Theorem Concerning Gauge Invariance in Quantum Electrodynamics

I. Bialynicki-Birula
Institute of Theoretical Physics, Warsaw University, Warsaw, Poland
(Received 3 October 1966)

It is pointed out that the theorem on gauge invariance of transition amplitudes in quantum electrodynamics given by Feynman and used by many others is incorrect. The invariance of physical scattering amplitudes under the gauge transformations of the photon propagator is briefly discussed.

In his classical paper Feynman proves the following theorem on the gauge invariance of the transition amplitudes. Transition amplitudes obtained directly from Feynman diagrams are not influenced on the mass shell by any terms proportional to either $k_\mu$ or $k_\nu$ in the photon propagator $D_{\mu\nu}(k)$. Since 1949, this theorem has been widely used by various authors.

The purpose of this article is to show that the theorem is false and to explain why all physical scattering amplitudes are invariant under gauge transformations of the photon propagator:

$$D_{\mu\nu}(k) \rightarrow D_{\mu\nu}(k) + k_\mu k_\nu f(k^2).$$

Feynman's proof is based on some identities, which are derived under a tacit assumption that none of the factors $(m - p_k)^{-1}$ becomes singular on the mass shell. This assumption is not valid whenever there are radiative corrections to external electron lines. Such corrections are present in all higher orders of perturbation theory, and therefore, the theorem is valid only in the lowest order of perturbation theory (for example, for Møller or Bhabha scattering).

As a simple example, consider the scattering of an electron in an external electromagnetic field $A_\mu$ in the second order of perturbation theory. In this case, the longitudinal part $k_\mu k_\nu f(k^2)$ of the photon propagator contributes on the mass shell, the following terms to the transition amplitude:

$$-\gamma^2 A_\mu (p_1 - p_2) \int d^4k \, f(k^2).$$

This difference from a zero result is to be expected on the basis of renormalization theory. Unrenormalized $S$-matrix elements obtained from Feynman diagrams have no direct physical meaning. They appear in physical-scattering amplitudes always multiplied by renormalization constants $Z_2$ and $Z_4$. While $Z_2$ is gauge invariant, $Z_4$ is gauge dependent. The unrenormalized transition amplitudes, proved erroneously in Ref. 1 to be invariant, must, therefore be, gauge-dependent in order to secure the gauge invariance of the product: renormalization constants times unrenormalized matrix elements. Gauge-transformation properties of the propagators in quantum electrodynamics enable one to prove directly that this invariance is, indeed, obtained.

Finally, we would like to mention that various proofs of the invariance of the unrenormalized $S$ operator under the gauge transformations of the photon propagator are also incorrect. All such proofs employ integration by parts and the boundary terms are not taken properly into account. When these terms are computed correctly, they contribute precisely the factors needed to make the renormalized $S$-matrix elements invariant under the gauge transformations of the photon propagator.


Reference 1, p. 781.

Reference 1, Fig. 4.

It can even be made finite in every order of perturbation theory by a proper choice of the longitudinal part of the photon propagator [I. Bialynicki-Birula, Nuovo Cimento 17, 122 (1960)].
