Propagation of Squeezing of the Electromagnetic Field

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Abstract Quantum field theoretic approach is used to describe the squeezing of the electromagnetic field in its most complete form that takes into account temporal and spatial characteristics of the squeezed state. This approach enables one to study the propagation of the “squeezing wave” in space-time. A simple example of a weak squeezing, that allows for all calculations to be done analytically, is discussed in detail.

1 Introduction

The standard description of squeezing of the electromagnetic field [1] employs the mode decomposition of the field operators. Subsequently only one or two modes are singled out for the analysis of the squeezing phenomena. This procedure gives an effective tool to explain most squeezing experiments. However, it will not be adequate if one attempts to describe the propagation of squeezing in space-time, when many modes are participating in the process and the state of the field exhibits strong correlations between different modes. In order to accommodate such situations, I shall use a description of squeezing started in Refs. [3–5] that employs formal methods of quantum field theory and enables one to produce compact and closed expressions for the space-time characteristics of the squeezing phenomena.

Since the electromagnetic field may be viewed as one multidimensional harmonic oscillator, many effects of the mode coupling can be clearly seen in the simpler case of an \( N \)-dimensional harmonic oscillator in quantum mechanics. Using this example, I shall introduce in Sec. 2 the necessary formalism and in Sec. 3 I shall describe the solution of the time evolution problem. Even though the electromagnetic field corresponds to an infinite-dimensional oscillator, there is a complete formal analogy between these two cases. In Sec. 4, the formalism developed for

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squeezed states of the quantum mechanical oscillator will be carried over to quantum electrodynamics. Finally, in Sec. 5 the problem of the propagation of squeezing of the electromagnetic field in space-time will be addressed.

2 Multidimensional harmonic oscillator in quantum mechanics

Squeezed states are usually defined in terms of annihilation and creation operators, but the description in terms of the wave function in coordinate representation or in momentum representation is very useful for a general analysis:

- It offers a geometric picture of the squeezing phenomenon that is easy to grasp intuitively
- It leads to compact formulas that exhibit the correlations between various oscillators
- It enables one to give a simple general description of time evolution
- It uses the smallest possible number of parameters to describe an arbitrary squeezed state

Of course, the formalism based on annihilation and creation is very useful and it will be also introduced and related to the wave function formalism.

The general Hamiltonian of an $N$-dimensional harmonic oscillator has the form

\[
H = \frac{1}{2}(g^{ij}p_ip_j + u_{ij}x^ix^j),
\]

(1)

where both matrices $g$ and $u$ are symmetric and positive definite. By an appropriate change of variables, we may transform $g^{ij}$ to the Kronecker delta.

\[
g^{ij} = \delta^{ij}.
\]

(2)

Note, that the new variables $x^i$ have the dimension of length times the square root of mass. By an additional orthogonal transformation, we may transform the potential matrix $u$ to a diagonal form. The eigenvalues $u_i$ of this matrix are multiples of the squares of the characteristic frequencies $\Omega_i$ of the oscillators $\Omega_i = \sqrt{u_i}$.

The ground state of the system with the Hamiltonian (1) is described by the following Gaussian wave function

\[
\psi_0(r) = C \exp\left(-\frac{1}{2\hbar}K^{0}_{ij}x^ix^j\right),
\]

(3)
where $K^0$ is the (positive) square root of the potential matrix $u$. In our diagonal representation $K^0$ has the form

$$K^0 = \Omega \equiv \begin{pmatrix} \Omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \Omega_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \Omega_n \end{pmatrix}.$$  \hspace{1cm} (4)

In addition to the ground state there exist other Gaussian wave functions. The most general Gaussian wave function of an $N$-dimensional harmonic oscillator is characterized by two real vectors $\xi^i$ and $\pi_i$ and two real symmetric $N \times N$ matrices forming one complex matrix $K_{ij} = a_{ij} + ib_{ij}$,

$$\psi(r) = C \exp\left[\frac{-1}{2\hbar} K_{ij}(x^i - \xi^i)(x^j - \xi^j) + i\pi_i x^i / \hbar\right], \hspace{1cm} (5)$$

where $C$ is the normalization constant.

Gaussian wave functions are special: the Gaussian form of the initial wave function remains Gaussian during the whole time evolution. This property has been recognized right after the discovery of wave mechanics [7]. In addition, the Gaussian wave functions saturate the Heisenberg uncertainty relations (for appropriate combinations of position and momentum variables) and also the entropic [6] uncertainty relations.

The existence of the ground state provides us with a standard against which we can measure the departures of the mean square deviations in positions and momenta from their normal values and define the notion of squeezing. Thus, the notion of a squeezed state always refers to a specified Hamiltonian. All Gaussian wave functions correspond to squeezed states, except the one describing the ground state for which $a_{ij} = K^0_{ij}$ and $b_{ij} = 0$. When $a$ is “larger” than $K^0$ then the state is squeezed in the position variables and when it is smaller, it is squeezed in the momentum variables. Of course, $K^0$ is a matrix so that the degree of squeezing may depend on the direction.

The energy of the system separates into the center of mass part and the internal motion part

$$E = E_{\text{cm}} + E_{\text{int}}, \hspace{1cm} (6)$$

$$E_{\text{cm}} = \frac{1}{2} g^{ij} \pi_i \pi_j + \frac{1}{2} u_{ij} \xi^i \xi^j, \hspace{1cm} (7)$$

$$E_{\text{int}} = \frac{\hbar}{4} \text{Tr}\{gba^{-1}b + ga + ua^{-1}\}. \hspace{1cm} (8)$$

The energy of the internal motion tends to zero in the classical limit, because the Gaussian shrinks to zero, when $\hbar \to 0$ and one is left with the motion of a classical point-like particle. It is worthwhile to observe that the Gaussian solutions of the
Schrödinger equation correspond to two independent autonomous mechanical systems with their Hamiltonians given by the expressions (7) and (8).

From the expression for $E_{\text{int}}$ one may obtain the following formulas for the first and second variation of the internal energy around the ground state

$$
\delta E_{\text{int}} = \hbar \frac{1}{4} \text{Tr}\{\delta a - ua^{-1}\delta aa^{-1}\} = 0,
$$
\((9)\)

$$
\delta^2 E_{\text{int}} = \hbar \frac{1}{2} \text{Tr}\{\delta b \Omega^{-1} \delta b + \delta a \Omega^{-1} \delta a\},
$$
\((10)\)

where I have used $g = 1$. The equilibrium condition derived from $\delta E_{\text{int}} = 0$ has the form ($\delta a$ is arbitrary)

$$
a^{-1}ua^{-1} = 1,
$$
\((11)\)

in accordance with the equations of motion that give the same formula for the time independent state. The quadratic form defining the second variation is clearly positive confirming the fact that the energy of the ground state is minimal.

The formula (6) can also be used to find the change in the energy of the ground state due to a change of the matrices $g$ and $u$ describing the system. Here we can see the advantage of keeping $g$ and $u$ arbitrary.

$$
\delta E = \hbar \frac{1}{4} \text{Tr}\{\delta g a + g \delta a + \delta u a^{-1} - ua^{-1} \delta aa^{-1}\} = \hbar \frac{1}{4} \text{Tr}\{\delta g a + \delta u a^{-1}\}.
$$
\((12)\)

Since for the ground state the variations $\delta a$ of the matrix $a$ cancel out, this formula may be viewed as an analog of the Feynman-Hellmann theorem [9,10].

Analogous results are obtained in momentum representation. The general Gaussian wave function in this representation is obtained by the Fourier transformation of the wave function (5) in coordinate representation and it reads

$$
\phi(r) = C \exp\left[-\frac{1}{2\hbar}(K^{-1})^{ij}(p_i - \pi_i)(p_j - \pi_j) - i\xi^i p_i / \hbar\right].
$$
\((13)\)

Also the formula (8) for the internal energy of the Gaussian state, despite its appearance, exhibits a symmetry under the interchange of positions and momenta. Denoting by $c$ and $d$ the real and the imaginary parts of the matrix $K^{-1}$,

$$
c = (a + ba^{-1}b)^{-1}, \quad d = -a^{-1}b(a + ba^{-1}b)^{-1},
$$
\((14)\)

we can rewrite (6) as

$$
E_{\text{int}} = \hbar \frac{1}{4} \text{Tr}\{udc^{-1}d + uc + gc^{-1}\}.
$$
\((15)\)
3 Time evolution of squeezed states

In order to satisfy the time-dependent Schrödinger equation, the vectors $\xi$, $\pi$, and the matrix $K$ must be taken as appropriate functions of time. Upon substituting (5) into the Schrödinger equation with the Hamiltonian (1), one finds the classical equations of motion for the position $\xi$ and momentum $\pi$ of a harmonic oscillator

$$\frac{d}{dt} \xi^i(t) = g^{ij} \pi_j(t), \quad \frac{d}{dt} \pi_i(t) = -u_{ij} \xi^j(t), \tag{16}$$

and the following nonlinear evolution equation for the matrix $K$

$$\frac{d}{dt} K(t) = -i K(t) g K(t) + i u. \tag{17}$$

The vectors $\xi$ and $\pi$ determine the center of mass position and the center of mass momentum of the Gaussian wave packet. The matrices $a$ and $b$ determine the shape of the wave function and the distribution of the probability current of the internal “tumbling” motion, respectively.

It follows from Eq.(17) and from the relation $dK^{-1}/dt = -K^{-1} dK/dt K^{-1}$ that the inverse matrix $K^{-1}$ satisfies the evolution equation in which the roles of $g$ and $u$ are interchanged,

$$\frac{d}{dt} K^{-1}(t) = -i K^{-1}(t) u K^{-1}(t) + i g, \tag{18}$$

as one would expect from the formula (13) and the symmetry of the Hamiltonian under the interchange of positions and momenta.

The nonlinear equation (17) is a matrix Riccatti equation and it may be converted into a set of linear equations by the following substitution ($g = 1$)

$$K(t) = D^{-1}(t) N(t) \Omega. \tag{19}$$

Eq. (17) is satisfied when two complex matrices $N$ and $D$ satisfy the set of equations

$$\frac{d}{dt} D(t) = i N(t) \Omega, \quad \frac{d}{dt} N(t) = i D(t) \Omega. \tag{20}$$

The solution of this linear set of equations

$$N(t) = N(0) \cos \Omega t + i D(0) \sin \Omega t, \quad D(t) = D(0) \cos \Omega t + i N(0) \sin \Omega t, \tag{21,22}$$

gives the following result for $K(t)$

$$K(t) = (\cos \Omega t + i K(0) \frac{\sin \Omega t}{\Omega})^{-1} (K(0) \cos \Omega t + i \Omega \sin \Omega t). \tag{23}$$
The order of matrices is here important since in general the initial value $K(0)$ and $\Omega$ do not commute. Actually, there are two equivalent forms of $K(t)$ resulting from the identity

$$
(\cos \Omega t + iK(0)\frac{\sin \Omega t}{\Omega})^{-1}(K(0)\cos \Omega t + i\Omega \sin \Omega t) = (\cos \Omega t + i\Omega \sin \Omega t)(\cos \Omega t + i\frac{\sin \Omega t}{\Omega}K(0))^{-1}.
$$

(24)

Since both sides of this equality are related by the matrix transposition, the symmetry of the matrix $K(t)$ is preserved during time evolution. For the ground state (and only for the ground state), $K$ is constant in time, even though both matrices $N$ and $D$ vary with time: $N(t) = D(t) = e^{i\Omega t}$. From Eq. (23) one obtains the following formula for the time evolution of $a$

$$
a(t) = D^{-1}(t)a_0D^{-1}(t),
$$

(25)

that shows explicitly that the positive definite character of $a$ is preserved during the time evolution.

The formula (23) describes completely the dynamics of squeezed states for an arbitrary harmonic oscillator. The time evolution of $K$ may get quite complicated for a multidimensional harmonic oscillator when the matrices $\Omega$ and $K(0)$ do not commute so that they cannot be simultaneously diagonalized. In the simple case, when the initial value of $K$ represents just a small perturbation of the ground state,

$$
K(0) = \Omega + \delta K(0),
$$

(26)

then the formula (23) simplifies as follows

$$
K(t) = \Omega + e^{-i\Omega t} \delta K(0)e^{-i\Omega t}.
$$

(27)

Owing to the correspondence between a set of harmonic oscillators and the electromagnetic field, an analogous approach to the one presented here works for the general squeezed states of the electromagnetic field.

4 Quantized electromagnetic field

In order to use the analogy with quantum mechanics of harmonic oscillators to its full extent, I shall employ the formulation of quantum electrodynamics in a representation which is the closest possible counterpart of the Schrödinger representation. In this representation the state is described by a functional $\Psi[A]$ of the vector potential. The vector potential operator $\hat{A}(r)$ and the magnetic induction operator $\hat{B}(r)$ acts on the state functional as a multiplication,

$$
\hat{A}(r)\Psi[A] = A(r)\Psi[A], \quad \hat{B}(r)\Psi[A] = (\nabla \times A(r))\Psi[A],
$$

(28)
while the electric displacement operator $\hat{D}(\mathbf{r})$ acts as the functional differentiation,

$$\hat{D}(\mathbf{r})\Psi[A] = i\hbar\frac{\delta}{\delta A(\mathbf{r})}\Psi[A].$$

(29)

The Hamiltonian in this representation takes on the form

$$H = \frac{1}{2} \int d^3r \left[ -\frac{\hbar^2}{\epsilon(\mathbf{r})} \frac{\delta^2}{\delta A(\mathbf{r})^2} + \frac{1}{\mu(\mathbf{r})} (\nabla \times A(\mathbf{r}))^2 \right].$$

(30)

The most general Gaussian functional is determined by two vector functions $A(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ and — the counterparts of $\xi$ and $\pi$ — and a complex symmetric kernel $K(\mathbf{r}, \mathbf{r}')$ — the counterpart of the matrix $\mathcal{K}$,

$$\Psi[A] = C \exp \left[ -\frac{1}{2\hbar} \int d^3r \int d^3r' (A(\mathbf{r}) - A(\mathbf{r}')) \cdot \mathcal{K}(\mathbf{r}, \mathbf{r}') \cdot (A(\mathbf{r}') - A(\mathbf{r}')) \right] + \frac{i}{\hbar} \int d^3r \mathbf{D}(\mathbf{r}) \cdot A(\mathbf{r})],$$

(31)

In addition to being a function of two vector arguments, the kernel $K$ is also a $3 \times 3$ matrix since it acts on the vector indices of $A_i$. The vector function $A(\mathbf{r})$, like the kernel $K$, is just a label characterizing the state. To distinguish it from the argument $A$ of the state functional, it is set in a different font. In order to secure gauge invariance, i.e., the invariance of $\Psi[A]$ under the gauge transformations $A(\mathbf{r}) \rightarrow A(\mathbf{r}) + \nabla \lambda(\mathbf{r})$, the kernel $K^{ij}(\mathbf{r}, \mathbf{r}')$ must obey the conditions

$$\partial_i K^{ij}(\mathbf{r}, \mathbf{r}') = 0 = K^{ij}(\mathbf{r}, \mathbf{r}') \partial'_j.$$

(32)

Thus, the kernel $K$ must be a double curl of some new kernel $\mathcal{W}$,

$$K^{ij}(\mathbf{r}, \mathbf{r}') = \epsilon^{ikm} \partial_k W_{mn}(\mathbf{r}, \mathbf{r}') \partial'_l \epsilon^{jln}.$$

(33)

This means that $\Psi[A]$ depends on $A(\mathbf{r})$ only through $\mathbf{B}(\mathbf{r}) = \nabla \times A(\mathbf{r})$.

The total energy of the electromagnetic field, like in the case of the quantum-mechanical oscillator, is made of two parts: the classical energy of the coherent field and the quantum correction due to squeezing

$$E = \frac{1}{2} \int d^3r \left( \frac{D^2(\mathbf{r})}{\epsilon(\mathbf{r})} + \frac{B^2(\mathbf{r})}{\mu(\mathbf{r})} \right)$$

(34)

$$+ \frac{\hbar}{4} \int d^3r' \int d^3r \delta_{kl} \delta(\mathbf{r} - \mathbf{r}') \epsilon^{mn} \partial_m a^{-1}_{ij}(\mathbf{r}, \mathbf{r}') \epsilon^{nl} \partial'_n$$

$$+ \frac{\hbar}{4} \int d^3r \int d^3r' \delta_{ij} \left( \int d^3r'' \epsilon^{kl} \mathcal{W}_{mn}(\mathbf{r}, \mathbf{r}'') \partial_m a^{-1}_{kl}(\mathbf{r}', \mathbf{r}'') \partial'_n + a_{ij}(\mathbf{r}, \mathbf{r}) \right),$$
where $a^{ij}$ and $b^{ij}$ are the real and the imaginary part of $K^{ij}$,

$$K^{ij}(r, r') = a^{ij}(r, r') + ib^{ij}(r, r')$$  \hspace{1cm} (35)

and the inverse of $a^{ij}$ is to be taken in the subspace of transverse kernels satisfying the conditions (32).

In order to satisfy the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi[t|\mathcal{A}] = H \Psi[t|\mathcal{A}],$$  \hspace{1cm} (36)

the functions $\mathcal{A}$ and $\mathcal{D}$ and the kernel $\mathcal{K}$ must depend on time, except for the ground state. Their time evolution can be derived by substituting the wave functional

$$\Psi[t|\mathcal{A}] = C \exp\left[-\frac{1}{2\hbar}\int d^3r \int d^3r' (\mathcal{A}(r) - \mathcal{A}(r, t)) \cdot \mathcal{K}(r, r', t) \cdot (\mathcal{A}(r') - \mathcal{A}(r', t)) + \frac{i}{\hbar} \int d^3r \mathcal{D}(r, t) \cdot \mathcal{A}(r)\right],$$  \hspace{1cm} (37)

into (36) and then by comparing the terms of the same form on both sides of this equation. In this way one obtains the Maxwell equations for $\mathcal{B} = \nabla \times \mathcal{A}$ and $\mathcal{D}$

$$\partial_t \mathcal{B}(r, t) = -\nabla \times \frac{\mathcal{D}(r, t)}{\epsilon(r)},$$  \hspace{1cm} (38)

$$\partial_t \mathcal{D}(r, t) = \nabla \times \frac{\mathcal{B}(r, t)}{\mu(r)},$$  \hspace{1cm} (39)

and the following nonlinear integro-differential equation for $\mathcal{K}$

$$\partial_t \mathcal{K}^{ij}(r', r'', t) = -i \int d^3r \mathcal{K}^{ik}(r', r, t) \frac{\delta_{jl}}{\epsilon(r)} \mathcal{K}^{jk}(r, r'', t) + i e^{imk} \partial_m \delta_{ijl} \frac{\delta_{ij}(r' - r'')}{\mu(r')} \frac{\partial_n}{\partial_n} e^{jn}. \hspace{1cm} (40)$$

In the simplest case, when the medium is homogeneous, the kernel $\mathcal{K}$ may depend only on the difference of coordinates and the time-independent solution of Eq. (41) can be easily solved by the Fourier transformation. The resulting expression has the form

$$\tilde{\mathcal{K}}^{ij}(k) = \sqrt{\frac{\epsilon}{\mu}} e^{imk} k_m \frac{\delta_{kl}}{|k|} k_n e^{jn}. \hspace{1cm} (41)$$

The coordinate representation of this expression is

$$\mathcal{K}^{ij}(r - r') = \frac{1}{4\pi^2} \sqrt{\frac{\epsilon}{\mu}} e^{ikm} \partial_k \frac{\delta_{mn}}{|r - r'|^2} \partial_l e^{jn}. \hspace{1cm} (42)$$
and the ground-state functional of the electromagnetic field in the homogeneous medium is

$$\Psi_0 [A] = C \exp \left[ -\frac{1}{4\pi^2\hbar} \sqrt{\frac{\epsilon}{\mu}} \int d^3r \int d^3r' B(r) \frac{1}{|r - r'|^2} B(r') \right],$$  \hspace{1cm} (43)

This formula in the case of the vacuum has been written down by Wheeler. Here I would like to study its significance in the presence of a medium. The existence of the ground state in any static medium enables one to define the ground state and then the notion of squeezing with respect to this ground state. One can see from (43) that the dielectrics decrease the fluctuations of the magnetic field, while the magnetics increase them.

In the case of the electromagnetic field there also exist an analog of the quantum-mechanical momentum representation. In this representation the state of the electromagnetic field is given as a functional of $D$ which is the canonically conjugate variable to the potential $A$. Owing to the known symmetry of Maxwell theory without sources, the formula for the ground state functional in the $D$ representation can be obtained by the replacements

$$D \to B, \quad B \to -D, \quad \epsilon \to \mu, \quad \mu \to \epsilon.$$  \hspace{1cm} (44)

and it has the form

$$\Psi_0 \left[ \tilde{A} \right] = C \exp \left[ -\frac{1}{4\pi^2\hbar} \sqrt{\frac{\mu}{\epsilon}} \int d^3r \int d^3r' D(r) \frac{1}{|r - r'|^2} D(r') \right].$$  \hspace{1cm} (45)

where $\tilde{A}$ is the vector potential for $D$, i.e. $D = \nabla \times \tilde{A}$.

For an inhomogeneous medium, one may easily find first-order changes in $\mathcal{K}$ by perturbation theory. The most interesting case is that of a change in the dielectric constant. The exact kernel for the ground state in a static medium satisfies the equation

$$\int d^3r \mathcal{K}^{ij}(r', r) \frac{\delta \epsilon(r)}{\epsilon(r)} \mathcal{K}^{ij}(r, r'') = \epsilon^{imk} \delta_{mn} \frac{\delta \epsilon(r' - r'')}{\mu(r')} \partial_n \epsilon^{jnl},$$  \hspace{1cm} (46)

from which the following equation is obtained by taking a variation due a small change in $\epsilon(r)$

$$\int d^3r \delta \mathcal{K}^{ik}(r', r) \frac{\delta \epsilon(r)}{\epsilon(r)} \mathcal{K}^{ij}(r, r'') + \int d^3r \mathcal{K}^{ik}(r', r) \frac{\delta \epsilon(r)}{\epsilon(r)} \delta \mathcal{K}^{ij}(r, r'') \quad = \quad \int d^3r \mathcal{K}^{ik}(r', r) \frac{\delta \epsilon(r)}{\epsilon(r)^2} \mathcal{K}^{ij}(r, r'').$$  \hspace{1cm} (47)

When the variation is taken around the vacuum state, the translational invariance of this state allows one to seek the change in $\mathcal{K}$ in the form

$$\delta \mathcal{K}^{ij}(r', r'') = \int d^3r \Gamma^{ij}(r' - r, r - r'') \delta \epsilon(r).$$  \hspace{1cm} (48)
Applying the Fourier transformation to Eq. (47), one finds the following Fourier transform of the kernel \( \Gamma^{ij} \)

\[
\hat{\Gamma}^{ij}(k_1, k_2) = (\delta^{mn}k_1^2 - k_1^i k_1^m) \frac{c \delta_{mn}}{|k_1| |k_2| (|k_1| + |k_2|)} (\delta^{ij}k_2^2 - k_2^i k_2^j),
\]

(49)

where \( c \) is the speed of light in the medium In the coordinate representation, the expression for \( \Gamma^{ij} \) reads

\[
\Gamma^{ij}(r_1, r_2) = (\delta^{ik} \Delta - \partial^i \partial^k_1) (2\pi)^3 \frac{\delta_{ij}}{|r_1| + |r_2|} \left[ \frac{1}{\sqrt{\epsilon \mu}} \right] (\delta^{jk} \Delta - \partial^j \partial^k_2).
\]

(50)

This expression shows the long range effect in the squeezing produced by a dielectric: when one moves away from the dielectric, the function \( \Gamma \) falls only as \( 1/r^4 \).

With the use of Feynmann-Hellmann theorem (12) it is also easy to find the first order correction to the ground state energy \( E_0 \) due to a small departure of the dielectric constant from its vacuum value \( \epsilon_0 \). To this end, one may use the following adaptation to the present case of the formula (12) for the harmonic oscillator

\[
\delta E_0 = -\frac{\hbar}{4} \int d^3r \int d^3r' \frac{\delta \epsilon(r)}{\epsilon_0} \delta_{ij} K^{ij}(r' - r, r - r')
\]

(51)

The factor of 2 comes from two polarization states of the photon. The integral over \( k \) represents the energy of all photons that are affected by the change in the dielectric constant. Obviously, the integration does not extend to infinity because all dielectrics behave like the vacuum for sufficiently large photon energy and that fact will introduce a cutoff. Therefore, a more appropriate way to write this formula (51) is

\[
\delta E_0 = -\frac{1}{4} \int d^3r \int d^3k \frac{\delta \epsilon_k(r)}{\epsilon_0} \frac{1}{(2\pi)^3} \hbar c_k |k|.
\]

(52)

where \( \epsilon \) and \( c \) are taken as slowly varying functions of the wave vector \( k \). Since \( \delta c = \delta (1/\sqrt{\epsilon \mu}) = - (\delta \epsilon/2\epsilon) c \), the expression (52) can be recognized as the variation of the zero point energy evaluated in the local approximation when the variation of \( \epsilon \) in space is very slow over the distance of a wave length \( 1/k \),

\[
\delta E_0 = \delta \left( \frac{1}{2} \int d^3r \int d^3k \frac{\delta \epsilon_k(r)}{2(2\pi)^3} \hbar c_k |k| \right).
\]

(53)

First order perturbation theory, described here as a simple illustration of the global approach to squeezing, is clearly not sufficient to give the Casimir force. This force can also be understood as coming from the change in the energy of the ground state but it requires the calculation of the energy at least up to second order.

Correlation functions for the electromagnetic field in a squeezed state can be introduced along the same lines as for the harmonic oscillator. There are again four kinds of correlation functions but there are relations among them since they can all be expressed in terms of the kernel \( K \).
5 Propagation of squeezing

Propagation of squeezing, in principle, is fully described by the time evolution equation (23) of the kernel $K$ but it is rather difficult to draw physical conclusions from this equation in the general case. However, for small perturbations, when the departure from the vacuum situation is given by (48) and (50), one can find an explicit expression for the time dependence of $K$. To this end, I shall use the first order formula (27) obtained for the harmonic oscillator. In the present case, this formula takes on a very simple when $K$ is taken as a function of wave vectors because in this representation the frequency $\Omega$ is diagonal. The application of the operations $\exp(-i\Omega t)$ in the formula (27) amounts simply to the multiplications by $\exp(-ic|k|t)$. This leads to the following formula for $\delta K(t)$

$$\delta K^{ij}(r_1, r_2, t) = \int d^3r \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{ik_1 \cdot (r_1 - r)} e^{ik_2 \cdot (r - r_2)} \delta \epsilon(r) e^{-ic|k_1|t} e^{-ic|k_2|t} \tilde{\Gamma}^{ij}(k_1, k_2)$$

This leads to the following formula for $\Gamma^{ij}(r_1 - r, r - r_2, t)$

$$\Gamma^{ij}(r_1 - r, r - r_2, t) = c(\delta^{ik} \Delta - \partial^i \partial^k) \delta \epsilon G(r_1 - r, r - r_2, t)(\delta^{\alpha \beta} \Delta - \partial^\alpha \partial^\beta)G(r_1 - r, r - r_2, t),$$

where

$$G(r_1, r_2) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{ik_1 \cdot r_1} e^{ik_2 \cdot r_2} e^{-ic|k_1|t} e^{-ic|k_2|t} \frac{|k_1||k_2|(|k_1| + |k_2|)}{r_1 r_2}.$$

The integration over $k_1$ and $k_2$ can be easily performed after the change of variables:

$$G(r_1, r_2, t) = \frac{1}{(2\pi)^3} \int_0^{\infty} dk \int_0^{\infty} dk' \frac{e^{-ik\cdot r_1}}{k} \frac{e^{-ik'\cdot r_2}}{k'} \frac{\sin(k_1 r_1)}{k_1} \frac{\sin(k_2 r_2)}{k_2}.$$

The integration over $k_1$ and $k_2$ can be easily performed after the change of variables:

$$G(r_1, r_2, t) = \frac{1}{(2\pi)^3} \int_0^{\infty} dk \frac{e^{-ik\cdot r_1}}{k} \frac{\sin(k r_2)}{r_2} \frac{\sin(k r_1)}{r_1}.$$
where $\theta(r-ct)$ is the step function. The function (57) describes fully the propagation of squeezing in the vacuum when the departure from the vacuum state is small (cf. Eq. (27)). Unfortunately, this approximation is not valid near the light cone in the variables $r$ and $t$ because of the singularities in Eq. (57). A better, nonperturbative approach is needed to determine the behavior of squeezing on the light cone. What does follow, however, from these considerations is that squeezing has very interesting propagation properties. In my simple example, the initial value of the squeezing kernel $K$ has been influenced by a change in the value of $\epsilon$ that took place in some region of space. This influence extends throughout space with the power-law falloff when moving away from the region where $\epsilon$ has been changed, as determined by the formula (50). Then, at $t = 0$, the vacuum value of $\epsilon$ is restored and the electromagnetic field begins its relaxation to the true vacuum state. The real part of $K$ relaxes to zero with the speed of light as seen from the presence of the step functions, while the imaginary part relaxes to zero much more slowly as determined by the logarithmic functions.

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References