Quantum State Reconstruction via Continuous Measurement

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Quantum Information Processing

Classical Input

\[ \psi_{in} \]

QUANTUM WORLD

Control

\[ \psi_{out} \]

State Preparation

Measurement

Classical Output
Evaluating Performance

• State Preparation
  How well did we prepare $\rho_0$?

• Control dynamics
  How well did we implement a map $\rho_t = M_t[\rho_0]$?

• Sense external fields:  Metrology
  Under what Hamiltonian $H(\Theta)$ did the system evolve?
Quantum State Estimation

- Fundamental problem for quantum mechanics: Measure state $\rho$

Finite dimensional Hilbert space, dim $d$
- Any measurement, at most $\log_2 d$ bits/system
  e.g. Spin 1/2 particle, $d=2$: one bit.
- Density operator, $d^2-1$ real parameters
  Hermitian matrix with unit trace.
- Required fidelity: $b(d^2-1)$ bits of information.
  $b$ bits/matrix-element.

Require many copies $\rho_N = \rho^\otimes N$
Spin-1/2:

**density matrix** 3 indep. real numbers

\[ \text{Tr}[\rho] = 1 \]
\[ \rho = \rho^+ \]
\[ \rho_{11} \quad \text{Re}[\rho_{12}] \]
\[ \rho_{22} \quad \text{Im}[\rho_{12}] \]

**Stern-Gerlach measurement:**

<table>
<thead>
<tr>
<th>Q-Axis</th>
<th>Apparatus</th>
<th>Observable</th>
<th>Information</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \hat{S}_z )</td>
<td>( \rho_{11}, \rho_{22} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{S}_x )</td>
<td>( \text{Re}[\rho_{12}] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{S}_y )</td>
<td>( \text{Im}[\rho_{12}] )</td>
</tr>
</tbody>
</table>

Similarly

Note: can equally well rotate system 3 ways & measure \( \hat{S}_z \) Heisenberg picture
Quantum State Reconstruction

Basic Requirements:

- “Informationally Complete” Measurements

\[
\{M^{(i)}\} \{m^{(i)}_j\ j = 1, 2, \ldots, j^{(i)}_{\text{max}}\} \{p^{(i)}_j\} \Leftrightarrow \rho
\]

- Assign probabilities based on sampling ensemble

  - Standard approach: Strong measurement

    Each ensemble \[ p^{(i)}_j = \frac{n^{(i)}_j}{N} \]

    with possible accuracy of \( N \log_2 d \) bits
Challenges for Standard Reconstruction

Many measurements
- Each measurement on ensemble, \( d-1 \) independent results.
- Require at least \( d+1 \) measurements of different observables.

Strong backaction
- Projective measurements destroy state.
- Need to prepare identical ensembles for each \( M_i \)

Wasted quantum information
- Need only \( b \log_2 d \) bits. Wasteful when \( N \gg b \).
- Much more “quantum backaction” than necessary.
- Quantum State \textit{erased} after reconstruction.

Alternative -- Continuous measurement
Continuous Weak Measurement Approach

\[ \hat{\rho}_N = \hat{\rho}_0 \otimes N \]

- Ensemble identically coupled to probe (ancilla)

Measurement record

\[ M(t) = N\langle O \rangle_t + \sigma W \]

Signal

Noise

Probe noise: Gaussian variance

\[ \sigma^2 = \frac{1}{\kappa \Delta t} \]

Ultimate resolution -- Long averaging

\[ \delta M^2 = \frac{1}{\kappa T} \]

“Projection noise”

\[ N\Delta O^2 \ll \delta M^2 \]

No correlations
Advantages of Continuous Measurement

- Weak probes - NMR
  - Electron transport
  - Off resonance atom-laser interaction

- Real-time feedback control
The Basic Protocol

• Continuously measure observable $O$.

• Apply time dependent control to map new information onto $O$.

• Use *Bayesian* filter to update state-estimate given measurement record.

• Optimize information gain.
Mathematical Formulation

• Work in Heisenberg Picture (control independent of $\rho_0$)

$$\langle O \rangle_t = Tr(O(t) \rho_0) = \langle O(t) | \rho_0 \rangle$$

• Coarse-grain over detector averaging time.

$$O_i = \frac{1}{\Delta t} \int_{t_i}^{t_i + \Delta t} O(t') dt'$$

$$M_i = N \langle O_i | \rho_0 \rangle + \sigma W_i$$  

Measurement series

• Need to generate “complete set” of operators $O_i$ to find $\rho_0$

Time dependent Hamiltonian, $\{H_i\}$ generates $SU(d)$

Include decoherence (beyond usual Heisenberg picture)

• Stochastic linear estimation: Given $\{M_i\}$ find $\rho_0$
Bayesian Filter

Bayesian posterior probability

\[ P(\rho_0|\{M_i\}) = A P(\{M_i\}|\rho_0) P(\rho_0) \]

Conditional probability

Single measurement Gaussian

\[ P(M_i|\rho_0) = C_i \exp \left\{ - \frac{(M_i - N\langle O_i |\rho_0 \rangle)^2}{2\sigma^2} \right\} \]

Multidimensional Gaussian

\[ P(\{M_i\}|\rho_0) = \prod_i P(M_i|\rho_0) = C \exp \left\{ - \frac{1}{2} \langle \delta \rho | R | \delta \rho \rangle \right\} \]

\[ R = \frac{N^2}{\sigma^2} \sum_i |O_i \rangle \langle O_i | \quad \delta \rho = \rho_{\{M_i\}} - \rho_0 \quad |\rho_{\{M_i\}} \rangle = \frac{N}{\sigma^2} \sum_i M_i R^{-1} |O_i \rangle \]

Least square fit
Information Gain

Optimize Entropy in Gaussian Distribution:

\[ S = -\frac{1}{2} \log(\det R ) = -\sum \log \sqrt{\lambda_\alpha} \]

Eigenvalues of \( R \) : \( \sqrt{\lambda_\alpha} = \text{SNR for measurement of observable along principle axis } \alpha \).

\( R \) full rank \( (d^2 - 1) \) \( \rightarrow \) Informationally complete
Physical System

Ensemble of alkali atoms. Total spin $F, \dim = 2F+1$

**Measurement:** Couple to off-resonant laser
Polarization dependent index of refraction: Faraday Rotation

$e_x = \sigma_+ + \sigma_-$

$e_{\theta} = \sigma_+ + e^{i\theta}\sigma_-$

Magnetically polarized atomic cloud

Measures average spin projection: $O = F_z$
Basic Tool:
Off Resonance Atom-Laser Interaction

\[ |e\rangle \, nP_{3/2} \quad |g\rangle \, nS_{1/2} \]

\[ \Delta = \omega_L - \omega_{eg} \]

Monochromatic Laser
\[ \text{Re}(E e^{-i\omega_L t}) \]

Alkali Atom
Hyperfine structure
\[ F = J + I \]

Tensor Interaction
\[ \hat{V} = -\hat{\alpha}_{ij} E_i^* E_j \]

Atomic Polarizability
\[ \frac{\hat{t}}{\hat{\alpha}} = -\sum_{F_e} \frac{\hat{d}_{ge} \hat{d}_{eg}}{\hbar(\Delta_{eg} - i\Gamma/2)} \]

Irreducible decomposition
\[ \hat{V} = -\hat{\alpha}^{(0)}|E|^2 - \frac{\hbar}{\alpha^{(1)}} \cdot \left( E^* \times E \right) - \frac{\hbar}{\alpha^{(2)}} \left( \frac{E_i^* E_j + E_j^* E_i}{2} \right) \]
## Irreducible Tensor Decomposition

<table>
<thead>
<tr>
<th>Irreducible Component</th>
<th>Interaction</th>
<th>Effect on Atoms</th>
<th>Effect on Photons</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>scalar</strong></td>
<td>$\hat{V}^{(0)} = c_0</td>
<td>\mathbf{E}</td>
<td>^2 \hat{I}$</td>
</tr>
<tr>
<td>$\hat{\alpha}^{(0)} \sim \hat{I}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>vector</strong></td>
<td>$\hat{V}^{(1)} = c_1(\mathbf{E}^* \times \mathbf{E}) \cdot \hat{F}$</td>
<td><strong>Zeeman-like, shift linear in m-level.</strong></td>
<td><strong>Faraday rotation about atomic spin.</strong></td>
</tr>
<tr>
<td>$\hat{\alpha}^{(1)} \sim \hat{F}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>tensor</strong></td>
<td>$\hat{V}^{(2)} = c_2\left(3</td>
<td>\mathbf{E} \cdot \hat{F}</td>
<td>^2 - E^2 \hat{F}^2 \right)$</td>
</tr>
<tr>
<td>$\hat{\alpha}^{(2)} \sim \left(\hat{F}<em>\pm \right)^2, \hat{F}</em>\pm \hat{F}_z$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\left(3\hat{F}_z^2 - \hat{F}^2\right).$</td>
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</tbody>
</table>

**Note:** For detunings $\Delta \gg \Delta E_{HF}$: $c_0, c_1 \sim 1/\Delta$  
$c_2 \sim \Delta E_{HF}/\Delta^2 \propto \Gamma_{\text{scat}}$
Signal -- Continuous observation of Larmor Precession

**Experiment:**
P.S. Jessen, U of A
G. Smith et al.

Collapse and revival of Larmor precession

\[
\hat{H} = g_F \mu B \hat{F}_y + c (\varepsilon_L \cdot \hat{F})^2 - \frac{i}{2} \hbar \gamma_s \hat{1}
\]
Enhanced Non-Linearity on Cs D1 Transition

Non-linear (tensor) light shift

\[ V_{NL} = \beta \hbar \gamma_s |\hat{\epsilon}_{probe} \cdot \mathbf{F}|^2 \]

- maximize \( \beta \propto \Delta_{HF} \) by probing on D1 line -

Collapse & revival in Larmor precession (D1 probe)
Controlability

Our example system effective Hamiltonian:

\[ H = \mathbf{B}(t) \cdot \mathbf{F} + h\gamma_s \beta F_x^2 \]

For magnetic fields alone uncontrollable (algebra closes)

\[ F_x, F_y, F_z : \quad [F_i, F_j] = i\varepsilon_{ijk} F_k \]

The nonlinearity allows full controllability:

\[ [F_x^2, F_y^n] \rightarrow F_x F_z F_y^{n-1} \]

But... Light shift nonlinearity is tied to decoherence.
Measurement Strategy

- Time dependent $B(t)$ to cover operator space, $\text{su}(2F+1)$.

- In given trial, system decays due to decoherence.

Errors along primary axis given by the eigenvalues of $R$

Maximize information gain in time before decoherence, subject to “costs”
Quantum State Reconstruction

In the Lab:

- use NL light shift + time varying B-field to implement required evolution
- measure Faraday signal $\propto F_z(t)$

Fix magnitude and $B_z=0$

$\mathbf{B}(t) = |\mathbf{B}| (\cos \theta(t), \sin \theta(t), 0)$

Choose 50 points along trajectory to specify angles.

Interpolate up to full sample rate.
Some atomic physics

- Off-resonance: \( \Delta \gg \Gamma = 2\pi \, 5.2 \, \text{MHz} \)
- Scattering rate: \( \gamma_s = \frac{I}{I_{sat}} \frac{\Gamma^2}{4\Delta^2} \)
- Nonlinear light shift requires detuning not large compared to hyperfine splitting.

\[ \begin{align*}
6S_{1/2} & \quad \Delta E_{HF} = 9.2 \, \text{GHz} \\
6P_{1/2} & \quad \Delta E_{HF} = 1.2 \, \text{GHz} \\
6P_{3/2} & \quad \Delta E_{HF} = 0.25, \ 0.20, \ 0.15 \, \text{GHz}
\end{align*} \]

- Scattering time: 1 ms
- Detuning: D1 \( \Delta = 6 \, \text{GHz} \), D2 \( \Delta = 4 \, \text{GHz} \)
- Measurement duration: 4 ms
- Integration steps: 1000, \( \tau_D = 4 \, \mu\text{s} \)
Example: D1, $F=3$, Stretched state $|F = 3, m = 3\rangle$
Example: D2, F=4, Cat state, \( \frac{1}{\sqrt{2}} (|F = 4, m = 4\rangle + |F = 4, m = -4\rangle) \)
Quantum State Reconstruction (Cs, $F = 3$)

Preliminary Result
Single Measurement Record

*signal (polarization rotation)*

- **theory**
- **experiment**

**% error**

- 0
- 10
- 20
- 30

**time [ms]**

0 0.5 1 1.5 2 2.5 3 3.5 4

**fidelity** = 0.62
(random guess = 0.38)
Quantum State Reconstruction (Cs, $F = 3$)

Preliminary Result
128 Average

![Graph showing signal (polarization rotation) and % error over time with 'theory' and 'experiment' labels.]

% error

systematic errors

Initial State

Reconstructed Density Matrix

fidelity = 0.82
(random guess = 0.38)
What if control parameters are not known exactly?

**Background or unknown B-field**

Assume a distribution $P(B_i(t))$ then

$$P(M(t)|\rho_0) \propto \sum_i P(B_i) e^{(\rho - \hat{\rho}_{0i} | R_i | \rho - \hat{\rho}_{0i})}$$

**Inhomogeneities in field intensity** - include in model.

Optimization naturally seeks “spin-echo” solution.
Robustness to Field Fluctuations

Use known state to estimate field and adjust simulation.
• **Improved optimization** (convex sets).

• **Generalized measurements**: Ellipticity spectroscopy.

• **Limited reconstruction**: e.g. second moments.
  - Observation of entangling dynamics.

• **Toward feedback control of quantum features.**
Conclusions

• Continuous weak measurement allows quantum state reconstruction using a single ensemble.

• Maximizes information gain/disturbance tradeoff.

• Can be useful when probes are too noisy for single quantum system strong-measurement but sufficient signal-to-noise can be seen in ensemble.

• State-estimation beyond the “Kalman filter”.

• New possibilities of quantum feedback control.

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