Torsion T_2 appears even in configuration spaces with low numbers of particles, when embedding a proper subcomplex in $D_n(\Gamma)$ to obtain T_1 is not possible. As an example of such a situation, consider graph Γ, which is constructed by joining two graphs K_5 with an edge (Fig. 5.12) and compute $H_3(D_4(\Gamma))$.

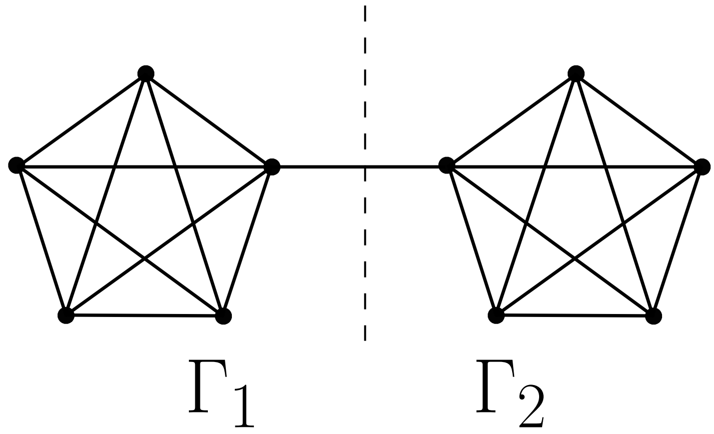


Figure 5.12: Two graphs K_5 connected by a single edge.

Because $H_3(D_3(\Gamma_i)) = 0$ and $H_2(D_2(\Gamma_i)) = 0$, recurrence (5.22) gives

$$\begin{aligned} H_3(D_4(\Gamma)) = H_3(D_4(\Gamma_1)) \oplus (H_2(D_3(\Gamma_1)) \otimes H_1(D_1(\Gamma_2))) \oplus \\ \oplus (H_1(D_1(\Gamma_1)) \otimes H_2(D_3(\Gamma_2))) \oplus H_3(D_4(\Gamma_1)) \oplus (T(H_1(D_2(\Gamma_1))) \otimes T(H_1(D_2(\Gamma_2)))) = \\ = \mathbb{Z} \oplus \left(\mathbb{Z}^{39} \otimes \mathbb{Z}^6\right) \oplus \left(\mathbb{Z}^6 \otimes \mathbb{Z}^{39}\right) \oplus \mathbb{Z} \oplus \mathbb{Z}_2 = \mathbb{Z}^{470} \oplus \mathbb{Z}_2. \end{aligned}$$

The torsion component comes from T_2 and there is no T_1-torsion. Homology groups $H_2(D_3(\Gamma_i)) = \mathbb{Z}^{39}$ and $H_3(D_4(\Gamma_1)) = \mathbb{Z}$ were computed via the discrete Morse theory.

5.3.6. When is imδ nontrivial?

The main obstacle in the continuation of this section's approach in a rigorous way is the knowledge of cycles that do not belong to $\ker\delta$. In this subsection we provide an example of such cycles for $D_n(\Gamma)$. We conjecture, that the types of cycles described in this subsection are all possible cycles that do not belong to $\ker\delta$. The first class of cycles appears while considering a simultaneous exchange of particles in $D_k(\tilde{\Gamma}_1)$, $D_l(\tilde{\Gamma}_2)$ and on a Y-subgraph centered at vertex v, see Fig. 5.13. Such a cycle is isomorphic to the tensor product of cycles $z = c \otimes c' \otimes c_Y$, where $c \in \mathfrak{C}_p(D_k(\tilde{\Gamma}_1))$, $c' \in \mathfrak{C}_s(D_l(\tilde{\Gamma}_2))$ and

$$c_Y = \{e_v^{u'}, u\} + \{e_u^v, u'\} + \{e_v^{u''}, u'\} - \{e_v^{u'}, u''\} - \{e_u^v, u''\} - \{e_v^{u''}, u\}.$$

We will next show that the homology class $[z]$ is a nonzero element of coimδ'_{l+1} from the Mayer-Vietoris sequence describing complex X_{l+1} (defined as in section 5.3.3 and 5.3.4). We have

$$\delta'_{l+1}:\ H_{p+s+1}(X_{l+1}) \to H_{p+s}(D_{k+1}(\tilde{\Gamma}_1) \times D_{l+1}(\tilde{\Gamma}_2)).$$

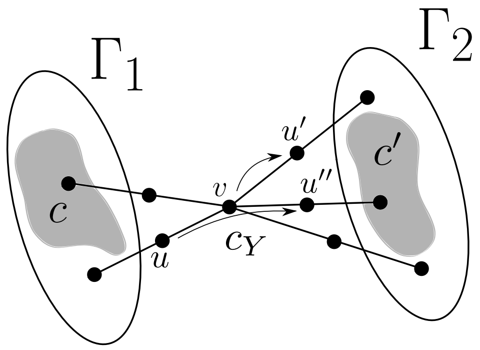


Figure 5.13: A cycle, for which $\delta[c \otimes c' \otimes c_Y] \neq 0$ for the boundary map in the proper Mayer-Vietoris seqience. Chain c_Y corresponds to the exchange of two particles on the Y-subgraph centered at v and spanned on vertices u, u', u''. Chains c, c' are cycles of arbitrary dimensions that are contained in $D_k(\tilde{\Gamma}_1)$ and $D_l(\tilde{\Gamma}_2)$ respectively.

The decomposition $z = x + y$, $x \in \mathfrak{C}_{p+s+1}(A'_k)$, $y \in \mathfrak{C}_{p+s+1}(B'_k)$ yields

$$x = c \otimes c' \otimes \left(\{e_v^{u'}, u\} - \{e_v^{u''}, u\}\right) \in \mathfrak{C}_{p+s+1}(D_{k+1}(\tilde{\Gamma}_1) \times D_{l+1}(\Gamma_2)),$$

$$y = c \otimes c' \otimes \left(\{e_u^v, u'\} + \{e_v^{u''}, u'\} - \{e_v^{u'}, u''\} - \{e_u^v, u''\}\right) \in \mathfrak{C}_{p+s+1}(D_{k+1}(\Gamma_1) \times D_{l+1}(\tilde{\Gamma}_2)).$$

For the boundary of a 1-cell $\partial\{e_a^b, c\} = \{b, c\} - \{a, c\}$, we have $\partial' x = -\partial' y = c \otimes c' \otimes (\{u', u\} - \{u'', u\}) \neq 0$.

We conjecture the only nontrivial elements from imδ are the ones that involve exchanges of particles on Y-subgraphs centred at vertex v, as described above. Based on such a conjecture, the methods from this section can be used *mutatis mutandis* to express $H_m(D_n(\Gamma))$ of an arbitrary graph of connectivity one by the homology groups of its higher-connected components.

5.4. Wheel graphs

In this section, we deal with the class of wheel graphs. A wheel graph of order m is a simple graph that consists of a cycle on $m - 1$ vertices, whose every vertex is connected by an edge (called a spoke) to one central vertex (called the hub). We provide a complete description of the homology groups of configuration spaces for wheel graphs. In particular, we show, that all homology groups are free. Therefore, in addition to tree graphs, wheel graphs provide another family of configuration spaces with a simplified structure of the set of flat complex vector bundles. The general methodology of computing homology groups for configuration spaces of wheel graphs is to consider only the product cycles and describe the relations between them. We justify this approach in subsection 5.4.3.

The simplest example of a wheel graph is graph K_4, which is the wheel graph of order 4. Let us next calculate all homology groups of graph K_4 and then present the general method for any wheel graph.

5.4.1. Graph K_4

Graph K_4 is shown on figure 5.14. It is the 3-connected, complete graph on 4 vertices.

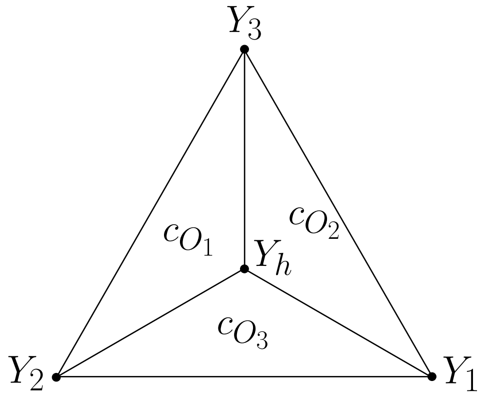


Figure 5.14: Graph K_4 and the relevant Y-subgraphs and cycles. We omit the subdivision of edges in the picture.

Second homology group

There are three independent cycles in K_4 graph. These are the cycles that contain the hub and two neighbouring vertices from the perimeter. However, any two such cycles always share some vertices. Hence, there are no tori that come from the products of c_O cycles. Hence, the product 2-cycles are either $c_Y \otimes c_O$ or $c_Y \otimes c_{Y'}$. There are four cycles of the first kind: $c_{Y_1} \otimes c_{O_1}$, $c_{Y_2} \otimes c_{O_2}$, $c_{Y_3} \otimes c_{O_3}$ and $c_{Y_h} \otimes c_O$, where c_O is the outermost cycle. However, cycle $c_{Y_h} \otimes c_O$ can be expressed as a linear combination of cycles $c_{Y_1} \otimes c_{O_1}$, $c_{Y_2} \otimes c_{O_2}$, $c_{Y_3} \otimes c_{O_3}$. Therefore, the second homology of the three-particle configuration space is

$$H_2(D_3(K_4)) = \mathbb{Z}^3.$$

If $n > 3$, there are still three independent $O \times Y$-cycles, as the differences between distributions of free particles in such cycles can always be expressed as combinations of $Y \times Y$-cycles. To see this, consider the following example. For $n = 4$, consider the $O \times Y$-cycles that involve cycle c_{O_1}, subgraph Y_1 and one of three possible free vertices (Fig. 5.15). The cycles are $c_{Y_1} \otimes c^u_{AB}$, $c_{Y_1} \otimes c^v_{AB}$, $c_{Y_1} \otimes c^w_{AB}$, where $c^v_{AB} := c_{O_1} \times v$. From (4.5) we have

$$c_{Y_2} \sim c_2 + c^v_{AB}, \; c_{Y_3} \sim c_2 + c^w_{AB}, \; c_{Y_h} \sim c_2 + c^u_{AB}.$$

Subtracting the above equations and multiplying the results by c_{Y_1}, we get

$$c_{Y_1} \otimes c_{Y_h} - c_{Y_1} \otimes c_{Y_2} \sim c_{Y_1} \otimes c^u_{AB} - c_{Y_1} \otimes c^v_{AB},$$
$$c_{Y_1} \otimes c_{Y_h} - c_{Y_1} \otimes c_{Y_3} \sim c_{Y_1} \otimes c^u_{AB} - c_{Y_1} \otimes c^w_{AB}.$$

This means that the differences between distribution of particles in AB-cycles can be expressed as combinations of $Y \times Y$ cycles. This fact generalises to $n > 4$ in a straightforward way.

Consider next all possible ways of choosing two Y-subgraphs. There are six $Y \times Y$-cycles modulo the distribution of free particles. Hence, if there are no free particles, i.e. when $n = 4$, we have

$$H_2(D_4(K_4)) = \mathbb{Z}^3 \oplus \mathbb{Z}^6.$$

If $n > 4$, we have to take into account the distribution of free particles in $\Gamma - (Y \cup Y')$. For a sufficiently subdivided graph one always ends up with two connected components (Fig. 5.16). A $Y \times Y$-cycle involves 4 particles, hence one has to calculate the number of

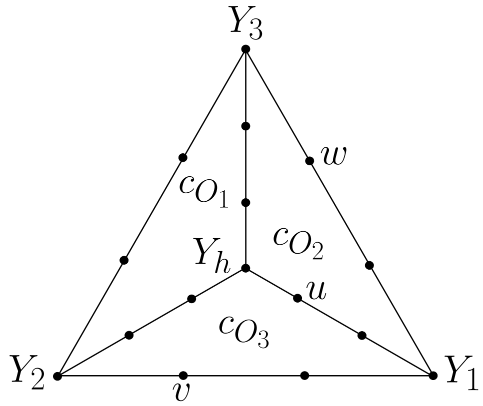


Figure 5.15: Graph K_4 subdivided for $n = 4$. Differences $c^u_{AB} - c^v_{AB}$ and $c^u_{AB} - c^w_{AB}$ are homologically equivalent to combinations of $Y \times Y$-cycles. $c_{Y_1} \otimes c_{Y_h} - c_{Y_1} \otimes c_{Y_2}$ and $c_{Y_1} \otimes c_{Y_h} - c_{Y_1} \otimes c_{Y_3}$ respectively.

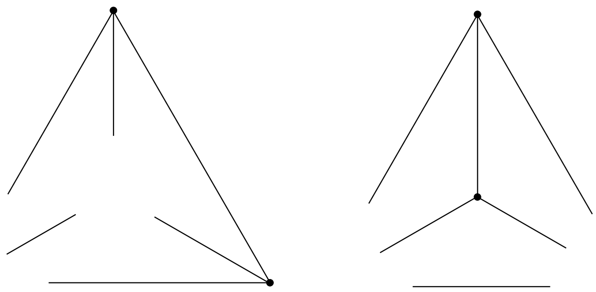


Figure 5.16: Graph K_4 after removing two Y-subgraphs.

all possible distributions of $n-4$ particles on those two components times the number of possible choices of the two Y-subgraphs. The number of all choices of the Y-subgraphs is $\binom{4}{2}$, while the number of possible distributions of $n-4$ particles on 2 components is $\binom{n-4+2-1}{2-1} = n-3$. Hence, the contribution from $Y \times Y$ cycles reads

$$\binom{4}{2}(n-3) = 6(n-3), \ n \geq 4.$$

Adding the contribution from $O \times Y$-cycles, the rank of the second homology group is then given by

$$\beta_2(C_n(K_4)) = 3 + 6(n-3) = 6n - 15, \ n \geq 3.$$

Higher homology groups

The product generators of higher homologies are even simpler than in the case of the second homology. There are only basis cycles of $Y \times Y \times \cdots \times Y$-type. After removing three and four Y-graphs, K_4 graph always disintegrates into 4 and 6 parts respectively. Taking into account the distributions of free particles, we get the following formulae for the Betti numbers.

$$\beta_3(C_n(K_4)) = \binom{4}{3}\binom{n-6+4-1}{4-1} = 4\binom{n-3}{3}, \ n \geq 6$$
$$\beta_4(C_n(K_4)) = \binom{4}{4}\binom{n-8+6-1}{6-1} = \binom{n-3}{5}, \ n \geq 8.$$

Because there are maximally four Y-graphs, group $H_5(C_n(K_4), \mathbb{Z})$ is zero.

5.4.2. Other wheel graphs

In Table 5.3 we list Betti numbers of configuration spaces of wheel graphs of order 5, 6 and 7 that were calculated using the discrete Morse theory.

Γ	n	$\beta_2(D_n(\Gamma))$	$\beta_3(D_n(\Gamma))$	$\beta_4(D_n(\Gamma))$
W_5	3	8	0	-
	4	22	0	0
	5	34	4	0
	6	46	30	0
	7	58	90	0
	8	70	196	13
W_6	3	15	0	-
	4	40	0	0
	5	60	15	0
	6	80	90	0
	7	100	250	5
W_7	3	24	0	-
	4	63	0	0
	5	93	36	0
	6	123	197	0
	7	153	527	24

Table 5.3: Betti numbers of configuration spaces for chosen wheel graphs computed using the discrete Morse theory. In all cases the calculated groups were torsion-free.

Second homology

Since there are no pairs of disjoint O-cycles in wheel graphs, we have

$$\beta_2(D_2(W_m)) = 0.$$

When $n = 3$, all product cycles are the $O \times Y$-cycles. Their number is $(m-1)(m-3)$, because there are $m-1$ choices of Y-subgraphs and $m-3$ cycles that are disjoint with a fixed Y-subgraph. Hence,

$$\beta_2(D_3(W_m)) = (m-1)(m-3).$$

When $n = 4$, we have to count the $Y \times Y$ cycles in. Let us divide the $Y \times Y$ cycles into two groups: i) cycles, where one of the subgraphs is Y_h and ii) cycles, where both subgraphs lie on the perimeter. There are no relations between the cycles within group i) and no relations between the cycles within group ii). However, there are some relations between the cycles of type i) and type ii). The relations occur between cycles $Y_h \times Y$ and $Y' \times Y$, when subgraphs Y_h and Y do not share any edges of the graph (like on Fig. 5.17b)). Then, as on Fig. 4.6, cycles c_{Y_h} and $c_{Y'}$ are in the same homology class in $D_2(W_m - Y)$, because they share the same O-cycle and they are connected by a path that is disjoint with Y. Therefore, by multiplying the relation by c_Y we get that

$$c_{Y_h} \times c_Y \sim c_{Y'} \times c_Y.$$

If $m > 4$, then for every pair $Y \times Y_h$ that does not share an edge, one can find subgraph

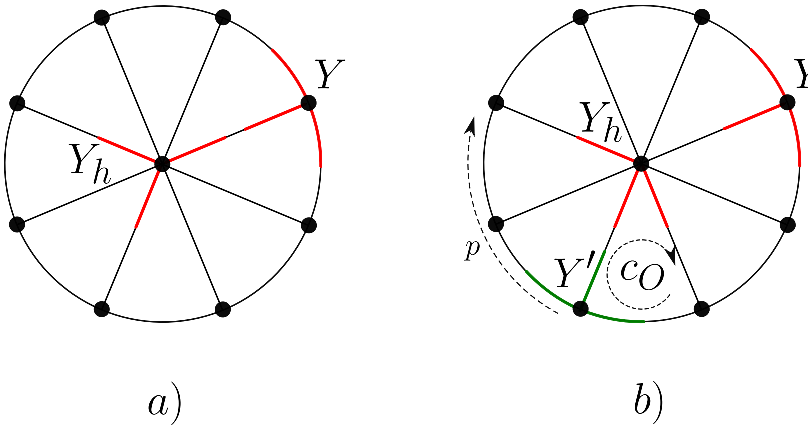


Figure 5.17: Relations between different pairs of $Y_h \times Y$-cycles in a wheel graph. a) Cycles, where Y_h and Y share an edge of the graph are independent. b) Cycle, where Y_h and Y do not share any edges is in the same homology class as cycle $Y' \times Y$.

Y' on the perimeter which gives rise to such a relation. There are $\binom{m-1}{2}$ tori coming from Y-subgraphs from the perimeter. For a fixed Y-subgraph, the contribution from $Y \times Y_h$-cycles turns out to be equal to the number of independent cycles in the fan graph, which is formed by removing subgraph Y from the wheel graph [20]. This number is equal to $m-3$. Hence,

$$\beta_2(D_4(W_m)) = 2(m-1)(m-3) + \binom{m-1}{2} = \frac{(m-1)(5m-14)}{2}.$$

For numbers of particles greater than 4, we have to take into account the distribution of free particles. Removing two Y-subgraphs from the perimeter may result with the decomposition of the wheel graph into at most two components. This happens iff two neighbouring Y-subgraphs have been removed. The number of nonequivalent ways of distributing the particles is $n-3$. The number of ways one can choose two neighbouring Y-subgraphs from the perimeter is $m-1$. This gives us the contribution of $(n-3)(m-1)$. Furthermore, removing a Y-subgraph from the hub and a subgraph from the perimeter always yields two nonequivalent ways of distributing the free particles. The first one being the edge e joining the hub and the central vertex of Y, the second one being the remaining part of the graph, i.e. $W_m - (Y \sqcup Y_h \sqcup e)$. The contribution is $(n-3)(m-1)(m-3)$. Adding the contribution from $O \times Y$-cycles and from non-neighbouring $Y_p \times Y_p$-cycles, we get that the final formula for the second Betti number reads

$$\beta_2(D_n(W_m)) = (n-2)(m-1)(m-3) + (m-1)(n-4) + \binom{m-1}{2}, \; n \geq 4.$$

Higher homologies

In computing the higher homology groups, we proceed in a similar fashion as in the previous section. However, the combinatorics becomes more complicated and in most cases it is difficult to write a single formula that works for all wheel graphs. Let us start with an example of $H_3(D_n(W_5))$. The possible types of product cycles are $O \times Y \times Y'$ and $Y \times Y' \times Y''$. Cycles of the first type arise in W_5 only when graphs Y and Y' are neighbouring subgraphs from the perimeter. There are four possibilities for such a choice of Y-subgraphs, hence

$$\beta_3(D_5(W_5)) = 4.$$

When $n > 5$, the free particles can be placed either on the edge joining the Y-subgraphs or on the connected part of W_5 that is created by removing subgraphs Y and Y'. By arguments analogous to the ones presented in section 5.4.1, the distribution of free particles on the connected component containing cycle O does not play a role. Hence, the contribution to β_3 is equal to the number of different distributions of free particles on the edge connecting Y and Y' and on the connected component. In other words, there are two bins and $n-5$ free particles. Hence, the total contribution from $O \times Y \times Y'$-cycles is $4(n-4)$. We split the contribution from $Y \times Y' \times Y''$-cycles into two groups. The first group consists of cycles only from perimeter ($Y_p \times Y_p' \times Y_p''$), for whom the combinatorial description is straightforward. The number of possible choices of Y-subgraphs is $\binom{4}{3}$ and it always results with the decomposition of W_5 into 3 components. Hence, with $n-6$ free particles the number of independent $Y_p \times Y_p' \times Y_p''$-cycles is $4\binom{n-4}{2}$. In order to determine the number of independent cycles $Y_p \times Y_p' \times Y_h$ (two subgraphs from the perimeter and one from the hub), one has to consider different graphs that arise after removing two Y-subgraphs from the perimeter of W_5. The number of independent Y_h-cycles for a fixed choice of Y_p and Y_p' is the same as in a certain fan graph, which is determined by the choice of the Y_p-subgraphs. Choosing Y_p and Y_p' to lie on the opposite sides of the diagonal of W_5, the resulting fan graph is the star graph S_4. The free particles outside Y_p and Y_p' can always be moved to the S_4-subgraph. Hence, the contribution from such cycles is given by the number of independent Y-cycles in S_4 for $n-4$ particles. We denote this number by $\beta_1^{(n-4)}(S_4)$. The last group of cycles that we have to take into account are $Y_p \times Y_p' \times Y_h$, where Y_p and Y_p' are neighbouring subgraphs. The resulting fan graph is shown on Fig. 5.18. The $n-4$ particles that do not exchange on the perimeter subgraphs are distributed between the fan graph and the edge joining Y_p and Y_p'. There have to be at least 2 particles exchanging on a Y_h-subgraph of the fan graph. The number of independent Y_h-cycles for $k+2$ particles on the fan graph is given in the caption under Fig. 5.18. After summing all the above contributions, the final formula for the third Betti number reads

$$\beta_3(D_n(W_5)) = 4(n-4)+4\binom{n-4}{2}+2\beta_1^{(n-4)}(S_4)+4\sum_{k=0}^{n-6}\left(\beta_1^{(k+2)}(S_3)+\binom{k+3}{k+1}-1\right).$$

The fourth Betti number is easier to compute, because removing three Y_p-subgraphs always results with the same type of fan graph. This fan graph has no cycles, hence there are no $O \times Y \times Y \times Y$-cycles. Moreover, there is only one possible choice of four Y-subgraphs from the perimeter. This always results with the decomposition of W_5 into 5 components. Choosing three Y-subgraphs from perimeter results with the decomposition of W_5 into 3 components: a fan graph and 2 edges. The number of independent Y_h cycles in the fan graph is the same as in S_4. Taking into account the distribution of $n-6$ particles between the two edges and the fan graph, we have

$$\beta_4(D_n(W_5)) = \binom{n-4}{4} + 4\sum_{k=0}^{n-8}(n-k-7)\beta_1^{(k+2)}(S_4),\ n \geq 8.$$

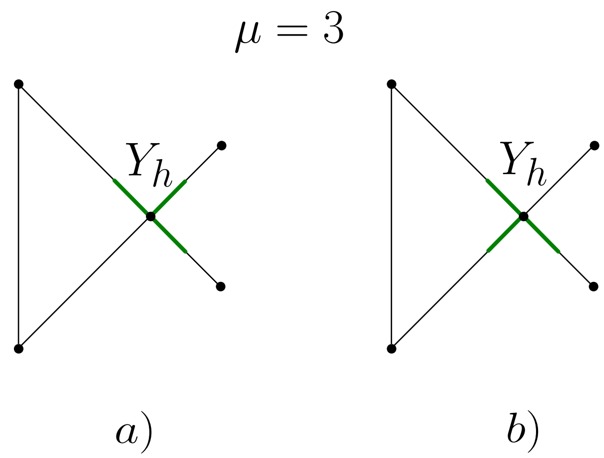


Figure 5.18: The fan graph that is created after removing two neighbouring Y-subgraphs from the perimeter of W_5. It has $\mu = 3$ leaves. There are two types of Y-cycles at the hub: a) cycles, where the Y-graph is spanned in three different leaves - the number of such cycles for $k+2$ particles is $\beta_1^{(k+2)}(S_3)$, b) cycles, where the Y-graph is spanned in two different leaves - the number of such cycles for $k+2$ particles is $\binom{k+3}{k+1} - 1$, see [20].

The top homology for $D_n(W_5)$ is H_5. Distributing $k+2$ particles on the central S_4 graph and the remaining particles on four free edges joining Y_p-subgraphs, we get

$$\beta_5(D_n(W_5)) = \sum_{k=0}^{n-10} \binom{n-k-7}{3} \beta_1^{(k+2)}(S_4), \ n \geq 10.$$

Let us next generalise the above procedure to an arbitrary wheel graph W_m. The dth Betti number is zero whenever the number of particles is less than $2(d-1)+1 = 2d-1$. If $n = 2d-1$ the only possible tori come from the products of $d-1$ Y-cycles and one O-cycle. The graph also cannot be too small, i.e. the condition $m-3 \geq d-1$ must be satisfied. Otherwise, there is no cycle that is disjoint with $d-1$ Y-subgraphs. Hence,

$$\beta_d(D_n(W_m)) = 0 \text{ if } n < 2d-1 \text{ and } \beta_d(D_{2d-1}(W_m)) = 0 \text{ if } m < d+2. \tag{5.23}$$

Otherwise, for $n = 2d-1$, if the graph is large enough, one has to look at all the possibilities of removing Y-subgraphs from the perimeter and what fan graphs are created. We are interested in the number of leaves (μ) of the resulting fan graph. The number of cycles in such a fan graph with μ leaves is $m-1-\mu$. It is a difficult task to list all possible fan graphs for any W_m in a single formula. The results for graphs up to W_7 are shown in Table 5.4. Using the notation from Table 5.4, the general formula for β_d reads

$$\beta_d(D_{2d-1}(W_m)) = \sum_{\mathbf{n}:|\mathbf{n}|=d-1} N_{\mathbf{n}}(m-1-\mu_{\mathbf{n}}), \tag{5.24}$$

where $|\mathbf{n}| := \sum_{i=1}^{l} n_i$.

Γ	Groups of Y-subgraphs - $\mathbf{n}$	Number of possible choices - $N_{\mathbf{n}}$	Number of leaves - $\mu_{\mathbf{n}}$
W_5	(1)	4	1
	(1,1)	2	4
	(2)	4	3
	(3)	4	4
	(4)	1	4
W_6	(1)	5	2
	(1,1)	5	4
	(2)	5	3
	(2,1)	5	5
	(3)	5	4
	(4)	5	5
	(5)	1	5
W_7	(1)	6	2
	(1,1)	9	4
	(2)	6	3
	(1,1,1)	2	6
	(2,1)	12	5
	(3)	6	4
	(2,2)	3	6
	(3,1)	6	6
	(4)	6	5
	(5)	6	6
	(6)	1	6

Table 5.4: The possibilities of choosing a number of Y-subgraphs from the perimeter of a wheel graph. The groups of Y-subgraphs are denoted by sequences $(n_1, n_2, \ldots, n_l)$, where $l + \sum_{i=1}^{l} n_i \leq m - 1$. A group n_i means that n_i neighbouring Y-subgraphs were chosen. The groups have to be separated by at least one spoke. For a fixed set of groups there are many possibilities for distributing the remaining Y-subgraphs. The number of possibilities is written in the third column. The number of leaves of the resulting fan graph is written in the fourth column. It is independent on the distribution of the remaining Y-subgraphs and is given by $\mu_{\mathbf{n}} = \min\left(m-1, l + \sum_{i=1}^{l} n_i\right)$.

eq:wheel-zeroeq:wheel-lowwheel-2dwheel-full For higher numbers of particles, one has to take into account the $Y \times Y \times \cdots \times Y$ cycles and distribution of free particles. If $n = 2d$, the free particles are only in $O \times Y \times Y \times \cdots \times Y$-cycles, where they are distributed between the edges that come from removing a group of Y-subgraphs. Group n_i gives $n_i - 1$ edges. Hence, groups $(n_1, \ldots, n_l)$ give $|\mathbf{n}| - l$ edges. The final formula reads

$$\beta_d(D_{2d}(W_m)) = \binom{m-1}{d} + \tag{5.25}$$
$$+ \sum_{\mathbf{n}:|\mathbf{n}|=d-1} N_{\mathbf{n}} \Big((m-1-\mu_{\mathbf{n}})(d-1-\#\mathbf{n}) + \beta_1^{(2)}(S_{\mu_{\mathbf{n}}}) + (\mu_{\mathbf{n}}-1)(m-1-\mu_{\mathbf{n}}) \Big),$$

where $\#\mathbf{n}$ is the number of groups in $\mathbf{n}$ (the length of vector $\mathbf{n}$). The contribution $\beta_1^{(2)}(S_{\mu_{\mathbf{n}}}) + (\mu_{\mathbf{n}} - 1)(m - 1 - \mu_{\mathbf{n}})$ comes from the number of independent Y_h-cycles in the relevant fan graph. The general formula when $n > 2d$ reads as follows.

$$\begin{aligned}\beta_d(D_n(W_m)) = &\sum_{\mathbf{n}:|\mathbf{n}|=d-1} N_{\mathbf{n}}(m-1-\mu_{\mathbf{n}})\binom{n-d-\#\mathbf{n}}{d-\#\mathbf{n}-1}+ \\ &+\sum_{\mathbf{n}:|\mathbf{n}|=d} N_{\mathbf{n}}\binom{n-d-\#\mathbf{n}}{d-\#\mathbf{n}}+ \\ +\sum_{\mathbf{n}:|\mathbf{n}|=d-1} N_{\mathbf{n}}\sum_{l=0}^{n-2d}&\left(\beta_1^{(l+2)}(S_{\mu_{\mathbf{n}}})+\left(\binom{l+\mu_{\mathbf{n}}}{l+1}-1\right)(m-1-\mu_{\mathbf{n}})\right)\times \\ &\times\binom{n-d-\#\mathbf{n}-l-2}{d-\#\mathbf{n}-2}.\end{aligned} \tag{5.26}$$

The first sum describes the $O \times Y \times Y \times \cdots \times Y$-cycles and the distribution of the free $n-2d+1$ particles. Second sum is the number of $Y \times Y \times \cdots \times Y$-cycles, where all Y-subgraphs lie on the perimeter - there are $n-2d$ free particles. The last sum describes the number of independent $Y_h \times Y_p \times \cdots \times Y_p$-cycles. Here we used the formula for the number of Y_h-cycles for n particles on a fan graph with μ leaves and $m-1$ spokes [20]

$$n_Y^{(n)}(\mu, m-1) = \beta_1^{(n)}(S_{\mu_{\mathbf{n}}}) + \left(\binom{n+\mu-2}{n-1}-1\right)(m-1-\mu).$$

Sometimes, in formula (5.26), we get to evaluate $\binom{0}{0}=1$, $\binom{0}{-1}=0$, $\binom{-1}{-1}=1$.

The highest non-vanishing Betti number is β_m and its value is the number of the possible distributions of $n-2m$ free particles between the central S_m graph and the free $m-2$ edges on the perimeter.

$$\beta_m(D_n(W_m)) = \sum_{k=0}^{n-2m}\binom{n-m-k-2}{m-2}\beta_1^{(k+2)}(S_{m-1}),\ n \geq 2m.$$

5.4.3. Wheel graphs via Świątkowski discrete model

In this chapter we show, that the homology of configuration spaces of wheel graphs is generated by product cycles. The strategy is to consider two consecutive vertex cuts that bring any wheel graph to the form of a linear tree.

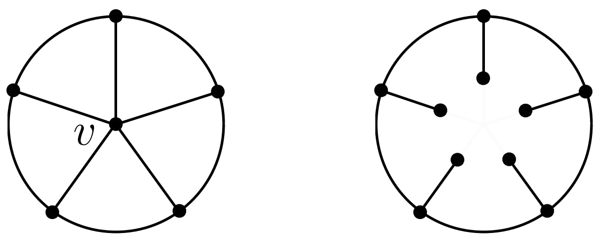


Figure 5.19: Vertex blowup at the hub of wheel W_{m+1} resulting with net graph N_m.

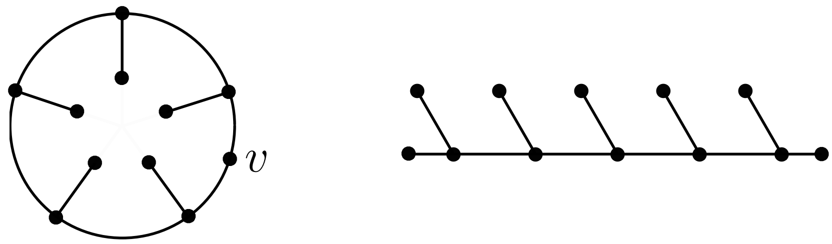


Figure 5.20: Blowup of a vertex in net graph N_m resulting with linear tree graph T_m.

Throughout, we use the knowledge of generators of the homology groups for tree graphs to construct a set of generators for net graphs and wheel graphs. Translating the results of section 5.3 to the Świątkowski complex, we have, that the generators of $H_d(S(T_m))$ are of the form

$$c_d = c_{Y_1} \dots c_{Y_d} v_1 \dots v_k e_1^{n_1} \dots e_l^{n_l},$$

subject to relations

$$c_d e \sim c_d v, \text{ if } e \cap v \neq \emptyset. \tag{5.27}$$

This means that computing the rank od $H_d(S(T_m))$ boils down to considering all possible distributions of $n - 2d$ free particles among the connected components of $T_m - (v_h(Y_1) \cup \dots \cup v_h(Y_d))$. By $v_h(Y_1)$ we denote the hub vertex of the Y-subgraph Y_i. Hence, $H_d(S(T_m))$ is freely generated by generators of the form

$$[Y_1, \dots, Y_d, n_1, \dots, n_{2d+1}], \; n_1 + \dots + n_{2d+1} = n - 2d, \tag{5.28}$$

where n_i is the number of particles on ith connected component of $T_m - (v_h(Y_1) \cup \dots \cup v_h(Y_d))$. In the first step, we connect two endpoints of T_m to obtain net graph N_m (Fig. 5.20).

Lemma 5.12. *The homology groups of $C_n(N_m)$ are freely generated by the product Y-cycles and the distributions of free particles on the connected components $N_m - (v_h(Y_1) \cup \dots \cup v_h(Y_d))$, which we denote by*

$$[Y_1, \dots, Y_d, n_1, \dots, n_{2d}], \; n_1 + \dots + n_{2d} = n - 2d. \tag{5.29}$$

The Betti numbers read

$$\beta_d(C_n(N_m)) = \binom{m}{d}\binom{n-1}{2d-1}.$$

Proof. Long exact sequence corresponding to vertex blow-up from figure 5.20 reads

$$\dots \xrightarrow{\Psi_{n,d+1}} H_d\left(S_{n-1}(T_m)\right) \xrightarrow{\delta_{n,d}} H_d\left(S_n(T_m)\right) \xrightarrow{\Phi_{n,d}} H_d\left(\tilde{S}_n^v(N_m)\right) \xrightarrow{\Psi_{n,d}}$$
$$\xrightarrow{\Psi_{n,d}} H_{d-1}\left(S_{n-1}(T_m)\right) \xrightarrow{\delta_{n,d-1}} H_{d-1}\left(S_n(T_m)\right) \xrightarrow{\Phi_{n,d-1}} \dots,$$

Let us next show that the connecting homomorphism δ is in this case injective. Map $\delta_{n,d}$ acts on generators (5.28) as

$$\delta_{n,d}([Y_1, \dots, Y_d, n_1, \dots, n_{2d+1}]) = [Y_1, \dots, Y_d, n_1 + 1, \dots, n_{2d+1}] +$$
$$-[Y_1, \dots, Y_d, n_1, \dots, n_{2d+1} + 1],$$

where n_1 and n_{d+1} are respectively the numbers of particles on the leftmost and on the rightmost connected component of $T_m - (v_h(Y_1) \cup \cdots \cup v_h(Y_d))$. One can check, that vectors $\{[Y_1, \ldots, Y_d, n_1+1, \ldots, n_{2d+1}] - [Y_1, \ldots, Y_d, n_1, \ldots, n_{2d+1}+1]\}$ corresponding to different choices of Y-subgraphs of T_m are linearly independent. Hence, any vector from $\mathrm{im}\delta_{n,d}$ can be uniquely decomposed in this basis and its preimage can be unambiguously determined by subtracting the particles from n_1 and n_{d+1}. By injectivity of δ,

$$H_d\left(\tilde{S}_n^v(N_m)\right) \cong \mathrm{coker}(\delta_{n,d}).$$

Hence, the rank of $H_d(S_n(N_m))$ is equal to $\mathrm{rk}(\mathrm{coker}_{n,d}) = \beta_d\left(S_n(T_m)\right) - \beta_d\left(S_{n-1}(T_m)\right)$. The Betti numbers of $S_n(T_m)$ can be computed by counting the distributions of $n-2d$ particles on $2d+1$ connected components multiplied by the number of d-subsets of Y-subgraphs of T_m. The result is

$$\beta_d\left(S_n(T_m)\right) = \binom{m}{d}\binom{n}{2d}.$$

The claim of the lemma follows directly from the above formula. The result is the same as the number of distributions of $n-2d$ particles on $2d$ connected components of $N_m - (v_h(Y_1) \cup \cdots \cup v_h(Y_d))$. □

Let us next consider the homology sequence associated with the vertex blow-up from W_{m+1} to N_m (fig. 5.19).

$$\ldots \xrightarrow{\Psi_{n,d+1}} \bigoplus_{h\in H(v)-\{h_0\}} H_d\left(S_{n-1}(N_m)\right) \xrightarrow{\delta_{n,d}} H_d\left(S_n(N_m)\right) \xrightarrow{\Phi_{n,d}} H_d\left(\tilde{S}_n^v(W_{m+1})\right) \xrightarrow{\Psi_{n,d}}$$
$$\xrightarrow{\Psi_{n,d}} \bigoplus_{h\in H(v)-\{h_0\}} H_{d-1}\left(S_{n-1}(N_m)\right) \xrightarrow{\delta_{n,d-1}} H_{d-1}\left(S_n(N_m)\right) \xrightarrow{\Phi_{n,d-1}} \ldots,$$

We next describe the kernel of map δ. Our aim is to show that it is free abelian, which in turn gives us that the short exact sequences for $H_d(S_n(W_{m+1}))$ split and yield $H_d(S_n(W_{m+1})) \cong \mathrm{coker}(\delta_{n,d}) \oplus \ker(\delta_{n-1,d})$. Map $\delta_{n,d}$ assigns to generators (5.29) of $H_d(S_n(W_{m+1}))$ the differences of generators derived from a given generator by adding one particle to a connected component of $N_m - (v_h(Y_1) \cup \cdots \cup v_h(Y_d))$. In order to write down the action of map δ, let us first establish some notation. The connected components of $N_m - (v_h(Y_1) \cup \cdots \cup v_h(Y_d))$ are either isomorphic to edges or to linear tree graphs. The number of connected components that are edges, which have one vertex of degree one in N_m is equal to d. The number of the remaining connected components is always equal to d, but their type depends on the distribution of subgraphs $Y_1, \ldots, Y_d$ in N_m. The situations, that are relevant for the description of $\ker\delta$ are those, where a particle is added by map δ to two connected components, which contain an edge, which before the blow-up was adjacent to the hub of W_{m+1}. There are at most $2d$ such components, as removing the hub-vertices of two neighbouring Y-subgraphs of N_m yields a connected component of the edge type, which is not adjacent to the hub of W_{m+1}. We label these components by numbers $1, \ldots, l$ (we always have $d \leq l \leq 2d$) and the occupation numbers of these components are $n_1, \ldots, n_l$. We choose component 1 to be the component adjacent to edge $e(h_0)$ and increase the labels in the clockwise direction from the component with label 1. The remaining components are labelled by numbers $l+1,\ldots,2d$. Map δ acts on basis elements of $\bigoplus_{h\in H(v)-\{h_0\}} H_d\left(S_{n-1}(N_m)\right)$ as follows.

$$\begin{gathered}\delta_{n,d}\left([Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i},\ldots,n_{2d}]_i\right)=\\ =[Y_1,\ldots,Y_d,n_1+1,\ldots,n_{p_i},\ldots,n_{2d}]-[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i}+1,\ldots,n_{2d}],\end{gathered}$$

where $[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i},\ldots,n_{2d}]_i$ is a generator corresponding to the h_i-component of $\bigoplus_{h\in H(v)-\{h_0\}} H_d\left(S_{n-1}(N_m)\right)$ and p_i is the label of the connected component, which contains edge $e(h_i)$. One easily observes, that if $p_i=1$, i.e. edges $e(h_0)$ and $e(h_i)$ belong to the same connected component of $N_m-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$, generator $[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i},\ldots,n_{2d}]_i\in\ker\delta_{n,d}$. Such an element is represented in $S_n(W_{m+1})$ as cycle $c_{Y_1}\ldots c_{Y_d}c_O$, where O is the cycle in W_{m+1}, which contains edges $e(h_0)$, $e(h_i)$ and the hub of W_{m+1}. Similarly, element

$$[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i},\ldots,n_{2d}]_i-[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_j},\ldots,n_{2d}]_j$$

is in the kernel of $\delta_{n,d}$, whenever edges $e(h_i)$ and $e(h_j)$ belong to the same connected component of $N_m-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$. The last type of elements of $\ker\delta_{n,d}$ are combinations of generators, that when acted upon by $\delta_{n,d}$, compose to the boundary of a Y-cycle centred at the hub of W_{m+1}. Such kernel elements correspond to cycles $c_{Y_1}\ldots c_{Y_d}c_{Y_h}$ in $S_n(W_{m+1})$, where Y_h is a Y-cycle, whose hub-vertex is the hub-vertex of W_{m+1}. The precise form of such kernel elements is the following.

$$\begin{gathered}[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i},\ldots,n_{p_j}+1,\ldots]_i-[Y_1,\ldots,Y_d,n_1+1,\ldots,n_{p_i},\ldots,n_{p_j},\ldots]_i+\\ +[Y_1,\ldots,Y_d,n_1+1,\ldots,n_{p_i},\ldots,n_{p_j},\ldots]_j-[Y_1,\ldots,Y_d,n_1,\ldots,n_{p_i}+1,\ldots,n_{p_j},\ldots]_j,\end{gathered}$$

where $i<j$. In order to manage the relations between the above kernel elements, we use the already mentioned fact, that they are in a one-to-one correspondence with 1-cycles (O-cycles and Y-cycles) in a configuration space of the disconnected graph $W_{m+1}-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$. More specifically, the disconnected graph $W_{m+1}-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$[2] is a disjoint sum of a number of edges and of one fan graph. We regard the 1-cycles (O-cycles or Y-cycles) at the hub as generators of the first homology group of the configuration space of the fan graph multiplied by different distributions of particles on the disjoint edge-components of $W_{m+1}-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$. Fan graphs are planar, hence by theorem 4.4 there is no torsion in $\ker\delta_{n,d}$. Hence, $H_d(S_n(W_{m+1}))$ is torsion-free and short exact sequence for $H_d(S_n(W_{m+1}))$ gives in this case

$$\beta_d(S_n(W_{m+1}))\cong\beta_d(S_n(N_m))-m\beta_d(S_{n-1}(N_m))+\mathrm{rk}(\ker\delta_{n,d})+\mathrm{rk}(\ker\delta_{n,d-1}).$$

The computation of ranks of kernels of maps $\delta_{n,d}$ is a combinatorial task, which has been accomplished using the correspondence with cycles in configuration spaces of fan graphs in subsection 5.4.2.

5.5. Graph $K_{3,3}$

Graph $K_{3,3}$ is shown on Fig. 5.21. We will draw graph $K_{3,3}$ in two ways: 1) immersion in $\mathbb{R}^2$, Fig. 5.21a), ii) embedding in $\mathbb{R}^3$, Fig. 5.21b).

[2] $W_{m+1}-(v_h(Y_1)\cup\cdots\cup v_h(Y_d))$ is a disconnected topological space. We give this space the structure of a graph by adding a vertex to the open end of each open edge.

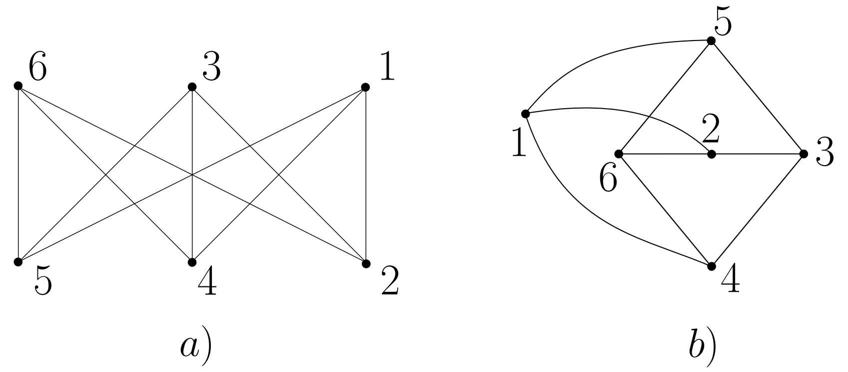


Figure 5.21: Graph $K_{3,3}$.

Graph $K_{3,3}$ has the property, that all its vertices are of degree three. High homology groups of graphs with such a property have been studied in [61]. In particular, we have the following result.

Theorem 5.13. *Let Γ be a simple graph, whose all vertices have degree 3. Denote by N the number of vertices of graph Γ and label the vertices by labels $1, \ldots, N$. Moreover, denote by $\mathcal{Y} = \{Y_1, \ldots Y_N\}$ the set of Y-subgraphs of Γ such, that the hub of Y_k is vertex k. Group $H_N(S_n(\Gamma))$ is freely generated by product cycles*

$$e_1^{n_1} \ldots e_K^{n_K} \bigotimes_{Y \in \mathcal{Y}} c_Y,\ n_1 + \cdots + e_K = n - 2N.$$

Group $H_{N-1}(S_n(\Gamma))$ is generated by product cycles of the form

$$e_1^{n_1} \ldots e_K^{n_K} v \bigotimes_{Y \in \tilde{\mathcal{Y}}} c_Y,\ n_1 + \cdots + e_K = n - 2(N-1),$$

where $\tilde{\mathcal{Y}} \subset \mathcal{Y}$ is such, that $|\tilde{\mathcal{Y}}| = N - 1$, and $v \in V(\Gamma)$ is the unique vertex, that satisfies $v \cap (\cup_{Y \in \tilde{\mathcal{Y}}} Y) = \emptyset$. The above generators are subject to relations

$$e_1^{n_1} \ldots e_j^{n_j} \ldots e_K^{n_K} v \bigotimes_{Y \in \tilde{\mathcal{Y}}} c_Y \sim e_1^{n_1} \ldots e_j^{n_j+1} \ldots e_K^{n_K} \bigotimes_{Y \in \tilde{\mathcal{Y}}} c_Y,$$

whenever $e_j \cap v \neq \emptyset$.

As we show in section 5.7, the second homology group of configuration spaces of such graphs is also generated by product cycles. Later in this section, by comparing the ranks of homology groups computed via the discrete Morse theory, we argue, that $H_4(C_n(K_{3,3}))$ is also generated by product cycles. Interestingly, in $H_3(C_n(K_{3,3}))$ there is a new non-product generator. Using this knowledge, we explain the relations between the product and non-product cycles that give the correct rank of $H_3(C_n(K_{3,3}))$.

Second homology group

There are no pairs of disjoint cycles in $K_{3,3}$, hence the product part for $n = 2$ is empty. When $n = 3$, there are 12 $O \times Y$-cycles. This can be seen by choosing the Y-graph centered at vertex 1 on Fig. 5.21b) - there are 2 cycles disjoint with such a Y-subgraph. There are 6 Y-subgraphs in $K_{3,3}$, hence we get the number of $O \times Y$-cycles. One checks by a straightforward calculation, that 8 of them are independent. Hence,

$$\beta_2(D_3(K_{3,3})) = 8.$$

When $n = 4$, there are new product cycles of the $Y \times Y$-type. There are $\binom{6}{2} = 15$ cycles of this type, however there are relations between them. Such relations between the $Y \times Y$-cycles arise, when one of the cycles is in relation with a different Y-cycle. This happens only when we have a situation as on Fig. 4.6. Therefore, cycles of the $Y \times Y$-type, where the hubs of the Y-subgraphs, are connected by an edge, are all independent (Fig. 5.22a)). The number of such cycles is 9. The relations occur between $Y \times Y$-cycles, where the hubs of the subgraphs are not connected by an edge (Fig. 5.22b)). There are 6 such cycles. The number of relations is 4. To see this, consider Y-subgraph, whose hub is vertex 1 (Fig. 5.21). Denote this subgraph by Y_1. It is straightforward to see that in graph $K_{3,3} - Y_1$ we have $c_{Y_3} \sim c_{Y_6}$. Hence,

$$(c_{Y_1} \otimes c_{Y_3}) \sim (c_{Y_1} \otimes c_{Y_6}).$$

Analogous relations for Y-subgrphs that lie on the same side of the $K_{3,3}$ graph as Y_1 (see Fig. 5.21a)) read

$$(c_{Y_3} \otimes c_{Y_1}) \sim (c_{Y_3} \otimes c_{Y_6}),\ (c_{Y_6} \otimes c_{Y_3}) \sim (c_{Y_6} \otimes c_{Y_1}).$$

From the above equations only two are independent. Similar situation happens for relations between pairs of graphs from the other side. The complete set of relations reads

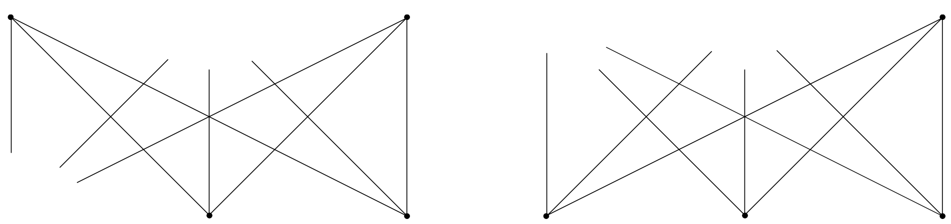
$$(c_{Y_1} \otimes c_{Y_3}) \sim (c_{Y_1} \otimes c_{Y_6}) \sim (c_{Y_3} \otimes c_{Y_6}),\ (c_{Y_2} \otimes c_{Y_4}) \sim (c_{Y_2} \otimes c_{Y_5}) \sim (c_{Y_4} \otimes c_{Y_5}).$$

Therefore,

$$\beta_2(D_4(K_{3,3})) = 8 + 9 + 2 = 19.$$

For $n > 4$, we have to take into account the distribution of free particles. Whenever two non-neighbouring Y-subgraphs are considered, all distributions of free particles are equivalent (Fig. 5.22b)). When the subgraphs are adjacent, there are two different parts of $K_{3,3}$, where the particles can be distributed, see Fig. 5.22a). This gives the formula

$$\beta_2(D_n(K_{3,3})) = 8 + 2 + 9(n-3) = 9n - 17,\ n \geq 4.$$



a) *b)*

Figure 5.22: Graph $K_{3,3}$ after removing two Y-subgraphs.

Higher homology groups

Let us first look at the third homology group. The are no product cycles for $n = 4$ however, from the Morse theory for the subdivided graph from Fig. 5.23 we have

$$\beta_3(D_4(K_{3,3})) = 1.$$

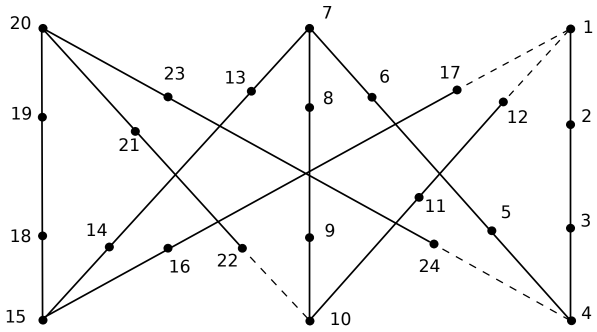


Figure 5.23: Graph $K_{3,3}$ sufficiently subdivided for $n = 4$. The deleted edges are marked with dashed lines.

The Morse complex has dimension 3. The generator of $H_3(D_4(K_{3,3}))$ is isomorphic to a closed 3-manifold with Euler characteristic $\chi = 11$. The cycle on the level of the Morse complex has the form

$$c = \left\{e_1^{12}, e_4^{24}, e_{10}^{22}, 2\right\} - \left\{e_1^{17}, e_4^{24}, e_{10}^{22}, 2\right\} - \left\{e_1^{12}, e_{10}^{22}, e_{15}^{18}, 16\right\} + \left\{e_1^{17}, e_4^{24}, e_{10}^{22}, 11\right\} +$$
$$\left\{e_1^{17}, e_7^{13}, e_{10}^{22}, 8\right\} + \left\{e_1^{12}, e_4^{24}, e_{15}^{18}, 16\right\} - \left\{e_4^{24}, e_7^{13}, e_{10}^{22}, 8\right\} - \left\{e_1^{12}, e_4^{24}, e_{10}^{22}, 5\right\}.$$

For $n = 5$, we have the $Y \times Y \times O$-cycles. These are the cycles, where the Y-subgraphs are adjacent. For every pair of adjacent Y subgraphs there is an unique O-cycle. An example of such a cycle is

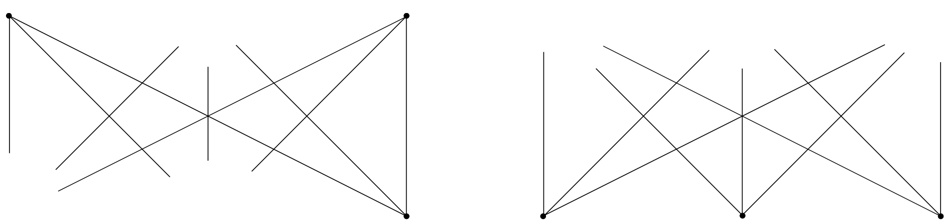
$$c_{Y_1} \times c_{Y_2} \times \left(\{e_3^4\} + \{e_4^6\} - \{e_5^6\} - \{e_3^5\}\right).$$

The number of all such cycles is equal to the number of pairs of adjacent Y-subgraphs, which is 9. Adding the properly embedded generator of $H_3(D_4(K_{3,3}))$, we get

$$\beta_3(D_5(K_{3,3})) = 10.$$

For $n \geq 6$, all $Y \times Y \times Y$-cycles are independent. Consider two ways of choosing three Y-subgraphs. The first way is to remove two Y-graphs from the same side and one from the opposite side. This results with the partition of $K_{3,3}$ into three components (Fig. 5.24a)). Removing three Y-graphs from the same side splits $K_{3,3}$ into three parts (Fig. 5.24b)). Therefore,

$$\beta_3(D_n(K_{3,3})) = 1 + 9(n-4) + \binom{6}{3}\binom{n-4}{2}, \quad n \geq 6.$$



a) *b)*

Figure 5.24: Graph $K_{3,3}$ after removing three Y-subgraphs.

The product contribution to higher homology groups requires considering different choices of Y-subraphs. There are no $Y \times Y \times \cdots \times Y \times O$-cycles in $H_p(D_n(K_{3,3}))$ for $p \geq 4$. As direct computations using discrete Morse theory show, there are also no non-product generators (see table 5.1). Therefore, only $Y \times Y \times \cdots \times Y$-cycles contribute to $H_p(D_n(K_{3,3}))$ for $p \geq 4$. Removing four Y-graphs from $K_{3,3}$ always results with the splitting into 5 parts, removing five Y-graphs gives 7 parts and removing all six Y-graphs gives 9 parts. Summing up,

$$\beta_4(D_n(K_{3,3})) = \binom{6}{4}\binom{n-4}{4}, \quad \beta_5(C_n(K_{3,3})) = \binom{6}{5}\binom{n-4}{6},$$
$$\beta_6(C_n(K_{3,3})) = \binom{n-4}{8}.$$

All homology groups higher than H_6 are zero for any number of particles.

5.6. Triple tori in $C_n(K_{2,p})$

In this chapter we study a family of graphs, where some cycles generating the homology groups of the n-particle configuration space are not product. This is the family of complete bipartite graphs $K_{2,p}$ (see figure 5.25a). The first interesting graph from this family is $K_{2,4}$. As we show below, its 3-particle configuration space gives rise to a 2-cycle, which is a triple torus. It turns out, that such triple tori together with products of Y cycles generate the homology groups of $C_n(K_{2,p})$. The most convenient discrete model for studying $C_n(K_{2,p})$ is the Świątkowski model. In fact, we study

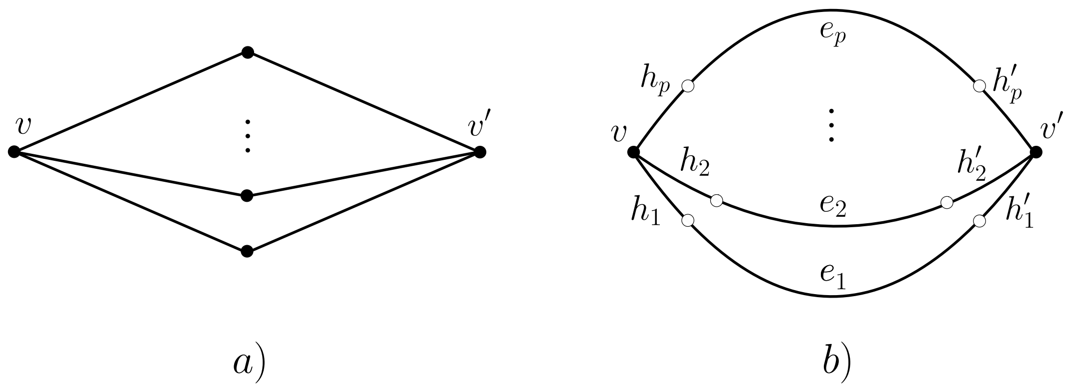


Figure 5.25: a) Graph $K_{2,p}$. b) Graph Θ_p.

the Świątkowski configuration space of graph Θ_p (see 5.25b), which is topologically equivalent to $K_{2,p}$, but it has the advantage, that its discrete configuration space is of the optimal dimension. Because there are no 3-cells in $S_n(\Theta_p)$, hence automatically we get, that

$$H_i(C_n(K_{2,p})) = 0 \text{ for } i \geq 3.$$

This in turn means, that $H_2(C_n(K_{2,p}))$ as the top homology group is a free group. The first homology group can be computed using the methods that we reviewed in section 4.4.

Lemma 5.14. *The first homology group of $C_n(K_{2,p})$ is equal to $\mathbb{Z}^{p(p-1)}$ for $n \geq 2$ and $p-1$ for $n = 1$.*

Proof. Graph $K_{2,p}$ is 2-connected, hence $H_1(C_n(K_{2,p}))$ stabilises. There is one 2-cut at vertices $\{v, v'\}$, which results with the decomposition of $K_{2,p}$ into p disconnected components. Hence, applying theorem 4.4 for $N_2 = \frac{1}{2}(p-1)(p-2)$ and $\beta_1(\Gamma) = p-1$, we get the desired result. □

By counting the number of 0-, 1- and 2-cells in $S_n(K_{2,p})$, we compute the Euler characteristic.

Lemma 5.15. *The Euler characteristic of $S_n(K_{2,p})$ for $n \geq 3$ and $p \geq 3$ is*

$$\chi = (p-1)^2\binom{n-3+p}{p-1} - 2(p-1)\binom{n-2+p}{p-1} + \binom{n-1+p}{p-1}.$$

On the other hand, $\chi(S_n(K_{2,p})) = 1 - \beta_1(S_n(K_{2,p})) + \beta_2(S_n(K_{2,p}))$. Therefore, we compute the second Betti number of $C_n(K_{2,p})$ as

$$\begin{aligned}\beta_2(C_n(K_{2,p})) = (p-1)^2\binom{n-3+p}{p-1} - 2(p-1)\binom{n-2+p}{p-1}+ \\ +\binom{n-1+p}{p-1} + \frac{p(p-1)}{2} - 1 \text{ for } n \geq 3 \text{ and } p \geq 3.\end{aligned} \tag{5.30}$$

In the remaining part of this section we describe the generators of $H_2(C_n(K_{2,p}))$ and the relations that lead to the above formula. We represent them in terms of 2-cycles in $S_n(\Theta_p)$.

Example 5.16. Generators of $H_2(S_n(\Theta_3))$. Group $H_2(S_n(\Theta_3))$ is generated by products of Y-cycles at vertices v and v'. More precisely, consider the following two Y-cycles

$$\begin{aligned}c_{123} &= e_1(h_2 - h_3) + e_2(h_3 - h_1) + e_3(h_1 - h_2),\\ c'_{123} &= e_1(h'_2 - h'_3) + e_2(h'_3 - h'_1) + e_3(h'_1 - h'_2).\end{aligned}$$

Group $H_2(S_n(\Theta_3))$ is freely generated by cycles

$$c_{123}c'_{123}e_1^{n_1}e_2^{n_2}e_3^{n_3}.$$

This can be seen by comparing the number of cycles of the above form with $\beta_2(S_n(\Theta_3))$ from formula 5.30. In both cases the answer is the number of distributions of $n-2$ particles among edges e_1, e_2, e_3 (the problem of distributing $n-2$ indistinguishable balls into 3 distinguishable bins), which is $\binom{n-2}{2} = \frac{1}{2}(n-2)(n-3)$.

From now on, we denote the Y-cycles as

$$\begin{aligned}c_{ijk} &= e_i(h_j - h_k) + e_j(h_k - h_i) + e_k(h_i - h_j),\ i<j<k,\\ c'_{ijk} &= e_i(h'_j - h'_k) + e_j(h'_k - h'_i) + e_k(h'_i - h'_j),\ i<j<k.\end{aligned} \tag{5.31}$$

Cycle c_{ijk} is the Y-cycle of the Y-subgraph, whose hub vertex is v and which is spanned on edges e_i, e_j, e_k. Cycle c'_{ijk} corresponds to an analogous Y-subgraph, whose hub is v'.

Example 5.17. The generator of $H_2(S_3(\Theta_4))$. Formula (5.30) tells us, that $\beta_2(C_3(K_{2,4})) = 1$. The corresponding generator in $S_3(\Theta_4)$ has the following form.

$$c_\Theta = -(h_1 - h_2)c'_{134} + (h_1 - h_3)c'_{124} - (h_1 - h_4)c'_{123}.$$

By expanding the Y-cycles, one can see, that the above chain is a combination of all 2-cells of $S_3(\Theta_4)$, hence, $C_n(K_{2,4})$ has the homotopy type of a closed 2-dimensional surface. Its Euler characteristic is equal to -4, hence this is a surface of genus 3. By the classification theorem of surfaces [74, 75], we identify $C_n(K_{2,4})$ to have the homotopy type of a triple torus (fig. 5.26).

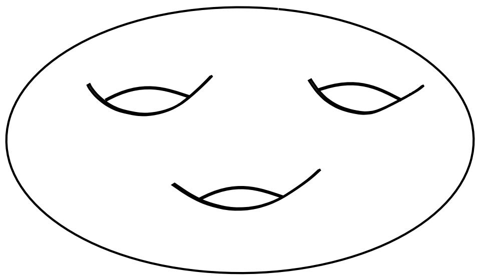


Figure 5.26: A triple torus.

From now on, we denote the Θ-cycles as

$$c_{ijkl} = -(h_i - h_j)c'_{ikl} + (h_i - h_k)c'_{ijl} - (h_i - h_l)c'_{ijk}, \; i < j < k < l. \qquad (5.32)$$

Cycle c_{ijkl} involves cells from $S_3(\Theta_4)$ for Θ_4 being the subgraph of Θ_p spanned on edges e_i, e_j, e_k, e_l. Using the notation set in equations (5.32) and (5.31), we propose the following generators of $H_2(S_n(\Theta_p))$.

$$c_{ijk}c'_{rst}e_1^{n_1}\dots e_p^{n_p}, \; i < j < k, \; r < s < t, \; n_1 + \dots + n_p = n - 4,$$
$$c_{ijkl}e_1^{n_1}\dots e_p^{n_p}, \; i < j < k < l, \; n_1 + \dots + n_p = n - 3.$$

Let us start with $n = 3$. The key to describe the relations between the Θ-cycles spanned on different Θ_4 subgraphs is to consider graph Θ_5.

Proposition 5.18. *The Θ-cycles in graph Θ_5 satisfy the following relation*

$$c_{1234} - c_{1235} + c_{1245} - c_{1345} + c_{2345} = 0. \qquad (5.33)$$

In graph Θ_p, many relations of the form (5.33) can be written down by choosing different Θ_5 subgraphs. The linearly independent ones are picked by choosing the corresponding Θ_5-subgraphs that are spanned on edge e_1 and some other four edges of Θ_p. Such a choice can be made in $\binom{p-1}{4}$ ways. Subtracting the number of linearly independent relations from the number of all Θ_4 subgraphs, we get

$$\beta_2(C_3(K_{2,p})) = \binom{p}{4} - \binom{p-1}{4} = \binom{p-1}{3}.$$

Increasing the number of particles to $n = 4$ introduces products of Y-cycles and new relations. First of all, by proposition 5.19 different distributions of additional particles in the Θ-cycle can be realised are combinations of different products of Y-cycles.

Proposition 5.19. *In graph Θ_4, we have the following relations*

$$(e_1 - e_2)c_{1234} = c_{124}c'_{123'} - c_{123}c'_{124},$$
$$(e_1 - e_3)c_{1234} = c_{123}c'_{134'} - c_{134}c'_{123},$$
$$(e_1 - e_4)c_{1234} = c_{124}c'_{134'} - c_{134}c'_{124}.$$

Hence, all Θ-cycles generate a subgroup of $H_2(S_n(\Theta_p))$, which is isomorphic to $\mathbb{Z}^{\binom{p-1}{3}}$. The last type of relations we have to account for[3] are the new relations between products of Y-cycles.

Proposition 5.20. *In graph Θ_5, products of Y-cycles satisfy*

$$c_{123}c'_{145} + c_{145}c'_{123} + c_{125}c'_{134} + c_{134}c'_{125} - (c_{124}c'_{135} + c_{135}c'_{124}) = 0.$$

Again, many relations of type 5.20 can be written by picking different Θ_5 subgraphs. Similarly as in the case of relations 5.33, the linearly independent ones are chosen by fixing e_1 to be the common edge of the Θ_5 subgraphs. Hence, the number of linearly independent relations is $\binom{p-1}{4}$. In particular, we have

$$\beta_2(C_4(K_{2,p})) = \binom{p-1}{3} + (\beta_1(C_2(S_p)))^2 - \binom{p-1}{4},$$

where $(\beta_1(C_2(S_p)))^2$ is the number of independent product cycles after taking into the account the relations within the two opposite star subgraphs. All the above relations are inherited by the cycles in $S_n(\Theta_p)$ after multiplying them by a suitable polynomial in the edges of Θ_p. In this way, they yield equation (5.30).

5.7. When is $H_2(C_n(\Gamma))$ generated only by product cycles?

In this section we prove the following theorem.

Theorem 5.21. *Let Γ be a simple graph, for which $|\{v \in V(\Gamma) : d(v) > 3\}| = 1$. Then group $H_2(C_n(\Gamma))$ is generated by product cycles.*

In the proof we use the Świątkowski discrete model. The strategy of the proof is to first consider the blowup of the vertex of degree greater than 3 and prove theorem 5.21 for graphs, whose all vertices have degree at most 3. For such a graph, we choose a spanning tree $T \subset \Gamma$. Next, we subdivide once each edge from $E(\Gamma) - E(T)$. We prove the theorem inductively by showing in lemma 5.22, that the blowup at an extra vertex of degree 2 does not create any non-product generators. The base case of induction is obtained by doing the blowup at every vertex of degree 2 in $\Gamma - T$. This way, we obtain graph, which is isomorphic to tree T and we use the fact, that for tree graphs the homology groups of $S_n(T)$ are generated by products of Y-cycles.

[3]We do not mention here the typical relations between different Y-cycles on Y-subgraphs of the S_p graphs, which are met while computing the first homology group of the configuration spaces of star graphs (see [20]). Such relations are also inherited by the products of Y-cycles.

Lemma 5.22. *Let Γ be a simple graph, whose all vertices have degree at most 3. Let T be a spanning tree of Γ. Let $v \in V(\Gamma)$ be a vertex of degree 2 and Γ_v the graph obtained from Γ by the vertex blowup at v. If $H_2(S_n(\Gamma_v))$ is generated by product cycles, then $H_2(S_n(\Gamma))$ is also generated by product cycles.*

Proof. Long exact sequence corresponding to the vertex blow-up reads

$$\ldots \xrightarrow{\Psi_{n,3}} H_2\left(S_{n-1}(\Gamma_v)\right) \xrightarrow{\delta_{n,2}} H_2\left(S_n(\Gamma_v)\right) \xrightarrow{\Phi_{n,2}} H_2\left(\tilde{S}_n^v(\Gamma)\right) \xrightarrow{\Psi_{n,2}}$$
$$\xrightarrow{\Psi_{n,2}} H_1\left(S_{n-1}(\Gamma_v)\right) \xrightarrow{\delta_{n,1}} H_1\left(S_n(\Gamma_v)\right) \xrightarrow{\Phi_{n,1}} \ldots .$$

We aim to show, that the corresponding long exact sequence

$$0 \to \operatorname{coker}\left(\delta_{n,2}\right) \to H_2\left(\tilde{S}_n^v(\Gamma)\right) \to \ker\left(\delta_{n,1}\right) \to 0$$

splits. To this end, we construct a homomorphism $f: \ \ker\left(\delta_{n,1}\right) \to H_2\left(\tilde{S}_n^v(\Gamma)\right)$ such, that $\Psi_{n,2} \circ f = id_{\ker(\delta_{n,1})}$. In the construction we use the explicit knowledge of elements of $\ker\left(\delta_{n,1}\right)$. Such a knowledge is accessible, as we know the generating set of $H_1\left(S_{n-1}(\Gamma_v)\right)$ - because all vertices of Γ_v have degree at most 3, it consists of Y-cycles and O-cycles, subject to the Θ-relations (equations (4.7) and (4.6)) and the distribution of free particles, which say, that $[ce] = [cv]$ whenever v is a vertex of e. Recall, that cycle c represents element of $\ker\left(\delta_{n,1}\right)$, whenever $[ce] = [ce']$, where e and e' are the edges incident to vertex v. This happens if and only if cycles ce and ce' are related by a Θ-relation or a particle-distribution relation. However, it is not possible to write the Θ relations in the form $ce - ce' = \partial(b)$ for any c. Hence, cycles ce and ce' must be related by the particle distribution, i.e. there exists a path in Γ_v, which is disjoint with $\mathrm{Supp}(c)$ and which joins edges e and e'. The desired homomorphism f is constructed as follows. For a generator c of $H_1\left(S_{n-1}(\Gamma_v)\right)$, find path $p(c)$, which joins e and e' and is disjoint with $\mathrm{Supp}(c)$. Having found such a path, we complete it to a cycle $O_{p(c)}$ in a unique way by adding to p vertex v and edges e, e'. From cycle $O_{p(c)}$ we form the O-cycle $c_{O_{p(c)}}$ (see definition 4.10). Homomorphism f is established after choosing the set of independent generating cycles and paths, that are disjoint with them. It acts as $f: \ [c] \mapsto [c \otimes c_{O_{p(c)}}]$. Clearly, we have $\Psi_{n,2}([c \otimes c_{O_{p(c)}}]) = [c]$ by extracting from $c_{O_{p(c)}}$ the part, which contains half-edges incident to v.

This way, we obtained, that $H_2\left(\tilde{S}_n^v(\Gamma)\right) \cong \ker\left(\delta_{n,1}\right) \oplus \operatorname{coker}\left(\delta_{n,2}\right)$ and that elements of $\ker\left(\delta_{n,1}\right)$ are represented by product $c_O \otimes c_Y$ cycles. By the inductive hypothesis, elements of $\operatorname{coker}\left(\delta_{n,2}\right)$ are the product cycles that generate $H_2\left(S_{n-1}(\Gamma_v)\right)$ subject to relations $ce \sim ce'$. □

The last step needed for the proof of theorem 5.21 is showing, that the blowup of Γ at the unique vertex of degree greater than 3 does not create any non-product cycles. Here we only sketch the proof of this fact, which is analogous to the proof of lemma 5.22. Namely, using the knowledge of relations between the generators of $H_1\left(S_{n-1}(\Gamma_v)\right)$, one can show, that the elements of $\ker\left(\delta_{n,1}\right)$ are of two types: i) the ones, that are of the form $\partial(c \otimes b_{p(c)})$, where $[c] \in H_1\left(S_{n-1}(\Gamma_v)\right)$ and $b_{p(c)}$ is the 1-cycle corresponding to path $p(c) \subset \Gamma_v$, which is disjoint with $\mathrm{Supp}(c)$ and whose boundary are edges incident to v, ii) pairs of cycles of the form $(c(e_j - e_0), c(e_0 - e_j))$, where e_0, e_i, e_j are edges incident to v and $[c] \in H_1\left(S_{n-2}(\Gamma_v)\right)$. Such pairs are mapped by $\delta_{n,1}$ to $c \otimes ((e_j - e_0)(e_0 - e_i) + (e_0 - e_i)(e_0 - e_j))$, which is equal to $\partial(c \otimes c_{0ij})$, where c_{0ij}

is the Y-cycle corresponding to the Y-graph in Γ centred at v and spanned by edges e_0, e_i, e_j. Next, in order to show splitting of the homological short exact sequences, we consider a homomorphism $f: \ker(\delta_{n,1}) \to H_2\left(\tilde{S}_n^v(\Gamma)\right)$, for which $\Psi_{n,2} \circ f = id_{\ker(\delta_{n,1})}$. Such a homomorphism maps $[c]$ to $[c \otimes c_{O_{p(c)}}]$, where $O_{p(c)}$ is the cycle, which contains path $p(c)$ and vertex v. Pairs $([c(e_j - e_0)], [c(e_0 - e_j)])$ are mapped by f to cycles $c \otimes c_{0ij}$. We obtain, that $H_2\left(\tilde{S}_n^v(\Gamma)\right) \cong \ker(\delta_{n,1}) \oplus \operatorname{coker}(\delta_{n,2})$, where the generators of $\ker(\delta_{n,1})$ are in a one-to-one correspondence with the product homology classes of $H_2\left(\tilde{S}_n^v(\Gamma)\right)$ described above. Elements of $\operatorname{coker}(\delta_{n,2})$ are also represented by product cycles. These cycles are the generators of $H_2(S_n(\Gamma_v))$ subject to relations $ce_0 \sim ce_i$, $i = 1, \ldots, d(v)$, where $e_0, e_1, \ldots, e_{d(v)}$ are edges incident to v.

The task of characterising all graphs, for which $H_2(S_n(\Gamma))$ is generated by product cycles requires taking into account the existence of non-product generators from section 5.6. As we show in section 5.6 the existence of pairs of vertices of degree greater than 3 in the graph implies that there may appear some multiple tori in the generating set of $H_2(C_n(\Gamma))$ stemming from subgraphs isomorphic to graph $K_{2,4}$. Furthermore, the class of graphs, for which higher homologies of $C_n(\Gamma)$ are generated by product cycles is even smaller. Recall graph $K_{3,3}$ whose all vertices have degree 3, but $H_3(C_n(K_{3,3}))$ has one generator, which is not a product of 1-cycles (see section 5.5).

Chapter 6

Summary

We summarise this thesis by outlining some possible paths of the future work that follow from the presented material and state some conjectural results concerning the behaviour of the homology groups of graph configuration spaces.

In the first part of this thesis, we explained, that the quantum statistics on a topological space X are classified by the conjugacy classes of unitary representations of the fundamental group of the configuration space $C_n(X)$. On the other hand, every such a unitary representation gives rise to a flat complex vector bundle over space $C_n(X)$. We interpret different isomorphism classes of flat complex vector bundles over $C_n(X)$ as fundamentally different families of particles. Among these families we find for example bosons and anyons (corresponding to the trivial flat bundle) and fermions, that correspond to a non-trivial flat bundle. We argue, that, the existence of more than only these two isomorphism classes is possible. However, the construction of non trivial flat bundles for $X = \mathbb{R}^2$ or $X = \mathbb{R}^3$ is difficult, hence some simplified mathematical models are needed. This motivates the study of configuration spaces of particles on graphs, which are computationally more tractable. Topological invariants, that give some information about the structure of the set of complex vector bundles over $C_n(X)$ are the homology groups of configuration spaces. In particular, Chern characteristic classes map the flat vector bundles to torsion components of the homology groups with coefficients in $\mathbb{Z}$. In the second part of this thesis, we compute homology groups of configuration spaces of certain families of graphs. We summarise the computational results as follows.

- This thesis contains first computation of the Betti numbers together with the explicit description of generators of the homology groups of the configuration spaces of i) tree graphs (subsection 5.3.3, equation (5.17)), ii) wheel graphs (subsection 5.4, equations (5.23, 5.24, 5.25, 5.26)), iii) graphs $K_{2,p}$ (section 5.6, equation (5.30)), iv) graph $K_{3,3}$ (subsection 5.5). Moreover, we provided a large family of simple graphs, for which the second homology group of $C_n(\Gamma)$ has a simplified structure, i.e. is generated by product cycles. These results were obtained by using the latest tools that were introduced to the mathematical community (discrete Morse theory [24] and the vertex blowup method [61]) and developing some original mathematical tools (product cycles, decomposition of the configuration spaces for graphs of connectivity one). In particular, we obtained, that configuration spaces of tree graphs, wheel graphs and complete bipartite graphs $K_{2,p}$ have no torsion in their homology. This means, that the set of flat bundles over configuration spaces of such graphs has a simplified structure, namely every flat vector bundle is stably equivalent to a trivial vector bundle. Hence, these families of graphs are good first candidates for studying the properties of non-abelian statistics on graphs.

- Computation of the homology groups of configuration spaces of some small canonical graphs via the discrete Morse theory shows, that in some cases there is a $\mathbb{Z}_2$-torsion in the homology. However, we were not able to provide an example of a graph, which has a torsion component different than $\mathbb{Z}_2$ in the homology of its configuration space.

- All results that we have obtained support the regularity conjecture of Betti numbers of graph configuration spaces.

  Conjecture 6.1. *Let Γ be a graph. For $n \geq 2p$, the behaviour of $\beta_p(D_n(\Gamma))$ becomes regular, i.e. $\beta_p(D_n(\Gamma))$ grows polynomially with n.*

- It is a difficult task to accomplish a full description of the homology groups of graph configuration spaces using methods presented in this work. One fundamental obstacle is that such a task requires the knowledge of possible embeddings of d-dimensional surfaces in $C_n(\Gamma)$, which generate the homology. Surfaces are fully classified up to dimension three. However, the asphericality of graph configuration spaces reduces the possible surfaces to the aspherical ones. This in particular means, that cycles generating the homology in dimension 2 have the homotopy type of tori or multiple tori. This allowed us to find all generators of the second homology group of configuration spaces of a large family of graphs in section 5.7.

In what follows, we give an exposition of some of the open problems related to quantum statistics on graphs, that are a natural continuation of the work presented in this thesis.

Problem 1. – K-theory for graph configuration spaces. With the knowledge of the homology groups of graph configuration spaces, one can compute the Grothendieck group via the Atiyah-Hirzebruch spectral sequence [11]. By computing torsion in the reduced Grothendieck group $\tilde{K}(C_n(\Gamma))$, one could extract some more information about the characteristic classes of flat complex vector bundles over $C_n(\Gamma)$. More precisely, we would like to know, whether torsion in $H_*(C_n(\Gamma), \mathbb{Z})$ is in the image of the characteristic map of some nontrivial flat bundles. In particular, if there is no torsion in $\tilde{K}(C_n(\Gamma))$, then we again have, that every flat bundle is stably equivalent to a trivial bundle.

Problem 2. – Unitary representations of graph braid groups - quantum computing perspective. Anyonic quantum computation is a dynamically developing field. It relies on constructing physical systems, where the excited states have anyonic statistics. Examples of such states are the excitations in the Kitayev toric model [77] or the quantum Hall states [76]. Quantum computation using anyons is appealing due to its intrinsic fault-tolerance [76]. Any quantum computation scheme that uses anyons, involves a set of unitary operators, which are assigned to generators of a braid group (the fundamental group of a configuration space of some manifold). Mathematically, they form a unitary representation of a braid group. The relevant representations are those, that are universal for quantum computation [80]. The problem of classification of unitary representations of braid groups is difficult and only sporadic results have been obtained [78, 79]. Perhaps tackling an analogous problem for graph braid groups [25] (fundamental groups of graph configuration spaces) could shed some new light on the non-abelian anyonic quantum computation. In particular, one can look for points in the moduli space of flat bundles over $C_n(\Gamma)$, that can be represented as universal quantum gates. Good candidates for such studies would be the graphs Γ, that do not have torsion in $H_*(C_n(\Gamma), \mathbb{Z})$, as in this case the moduli space of flat bundles has a simpler structure. However, one has to keep in mind, that we still lack a physical model of particles constrained to move on a graph, which has anyonic excitations.

Problem 3. – Inheritance of torsion from $H_*(C_n(\mathbb{R}^2), \mathbb{Z})$ by $H_*(C_n(\Gamma), \mathbb{Z})$. One possible way to construct examples of graphs, whose configuration space has torsion components different than $\mathbb{Z}_2$ in its homology, is to use the knowledge of generators of $H_*(C_n(\mathbb{R}^2), \mathbb{Z})$ [41, 42, 43] and the embedding $\Gamma \hookrightarrow \mathbb{R}^2$. More precisely, one would like to find a homomorphism $H_*(C_n(\mathbb{R}^2), \mathbb{Z}) \to H_*(C_n(\Gamma), \mathbb{Z})$, which does not map some of the torsion components of $H_*(C_n(\mathbb{R}^2), \mathbb{Z})$ to zero. This might be accomplished by embedding a (necessarily planar) graph Γ in $\mathbb{R}^2$, which induces embedding $C_n(\Gamma) \hookrightarrow C_n(\mathbb{R}^2)$, and considering for example the corresponding Mayer-Vietoris homological sequence. However, to accomplish this, one has to do an intermediate step, where the edges of Γ are thickened (are given a finite thickness), so that the resulting configuration space is a subset of $C_n(\mathbb{R}^2)$ of the full dimension. Denote the thickened graph by Γ_{thick}. By considering the continuous map, which changes the thickness of Γ_{thick} from ϵ to 0, we get a push-forward of homology $H_*(C_n(\Gamma_{thick}), \mathbb{Z}) \to H_*(C_n(\Gamma), \mathbb{Z})$. On the other hand, we have a proper embedding $C_n(\Gamma_{thick}) \hookrightarrow C_n(\mathbb{R}^2)$. Assume now, that Γ is a crate. Using the fact, that $H_*(C_n(\mathbb{R}^2), \mathbb{Z}) \cong H_*(C_n([0,1] \times [0,1]), \mathbb{Z})$, we have a decomposition of configuration spaces similar to the one we considered in section 5.3.1 while computing homology groups of graphs of connectivity one. This decomposition is induced by decomposing

$$[0,1] \times [0,1] = \Gamma_{thick} \cup \left(\sqcup_{\alpha \in A} D^2_\alpha \right), \tag{6.1}$$

where $\sqcup_{\alpha \in A} D^2$ is a disjoint union of disks indexed by A (see figure 6.1). Note, that $\Gamma_{thick} \cap (\sqcup_{\alpha \in A} D^2_\alpha)$ is a disjoint union of thick circles. To decomposition (6.1) we associate a decomposition of $C_n([0,1] \times [0,1])$ according to distributions of different numbers of particles on Γ_{thick} and $(\sqcup_{\alpha \in A} D^2)$. Then, by considering Mayer-Vietoris sequences we get homomorphisms that relate the homology of $C_n([0,1] \times [0,1])$ with the homology of $C_n(\Gamma_{thick})$ and $C_n((\sqcup_{\alpha \in A} D^2_\alpha))$. Note, that homology of $C_n((\sqcup_{\alpha \in A} D^2_\alpha))$ is a direct sum of homologies of configuration spaces of disks, which are the same as homology of $C_n([0,1] \times [0,1])$.

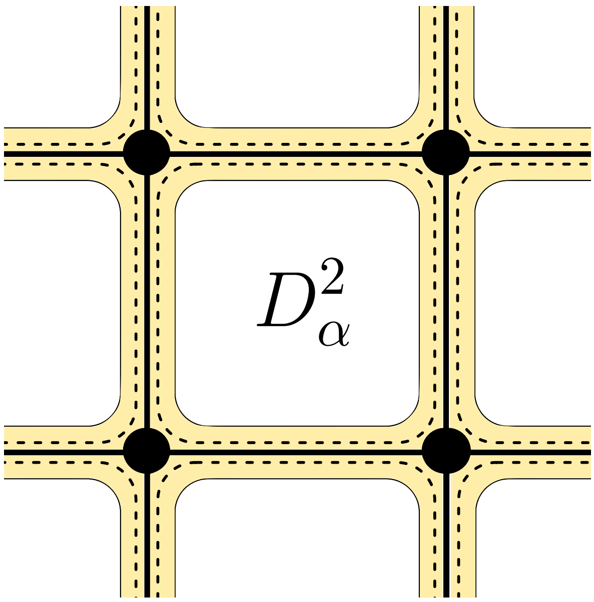


Figure 6.1: Decomposition of square $[0, 1] \times [0, 1]$ into a thick crate (yellow color) and a number of disjoint disks $\{D_\alpha^2\}_{\alpha \in A}$ (bounded by dashed lines).

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