Negativity of the Wigner function as an indicator of non-classicality

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Abstract
A measure of non-classicality of quantum states based on the volume of the negative part of the Wigner function is proposed. We analyse this quantity for Fock states, squeezed displaced Fock states and cat-like states defined as coherent superposition of two Gaussian wavepackets.

Keywords: quasiprobability distribution functions, non-classicality, negativity of Wigner functions, Fock states, squeezed states, generalized Fock states, Schrödinger cat state

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Analysing pure quantum states in an infinite dimensional Hilbert space it is useful to distinguish a family of coherent states, localized in the classical phase space and minimizing the uncertainty principle. These quantum analogues of points in the classical phase space are often considered as ‘classical’ states. For an arbitrary quantum state one may pose a natural question, to what extent is it ‘non-classical’ in a sense that its properties differ from that of coherent states? In other words, is there any parameter that may legitimately reflect the degree of non-classicality of a given quantum state? This question was motivated by the first observation of non-classical features of electromagnetic fields such as sub-Poissonian statistics, antibunching and squeezing. Additionally, it is well known that the interaction of (non)linear devices with quantum states may cause them to flip from one state to another; for instance, nonlinear devices may produce non-classical states from their interaction with the vacuum or a classical field. A systematic survey of non-classical properties of quantum states would be worthwhile because of the current ever increasing number of experiments in nonlinear optics. An earlier attempt to shed some light on the non-classicality of a quantum state was pioneered by Mandel [1], who investigated radiation fields and introduced a parameter \(q\) to measure the deviation of the photon number statistics from the Poissonian distribution, characteristic of coherent states.

In general, to define a measure of non-classicality of quantum states one can follow several different approaches [2]. Distinguishing a certain set \(C\) of states (e.g. the set of coherent states \(|\alpha\rangle\)), one looks for the distance of an analysed pure state \(|\psi\rangle\) to this set, by minimizing a distance \(d(|\psi\rangle, |\alpha\rangle)\) over the entire set \(C\). Such a scheme based on the trace distance was first used by Hillery [3, 4], while other distances (the Hilbert–Schmidt distance [5, 6] or the Bures distance [7, 8]) were later used for this purpose. The same approach is also applicable to characterize mixed quantum states: minimizing the distance of the density \(\rho\) to the set of coherent states is related [6, 9] to the search for the maximal fidelity (the Hilbert–Schrödinger fidelity \(\text{Tr}(\rho\sigma)\)) or the Bures–Uhlmann fidelity (\(\text{Tr} \sqrt{\rho^{1/2}\sigma\rho^{1/2}}\)) with respect to any coherent state, \(\sigma = |\alpha\rangle\langle\alpha|\). On the same footing, the Monge distance introduced in [10, 11] may be applied to describe to what extent a given mixed state is close to the manifold of coherent states.

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Yet another way of proceeding is based on the generalized (Cahill) phase space representation $R_{\tau}$ of a pure state, which interpolates between the Husimi ($Q$), the Wigner ($W$) and the Glauber–Sudarshan ($P$) representations. The Cahill parameter $\tau$ is proportional to the variance of a Gaussian function one needs to convolute with $P$-representation to obtain $R_{\tau}$ [12]. In particular for $\tau = 1, 1/2, 0$ one obtains the $Q$, $W$- and $P$-representations, respectively. By construction the $Q$ representation is non-negative for all states, while the Wigner function may also admit negative values, and the $P$ representation may be singular or may not exist.

The smoothing effect of $R_{\tau}$ is enhanced as $\tau$ increases. If $\tau$ is large enough so that $R_{\tau}$ becomes a positive definite regular function, thus acceptable as a classical distribution function, then the smoothing is said to be complete. The greatest lower bound $\tau_{0}$ for the critical value was adopted by Lee [13, 14], as the non-classical depth of a quantum state, and this approach was further developed in [15–17]. The limiting value, $\tau_{m} = 1$, corresponds to the $Q$ function which is always acceptable as a classical distribution function. The lowest value, $\tau_{m} = 0$, is ascribed to an arbitrary coherent state because its $P$ function is a Dirac delta function, so its $\epsilon$-smoothing becomes regular. The range of $\tau_{m}$ is thus $\tau_{m} \in [0, 1]$.

If the Husimi function of a pure state admits at least one zero $Q(0) = 0$, then a Cahill $R_{\tau}$ distribution with a narrower smearing, $\tau < 1$, becomes negative in the vicinity of 0. Therefore the classical depth for such quantum states is maximal, $\tau_{m} = 1$ [15]. The only class of states for which $Q$ representation has no zeros are the squeezed coherent states for which $\tau_{m}$ is a function of the squeezing parameter $s$. In the limiting case $s = 0$ one obtains the standard coherent state for which the $R_{0} = P$ distribution is a Dirac delta function, that is $\tau_{m} = 0$.

A possible way to distinguish a classical state is to require that its $P$-representation exists and is everywhere non-negative. Such an approach was advocated in [18] and further explored in [19], while a recent work [20] establishes a link between the task of classifying all states with positive $P$-representation and the 17th Hilbert problem concerning positive polynomials.

A closely related approach to characterizing quantum states is based on properties of their Wigner functions in phase space $(p,q)$. One can prove that the Wigner function is bounded from below and from above [12]. In the normalization $\int \int W(q,p) \, dq \, dp = 1$ used later in this work, such a bound reads $|W(q,p)| \leq 1/\pi \hbar$. Further bounds on integrals of the Wigner function were derived in [21], while an entropy approach to the Wigner function was developed in [22, 23].

In order to interpret the Wigner function as a classical probability distribution one needs to require that $W$ is non-negative. As found by Hudson in 1974 [24], this is the case for coherent or squeezed vacuum states only. A possible measure of non-classicality may thus be based on the negativity of the Wigner function which may be interpreted as a signature of quantumness.

The negativity of the Wigner function has been linked to non-locality, according to the Bell inequality [25], while investigating the original Einstein–Podolsky–Rosen (EPR) state [26]. In fact Bell argued that the EPR state will not exhibit non-local effects because its Wigner function is everywhere positive, and as such will allow for a hidden variable description of correlations. However, it is now demonstrated [27, 28] that the Wigner function of the EPR state, though positive definite, provides direct evidence of non-locality. This violation of Bell’s inequality holds true for the regularized EPR state [29] and also for a correlated two-mode quantum state of light [30].

It is also worth recalling that the Wigner function can be measured experimentally [31], including the measurements of its negative values [32]. The interest put on such experiments has triggered a search for operational definitions of the Wigner functions, based on experimental setup [33, 34].

The aim of this paper is to study a simple indicator of the non-classicality, which depends on the volume of the negative part of the Wigner function. To demonstrate a potential use of such an approach we investigate certain families of quantum states. The non-classicality indicator is defined in section 2. The Schrödinger cat state, being constructed as coherent superposition of two Gaussian wavepackets, is analysed in section 3 while section 4 is devoted to Fock states and to the squeezed displaced Fock states. Finally in section 5, a brief discussion of results and perspectives is given.

2. The non-classicality indicator

The Wigner function of a state $|\psi\rangle$ defined by [35, 36]

$$W_{\psi}(q,p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \left( q - x \right) |\psi(q + x/2)| \exp(ipx)$$

(1)

satisfies the normalization condition $\int \int W_{\psi}(q,p) \, dq \, dp = 1$. Hence the doubled volume of the integrated negative part of the Wigner function may be written as

$$\delta(\psi) = \int \int [W_{\psi}(q,p) - W_{\psi}(q,p)] \, dq \, dp$$

(2)

By definition, the quantity $\delta$ is equal to zero for coherent and squeezed vacuum states, for which $W$ is non-negative. Hence in this work we shall treat $\delta$ as a parameter characterizing the properties of the state under consideration. Similar quantities related to the volume of the negative part of the Wigner function were used in [37–39] to describe the interference effects which determine the departure from classical behaviour.

Furthermore, a closely related approach was recently advocated by Benedict and collaborators [40, 41]. Their measure of the non-classicality of a state $|\psi\rangle$ reads

$$\nu(\psi) = 1 - \frac{L_{+}(\psi) - L_{-}(\psi)}{L_{+}(\psi + i\epsilon) + L_{-}(\psi)}$$

(3)

where $L_{+}(\psi)$ and $L_{-}(\psi)$ are the moduli of the integrals over those domains of the phase space where the Wigner function is positive and negative, respectively. The normalization condition implies $L_{+} - L_{-} = 1$, so that $\nu = 2L_{-}/(2L_{+} + 1)$ leads to $0 \leq \nu < 1$. Using this notation we may rewrite (2) as

$$\delta = L_{+} + L_{-} - 1 = 2L_{-}$$

and obtain a simple relation between both quantities

$$\nu = \frac{2L_{-}}{1 + 2L_{-}} = \frac{\delta}{1 + \delta}$$

(4)
The Wigner function, in particular, if we demonstrate that the parameter \( \delta \) does not spatially separate.

Words we measure the size of the product wavefunction of such a state reads in the position representation

\[
\Psi(q) = \frac{N}{\sqrt{2}} [\phi_+(q) + \phi_-(q)]
\]

where

\[
\phi_{\pm}(q) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left( \frac{m\omega}{2\hbar} q^2 \pm q_0^2 \right) \left( \frac{\hbar}{2} q \pm q_0 \right).
\]

From now on atomic units are used \((m = \hbar = \omega = 1)\). In other words we measure the size of the product \(pq\) in units of \(\hbar\). The classical limit \(\hbar \to 0\) means the action \(pq\) characteristic of the system is many orders of magnitude larger than \(\hbar\). A glance at equation (6) reveals that the phase, governed by \(p_0\), is of great importance in that it induces oscillations on the wavefunction, as can be seen in figure 1. Note that the normalization constant \(N\) depends on the location of the centres \((q_0, p_0)\) of both coherent states that make up the cat state. Therefore one sees that the Wigner function may depend not only on the distance \(2q_0\) between both states, but also on their momentum, \(p_0\). So far, the studies on the cat states [34] have usually been restricted to the case of standing cats, \(p_0 = 0\). In this paper we demonstrate that the parameter \(p_0\) influences the shape of the Wigner function, in particular, if \(q_0 \sim 1\) and both packets are not spatially separated.

Inserting (5) into the Wigner function (1) one obtains

\[
W_{\Psi}(q, p) = W_+(q, p) + W_-(q, p) + W_{\text{int}}(q, p).
\]

Here

\[
W_{\pm}(q, p) = \frac{N^2}{2\pi} \exp(-q^2 \pm q_0^2) \left( \frac{\hbar}{2} q \pm q_0 \right).
\]

Figure 1. Schrödinger cat states wavefunctions plotted with \(p_0 = 0\) (left) and with \(p_0 = 4\) (right). Dashed and solid curves represent the imaginary and the real parts of the wavefunction, respectively. Notice that the envelopes of both wavefunctions do coincide.

The Schrödinger cat state is then defined as a superposition of two such states [37]. In our work we construct similar ‘cat states’ by choosing two coherent states \(\phi_{\pm}\) localized in two distant points of the configuration space, \(\pm q_0\). The wavefunction of such a state reads in the position representation

\[
\Psi(q) = \frac{N}{\sqrt{2}} [\phi_+(q) + \phi_-(q)]
\]

represents two peaks of the distribution centred at the classical phase space points \((\pm q_0, p_0)\), while

\[
W_{\text{int}}(q, p) = \frac{N^2}{\pi} \cos(2pq_0) \exp(-q^2 - (p - p_0)^2)
\]

stands for the interference structure which appears between both peaks. Normalizing (5) yields

\[
N = (1 + \cos(2pq_0) \exp(-q_0^2))^{1/2}.
\]

Making use of the formula (7) for the Wigner function of the cat state \(|\Psi\rangle\), its non-classicality parameter

\[
\delta(\Psi) = \int \int |W_+(q, p) + W_{\text{int}}(q, p) + W_-(q, p)| dq dp - 1
\]

may be approximated by

\[
\delta(\Psi) \approx N^2 \left[ 1 + \int \frac{dp}{\sqrt{\pi}} \cos(2pq_0) \exp(-(p - p_0)^2) \right] - 1.
\]

Strictly speaking the right-hand side of equation (12) forms an upper bound for \(\delta(\Psi)\), which may be practically used as its fair approximation. Because of the oscillations of the absolute value of cosine, it is difficult to perform the integration analytically. In the special case \(q_0 = 0\), the supersposition of coherent states (5) reduces to a single coherent state and correspondingly (12) leads to \(\delta(\Psi) = 0\).

Figure 2 shows plots of the Wigner function of the cat states for several values of the separation \(q_0\) and the momentum \(p_0\). One clearly sees the formation of the quantum interference structure halfway between the two humps as the separation distance \(q_0\) increases. The frequency of the interference structure increases with the separation [34].

For intermediate separations \((0 < q_0 \leq 4)\), the Wigner function changes its structure with \(p_0\), see figures 2(b) and (d). However, for a larger separation distance, \(q_0 > 4\), the Wigner function for \(p_0 = p_1 \neq 0\) may be approximated by the Wigner function for the state with \(p_0 = 0\) translated by a constant vector \(\Delta p = p_1\).
Figure 2. Plots of the Wigner functions of the Schrödinger cat states (7). Each panel is labelled by the separation distance $q_0$, the momentum $p_0$ and the resulting indicator $\delta$. Observe that for intermediate separations, $q_0 \sim 1$, the indicator $\delta$ changes with $p_0$. The left column shows the ‘standing cats’ ($p_0 = 0$) while the cats in motion ($p_0 = 4$) are represented in the right column.

In the case of ‘standing cats’, ($p_0 = 0$), the indicator $\delta$ increases monotonically with the separation $q_0$, and reflects the presence of the interference patterns at $q = 0$—see figure 3(k). The growth of the non-classicality saturates at $q_0 \approx 4$, as the interference patterns become practically separated from both peaks, and the parameter $\delta$ tends to the limiting value, $\delta_{\text{max}} \approx 0.636$. In the limit $q_0 \rightarrow \infty$ the oscillations of the cosine term in equation (12) become rapid and a crude approximation $|\cos(q_0 p_0)| \approx 1$ gives an explicit upper bound $\delta \leq 2N^2 - 1 \approx 1$.

This picture gets more complicated for the states with $p_0 \neq 0$, in particular for a small separation distance, ($0 < q_0 \leq 4$). In this case, $\delta$ exhibits oscillations as shown in figure 3. Notice that $\delta$ does not become zero; this is in contrast to what the eye test tends to show in figure 3. To shed some light on this behaviour we have chosen to plot in figure 4 the Wigner function for which $\delta(q_0)$ achieves extremal values. For instance, $\delta$ at $q_0 = 0.725$ (figure 4(h)) is smaller than at $q_0 = 0.4$ (figure 4(g)) or $1.175$ (figure 4(i)). This is due to the interference structure, which is not symmetric with respect to the reflection $p \rightarrow -p$, in contrast to the case of cats with $p_0 = 0$.

As shown in figure 3, the frequency of oscillations increases with $p_0$, but the limiting value $\delta(q_0 \rightarrow \infty)$ does
Figure 3. Indicator δ of the Schrödinger cat state |ψ⟩ as a function of the separation distance q₀ and several values of p₀ as labelled on each panel. Grey dots ((a)–(f)) refer to labels of individual panels of figure 2 while grey dots ((g)–(i)) refer to that of figure 4.

not depend on the initial momentum p₀. This can also be demonstrated investigating the dependence of the quantity δ as a function of p₀. As follows from equation (12), the indicator δ displays regular oscillations with the period posc = π/q₀—see figure 5. In other words a non-zero separation parameter q₀ breaks the translational invariance in momentum and introduces a characteristic momentum scale posc ~ 1/q₀. Note that the amplitudes of the oscillations decrease fast with q₀, so that for well separated cats with q₀ > 4 the quantity δ is practically independent of p₀.

4. Generalized Fock states

Let us consider the squeezed displaced Fock state defined by

\[ |β, η, n⟩ = D(β)S(η)|n⟩, \]  

where \(|n⟩\) is the original Fock state and \(n = 0, 1, 2, \ldots\). The displacement \(D(β)\) and the squeezed \(S(η)\) operators are defined by [34, 43]

\[ D(β) := \exp(βa^† − β^∗a) \]  
\[ S(η) := \exp(\frac{1}{2}(η^∗a^2 − ηa^†2)), \]

where \(a\) and \(a^†\) are usual photon annihilation and creation operators, respectively. The complex variable \(β\) represents the magnitude and angle of the displacement. Similarly, writing the complex number in its polar form, \(η = s \exp(iφ)\), it is easy to see that the radius \(s\) plays the role of the squeezing strength while the angle \(φ\) indicates the direction of squeezing. It was shown in [12] that the displacement operators \(D(β)\) form a
function by performing the inverse Fourier transform as can be expressed in the form of the operators function for the expansion of the density operator in terms so that the Wigner function may be interpreted as a weight distribution function can now be written as follows

Making use of the parity operator \((-1)^{q_1} = \exp(i\pi q_1 a)\), one finally shows that

\[
W(\alpha) = 2 \sum_{n=0}^{\infty} (-1)^n (\alpha D^{-1}(\alpha) \rho D(\alpha)) |n\rangle \langle n| ,
\]

since \(\exp(i\pi q_1 a) |n\rangle = \exp(i\pi n) |n\rangle\), with \(n\) being the photon number.

In the case of the squeezed displaced Fock states, \(\rho = |\beta, \eta, n\rangle \langle \beta, \eta, n|\), explicit calculations of matrices elements \([44]\) generated from (20) provides us with the following expression of the Wigner function

\[
W_n(\alpha) = \frac{2}{\pi} (-1)^n \exp(-2|b|^2) L_n(4|b|^2) ,
\]

with \(b = \cosh(s)(\alpha^* - \beta^*) + \exp(-i\phi) \sinh(s)(\alpha - \beta)\). Here \(L_n\) denotes the Laguerre polynomial of the \(n\)th order.

The Wigner function (21) allows us to compute the non-classicality parameter \(\delta(\beta, \eta, n)\) for a given displaced squeezed Fock state \(|\beta, \eta, n\rangle\). In what follows certain special cases will be investigated such as squeezed displaced vacuum states, pure Fock states and squeezed displaced Fock states. It will be therefore convenient to represent the complex variable \(\alpha\) by the position and momentum coordinates, \(\alpha = \frac{1}{\sqrt{2}}(q + ip)\), and treat likewise the displacement operator, \(\beta = \frac{1}{\sqrt{2}}(q_0 + ip_0)\).

Substituting \(\beta = \eta = 0\) in equation (21) yields the Wigner function for the Fock state \(|n\rangle\),

\[
W_n(q, p) = (-1)^n \exp[-(q^2 + p^2)] L_n(2(q^2 + p^2)) .
\]

This allows us to evaluate analytically the indicator \(\delta(n)\), for \(n = 1, 2, 3, 4\)

\[
\delta(0) = 0 \quad \text{(vacuum)}
\]

\[
\delta(1) = \frac{4}{e^{1/2}} - 2 \approx 0.4261226
\]

\[
\delta(2) = 4((2 + \sqrt{2})e^{-1/16} + (-2 + \sqrt{2})e^{-1/16}) \approx 0.72899
\]

\[
\delta(3) \approx 0.97667
\]

\[
\delta(4) \approx 1.19138.
\]

since the zeros of the Laguerre polynomials are available up to the fourth order. For larger \(n\) we computed the quantity \(\delta(n)\) numerically and plotted these in figure 6. The indicator \(\delta\) grows monotonically with \(n\) as the number of zeros of the Laguerre polynomial \(L_n\) increases with \(n\). For \(n \in [1, 250]\) this dependence may be approximated by \(\delta \approx \sqrt{n}\). Hence, the larger the quantum number \(n\), the less the Wigner function \(W_n\) can be interpreted as a classical distribution function.

Setting \(n = 0\) in (21) one obtains a squeezed coherent state or squeezed vacuum state. Choosing the squeezing angle \(\phi = 0\), one sees that the Wigner function is a Gaussian centred at the displacement vector \((q_0, p_0)\) with the shape determined by the squeezing parameter \(\beta\),

\[
W_0(q, p) = \frac{1}{\pi} \exp\left(-\frac{e^{2\|q - q_0\|^2 - 1}{e^{\|p - p_0\|^2}}\right) .
\]

In such a case the Wigner function remains everywhere non-negative for any choice of the squeezing and displacement
parameters [24], so that the non-classicality indicator vanishes, \( \delta(|\beta, s, 0\rangle) = 0 \). Note that the displacement of any state in phase space does not change the shape of the Wigner function, so the quantity \( \delta \) is independent of the displacement operator \( D(\beta) \).

Furthermore, the squeezing operator \( S(\eta) \) influences the shape of the Wigner function, but does not lead to a change in the volume of its negative part. Therefore, the parameter \( \delta \) does not also depend on the squeezing. As an illustration we have chosen the squeezed (\( |\alpha| = s, \phi = \pi/6 \)) displaced \( (\beta = 0) \) third photon \( (n = 3) \) state, \( |0, s \exp(i\pi/6), 3\rangle \). The contour plots of the Wigner function of such a state are shown in figure 7 for some values of the squeezing parameter \( s \). The indicator \( \delta \) is equal to 0.9762, irrespective to the squeezing strength. If squeezing is strong enough, the ring-like Wigner function collapses to a quasi one dimensional object with a cigar form.

The squeezed vacuum is often described as a non-classical state [34]. Since the quantity \( \delta \) does not depend on squeezing, it should not be interpreted as the only parameter which characterizes the non-classicality. To describe the non-classical features of the squeezed states one may use, for instance, the non-classical depth [13, 15, 17].

5. Concluding remarks

In this work we have proposed a simple indicator of non-classicality which measures the volume of the negative part of the Wigner function. Although the proposed coefficient \( \delta \) is a function of the related quantity \( \nu \), recently introduced by Benedict, Czirják et al [40, 41], it is much easier to compute numerically.

The quantity (2) was used to analyse exemplary quantum states, including the Schrödinger cat states. The non-classicality \( \delta \) increases with the separation between the classical points defining the cat state. This growth saturates, if the separation distance is so large that the quantum interference

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**Figure 6.** The non-classicality indicator \( \delta(n) \) of the Fock states versus the quantum number \( n \leq 250 \) (solid curve). The dashed curve represents \( \frac{1}{\sqrt{n}} \) plotted for comparison.

**Figure 7.** Contour plots of the Wigner functions of the squeezed Fock states \( |0, s \exp(i\pi/6), 3\rangle \) labelled by the squeezing strengths \( s \). Irrespective of \( s \) the indicator \( \delta \approx 0.9762 \).
patterns are well isolated from both main peaks of the distributions. Moreover, for a non-zero momentum \( p_0 \neq 0 \), the quantity \( \delta \) undergoes oscillations until the separation distance becomes so large that both packets are separated from each other, the quantity \( \delta \) decreases to a constant value, \( \delta_{\text{max}} \approx 0.636 \).

In the case of Fock states \( |n \rangle \), the quantity \( \delta \) equals zero for the coherent vacuum state \( |0 \rangle \) and grows monotonically with the quantum number \( n \). If a quantum state is displaced by the Glauber operator \( D(\beta) \), the shape of the Wigner function and the non-classicality parameter do not change. Although the squeezing operator \( S(\eta) \) changes the shape of the Wigner function, our results obtained for the squeezed Fock states show that the non-classicality \( \delta \) does not depend on squeezing. Since the non-classical depth of a state is a function of the squeezing strength, it is clear that there is no direct relation between \( \tau_m \) and \( \delta \), so both quantities characterizing quantum states may be regarded as complementary.

The results presented in this work were obtained for pure states of infinite dimensional Hilbert space with use of the standard harmonic oscillator coherent states. It is worth emphasizing that our approach is also suited to analysing mixed quantum states. Furthermore, one may study the similar problem for quantum states of a finite dimensional Hilbert space, which was originally tackled in [40]. In such a case one defines the Husimi function with the help of the SU(2) spin coherent states, while the Wigner functions may be obtained by expanding the density matrix in the complete basis of the rotation operators [45–47]. The Wigner function for finite dimensional systems may also be defined in alternative ways—see [48–54] and references therein. Studying the volume of the negative part of the Wigner function, defined according to [30], one may get interesting information concerning the non-classical properties of the state analysed. For instance some recent attempts [55, 39, 56] try to link the negativity of the Wigner function with the entanglement of analysed quantum states defined on a composed Hilbert space, or with the violation of the Bell inequalities.

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