Spatial Quantum State Tomography for Photons

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Isaac Newton (Cambridge, 1672) published *A Theory Concerning Light and Colors* - interpreted experiments as evidence that light is composed of *particles*, which bend their path according to their color.
Thomas Young (Cambridge, UK 1799)
- observed the pattern of dark and light bands seen on a screen behind an opaque plane containing one or two slits.
- strong evidence for a wave nature of light

Light passing through a single slit onto a screen
Light passing through two slits onto a screen
James Clerk Maxwell (Edinburgh, 1862-64) published *On Physical Lines of Force*

His new equations predicted the existence of electromagnetic waves, which travel at a fixed speed.

“This velocity is so nearly that of light that it seems we have strong reason to conclude that light itself is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field.”
Special Relativity

*Albert Einstein* (1905)

- the laws of physics look the same in any moving frame.
- light travels at the same speed $c$ in any frame.
- the energy of a particle with rest mass $m$, moving with momentum $p$ is:

$$ E = \sqrt{(mc^2)^2 + (cp)^2} $$

- for slow particles:

$$ E \approx mc^2 + \frac{p^2}{2m} + ... $$

_Einstein rest energy_  
_Newton kinetic energy_
Photoelectric effect (1900) - W. Hallwachs and P. Lenard

- Einstein hypothesized that light is made of particle-like objects called quanta, and later “photons.” Each photon has energy $E = h\nu$.

Einstein (1924): “There are therefore now two theories of light, both indispensable, and - as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists - without any logical connection.”
the modern view:
a photon is an excitation (that is, a state) of a quantum field.

still:

it is tempting to think of a photon as a kind of particle.

question:

if a photon is a particle, what is its quantum wave function, and what wave equation does it obey?
What are the differences between electrons and photons?

Electron has:
- nonzero mass
- any speed < c
- Spin = 1/2 (two possible projections along any chosen axis, +1/2, -1/2)
- \( \rightarrow \) two-component \textit{wave function} \( \Psi^{(2)} \)

obeying the Schrödinger equation:

\[
\frac{i\hbar}{\partial t} \Psi^{(2)} = -\frac{\hbar^2}{2m} \nabla^2 \Psi^{(2)}
\]

Photon has:
- zero mass
- speed = c
- Spin = 1 (two possible projections along propagation axis, +1, -1)
- \textit{wave function} obeying what equation? how many components?
Paul Dirac (Cambridge, 1928)
-relativistic quantum theory of electron

\[ E = \sqrt{(mc^2)^2 + (cp)^2} \]
\( m = \text{mass} \)
\( p = \text{momentum} \)

\[ i\hbar \frac{\partial}{\partial t} \Psi = \sqrt{(mc^2)^2 + c^2(-i\hbar \nabla)^2} \Psi \]

**Dirac Equation**

\[ i\hbar \frac{\partial}{\partial t} \Psi = cm\beta \Psi - i\hbar c (\vec{\alpha} \cdot \nabla) \Psi \]

\( v << c \)

**Schrödinger Equation**

\[ i\hbar \frac{\partial}{\partial t} \Psi^{(2)} \cong -\frac{\hbar^2}{2m} \nabla^2 \Psi^{(2)} \]

2 components
(spin up, down)
A portion of the family tree of the Photon

1905
Einstein

1930s
Landau, Peierls

1940s

1950s

1960s

1970s

1980s

1990s

2000s

Rochester

Marshak

Emil Wolf

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Derivation of Maxwell's Equations

\[ E = \sqrt{(cp)^2} \]

Photon, \( m=0, \ s=1, \ 3 \) components

\[ E \tilde{\psi}(\vec{p}, E) = c \sqrt{p \cdot \vec{p}} \tilde{\psi}(\vec{p}, E) \]

\[ \hat{O} = \sqrt{p \cdot \vec{p}} = ip \times \]

\[ \hat{O} \tilde{\psi} = i\vec{p} \times (i\vec{p} \times \tilde{\psi}) = (p \cdot \vec{p})\tilde{\psi} - \vec{p}(p \cdot \tilde{\psi}) = (p \cdot \vec{p})\tilde{\psi} \]

\[ E \tilde{\psi}_T(p, E) = c i \vec{p} \times \tilde{\psi}_T(p, E) \]

\[ \tilde{\psi}(\vec{r}, t) = \int \int dE d^3p \ \delta(E - c|\vec{p}|) \exp(-iEt / h + i\vec{p} \cdot \vec{r} / h) f(E)\tilde{\psi}(\vec{p}, E) \]

\[ i \frac{\partial}{\partial t} \tilde{\psi}(\vec{r}, t) = c \vec{\nabla} \times \tilde{\psi}(\vec{r}, t) \]

\[ \tilde{\psi}(\vec{r}, t) = \vec{E} + i\vec{B} \]

Riemann-Silberstein vector

\[ \frac{\partial}{\partial t} \vec{B} = -c \vec{\nabla} \times \vec{E} \]
\[ \frac{\partial}{\partial t} \vec{E} = c \vec{\nabla} \times \vec{B} \]
For a single-photon field, the complex electromagnetic field \((E+iB)\) is the quantum wave function of the photon.

\[
\psi(\vec{r}) = \begin{pmatrix} E_x(\vec{r}) + iB_x(\vec{r}) \\ E_y(\vec{r}) + iB_y(\vec{r}) \\ E_z(\vec{r}) + iB_z(\vec{r}) \end{pmatrix}
\]

I. Bialynicki-Birula (Ustron1993)
J. Sipe (PRA 1995)

Maxwell, in 1862, discovered a relativistic, quantum mechanical theory of a single photon.
Is it possible to determine the quantum wave function of a photon by making measurements?

“To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements.”

- John Bell (1966)

Converse--

To know the statistical restrictions on the results of all possible measurements is to know the quantum mechanical state of a system.
QUANTUM STATE TOMOGRAPHY

measure many different quantities

ensemble of particles

EM Wave Fronts (paraxial)

\[ \vec{E}(x) = \exp(ik_z z) \vec{e} \, E(x) \]

\[ p_x = \hbar k_x \]

Light Source

Near Field

Far-Field Diffraction
Wigner Function

\[ W(p_x, x) = \int \psi(x + x')\psi^*(x - x')e^{-2ip_x x'/h} \, dx' \]

The Wigner Function is uniquely related to the Wave Function, so its measurement reveals \( \psi(x) \) not just \( |\psi(x)|^2 \)

for photon: \[ \psi(x) \approx \tilde{\epsilon}(E_x(x) + iB_x(x)) \]
Transverse Wigner Function

- a Quasi-joint probability function for $x, k_x$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(x, k_x) dk_x dx = 1$$

$$\Pr(x) = \int_{-\infty}^{+\infty} W(x, k_x) dk_x = |\psi(x)|^2$$  \text{Near Field}

$$\Pr(k_x) = \int_{-\infty}^{+\infty} W(x, k_x) dx = |\tilde{\psi}(k_x)|^2$$  \text{Far Field Diffraction}
Measuring the Wigner Function of Photons by means of parity-inverting optics

Clockwise (CW) Beam

Counter-Clockwise (CCW) Beam

Diagram by Brian Smith
Parity-Inverting Sagnac Interferometer

Interference Pattern
Measuring the Wigner Function of Photons with a parity-inverting interferometer

\[ \psi(x') = \psi_0(x' + x)e^{ik_x x'} \]

Diagram by Bryan Killet
Output Field

\[ E_{\text{Output}}(x') = E_{\text{CW}}(x') + E_{\text{CCW}}(x') \]

\[ E_{\text{Output}}(x') = \frac{1}{\sqrt{2}} \left\{ E_0(-x' + x)e^{-ik_xx'} + E_0(x' + x)e^{ik_xx'} \right\} \]

The average photon count rate equals the intensity integrated over the whole detector surface.

\[ I \propto \int \left| E_{\text{output}}(x') \right|^2 dx' \equiv C_1 + C_2 \cdot \int_{-\infty}^{+\infty} E_0(x + x')E_0^*(x - x')e^{-2ik_xx'} dx' \]
Direct measurement of Wigner Function

The average photon count rate equals:

\[
I \equiv C_1 + C_2 \cdot \int_{-\infty}^{+\infty} \psi_0(x + x')\psi_0^*(x - x')e^{-2ik_xx'}dx'
\]

\[
I \equiv C_1 + C_2 \cdot W(x, k_x)
\]
Results – Gaussian Beam

photodiode detector

\[ W(x, k_x) \]
Single Slit - photon-counting detector

\[ W(x, k_x) \]

Measured Wigner function for ensemble of single photons.

\[ x = \text{position} \]

\[ k_x = \text{transverse momentum} \]
Double Slit - photon-counting detector

\[ W(x, k_x) \]

Measured Wigner function for ensemble of single photons.

\[ x = \text{position} \]

\[ k_x = \text{transverse momentum} \]
Wigner distribution of helium atoms after passing a double slit.

Two-photon Maxwell’s equations
- two possible forms

one-time:
\[
\frac{i}{c} \frac{\partial}{\partial t} \psi(r_1, r_2, t) = \vec{\nabla}_1 \times \overline{\psi}(r_1, r_2, t) + \vec{\nabla}_2 \times \overline{\psi}(r_1, r_2, t)
\]

two-time:
\[
\frac{i}{c} \frac{\partial}{\partial t_1} \psi(r_1, t_1; r_2, t_2) = \vec{\nabla}_1 \times \overline{\psi}(r_1, t_1; r_2, t_2)
\]
\[
\frac{i}{c} \frac{\partial}{\partial t_2} \psi(r_1, t_1; r_2, t_2) = \vec{\nabla}_2 \times \overline{\psi}(r_1, t_1; r_2, t_2)
\]

These are equivalent under the measurement-collapse hypothesis.
one-time Max.Eqn. \[ i \frac{\partial}{c \partial t} \psi(r_1, r_2, t) = \nabla_1 \times \psi(r_1, r_2, t) + \nabla_2 \times \psi(r_1, r_2, t) \]

solution: \[ \psi(r_1, r_2, t) = \sum_j C_j \psi_j(r_1, t) \otimes \phi_j(r_2, t) \]

at time $T_1$, measure $\vec{r}_1$, obtain value $\vec{R}_1$

\[ \psi(\vec{r}_1, \vec{r}_2, t) \Rightarrow \psi(\vec{R}_1, \vec{r}_2, t_2) = \sum_j \left[ C_j \psi_j(\vec{R}_1, T_1) \right] \otimes \phi_j(\vec{r}_2, t_2) \]

same form as two-time wave function before measurement:

\[ \psi(\vec{r}_1, t_1; \vec{r}_2, t_2) = \sum_j C_j \psi_j(\vec{r}_1, t_1) \otimes \phi_j(\vec{r}_2, t_2) \]
Synopsis

- Particle kinematics -> Single-photon Maxwell Equation for quantum wave function. (But there are some subtleties with Lorentz covariance of scalar product, etc.)
- Direct measurement of Wigner distribution for transverse state of photon.
- Two-photon wave function.
- Application: characterization of quantum channels and quantum logic gates
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