

# On the Structure of the Observable Algebra for QED on the Lattice

J. KIJOWSKI<sup>1</sup>, G. RUDOLPH<sup>2</sup> and C. ŚLIWA<sup>1</sup>

<sup>1</sup>*Center for Theoretical Physics, Polish Academy of Sciences, al. Lotników 32/46, 02-668 Warsaw, Poland*

<sup>2</sup>*Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany. e-mail: rudolph@rz.uni-leipzig.de*

(Received: 23 April 1997; revised version: 3 September 1997)

**Abstract.** We prove that the matter field subalgebra of the observable algebra for QED on a finite lattice is isomorphic to the enveloping algebra of the Lie algebra  $\mathfrak{sl}(2N, \mathbb{C})$ , factorized by a certain ideal. Using this result, we give a new proof of the decomposition of the physical Hilbert space into charge superselection sectors.

**Mathematics Subject Classifications (1991).** 81Q05, 81T13.

**Key words:** gauge invariants, observable algebra, charge superselection sectors.

## 1. Introduction

In [1] and [2], we have shown that the functional integrals of QED and QCD can be reformulated in terms of local gauge invariant quantities. The constructions presented in these papers were on the level of ‘classical’ Grassmann algebra-valued quantities. In [3] a similar construction of gauge-invariants on the level of field operators was used to formulate QED on a finite lattice in the Hamiltonian approach. The main result of this Letter is a complete description of the observable algebra and an explicit construction of the physical Hilbert space as a direct sum of charge superselection sectors. A similar programme for the continuum theory was formulated a long time ago. Since the problems arising for the continuum case are extremely difficult, only partial results have been obtained until now (see [4]–[10]). We shall comment on some relations to our work in Section 5.

The goal of this Letter is to study the structure of the subalgebra of observables being gauge-invariant bilinear combinations of fermionic fields. It turns out that this subalgebra carries the whole information about the fermionic sector (the bosonic sector consists of electric and magnetic field operators, carrying the well-known structure of a finitely generated Heisenberg algebra).

The bilinear fermionic invariants are

$$\hat{W}_\gamma^{ba} = \hat{\psi}^{b*}(y) \exp\left(-ig \int_\gamma \hat{A}\right) \hat{\psi}^a(x),$$

where  $\gamma$  is an oriented path starting at  $x$  and ending at  $y$ . Dividing the bi-spinor field  $\hat{\psi}^a$  into the positron degrees of freedom  $\hat{\phi}^K$  and the electron degrees of freedom  $\hat{\phi}_L^*$ ,  $K, L = 1, 2$ , we consider, in particular, the ‘pair annihilation operators’:

$$\hat{\mathcal{M}}_{\gamma;L}^K = \hat{\phi}_L(y) \exp\left(-ig \int_{\gamma} \hat{A}\right) \hat{\phi}^K(x), \quad (1)$$

(they annihilate a positron with helicity  $K$  at  $x$  and an electron with helicity  $L$  at  $y$ ). They are sufficient (see [3]) to reconstruct all operators  $\hat{W}$  defined above.

It follows from (1) that if two paths  $\gamma$  and  $\beta$  have the same endpoints  $x$  and  $y$ , then  $\hat{\mathcal{M}}_{\gamma;L}^K = \exp(-ig\hat{B}_{\gamma\beta^{-1}})\hat{\mathcal{M}}_{\beta;L}^K$ , where  $B_{\gamma\beta^{-1}}$  denotes the magnetic flux through a surface spanned by the closed contour  $\gamma\beta^{-1}$ . This means that the whole observable algebra may be reconstructed if we know (a) the electromagnetic field operators and (b) the operators  $\hat{\mathcal{M}}$  for a single path  $\gamma$  assigned to each pair of endpoints  $x$  and  $y$ . We call such an assignment ‘the choice of a *tree*’.

In the lattice version, we have a finite number of sites. Together with two possible helicities, they may be organized into a finite number of positron (respectively electron) degrees of freedom  $i = (x, K)$ ,  $j = (y, L)$ ,  $i, j = 1 \dots N$ . In this notation, the invariants (1) are written as  $m_{ij} \equiv \hat{\mathcal{M}}_{\gamma;L}^K$ , where  $\gamma$  is the path assigned to the pair  $(x, y)$ . Here we assume that the assignment  $\gamma(x, y)$  is defined by a set of *on-tree links*, i.e.  $\gamma(x, y)$  is the unique *on-tree path* from  $x$  to  $y$ .

The operators  $m_{ij}$  satisfy a number of axioms (see [3]) leading to a nontrivial structure of the observable algebra. In particular, they are nilpotent and satisfy some trilinear commutation relations, which we analyze in Section 2. We denote the associative  $*$ -algebra generated by these invariants by  $\mathcal{O}_m$  (Definition 1 in Section 3). In this Letter, we prove that it is isomorphic to the enveloping algebra of  $\mathfrak{sl}(2N, \mathbb{C})$ , divided by a certain ideal  $I$ . This gives a deeper insight into the structure of  $\mathcal{O}_m$ . In particular, using this result, we are able to find all its irreducible representations. This way we obtain a much simpler proof (comparing with [3]) of the decomposition of the physical Hilbert space into charge superselection sectors.

## 2. Lie Algebra Structure in $\mathcal{O}_m$

Let  $\mathfrak{g}$  denote the Lie algebra  $\mathfrak{sl}(2N, \mathbb{C})$  of traceless complex matrices of dimension  $2N$ . It is convenient to consider matrix indices running through the set  $\mathcal{I} := \{1, 2, \dots, N, \bar{1}, \bar{2}, \dots, \bar{N}\}$ . We denote  $E_{\alpha\beta} := (\delta_{\alpha\xi}\delta_{\beta\eta})_{\xi\eta} \in M_{2N}(\mathbb{C})$  (the matrix with entries equal to one at  $(\alpha, \beta)$  and zero elsewhere). We fix an order of  $\mathcal{I}$ , and define  $\alpha + 1$  as the successor of  $\alpha \in \mathcal{I}$ .

**THEOREM 1.** *If elements  $m_{ij}, m_{ij}^*$  ( $i, j = 1, 2, \dots, N$ ) of some Lie algebra satisfy relations*

$$[m_{ij}, m_{kl}] = 0, \quad (2)$$

$$[m_{ij}^*, m_{kl}^*] = 0, \quad (3)$$

$$[m_{ij}, [m_{kl}^*, m_{rs}]] = \delta_{ki}\delta_{ls}m_{rj} + \delta_{kr}\delta_{lj}m_{is}, \quad (4)$$

$$[m_{ij}^*, [m_{kl}, m_{rs}^*]] = \delta_{ki}\delta_{ls}m_{rj}^* + \delta_{kr}\delta_{lj}m_{is}^*, \quad (5)$$

then the Lie subalgebra generated by  $m_{ij}, m_{ij}^*$  is either isomorphic to  $\mathfrak{sl}(2N, \mathbb{C})$  or zero (all  $m_{ij}, m_{ij}^*$  are equal to zero).

There exists a Lie algebra with nonzero elements  $m_{ij}, m_{ij}^*$ , satisfying the above relations.

*Proof.* Let  $\mathfrak{h}$  be the Lie subalgebra generated by  $m_{ij}, m_{ij}^*$  and  $d_{ijkl} := [m_{ij}, m_{kl}^*]$ . Obviously, the linear subspace spanned by elements  $m_{ij}, m_{ij}^*, d_{ijkl} \in \mathfrak{h}$  is closed under the Lie bracket.

The only nonzero elements among  $d_{ijkl}$  are  $d_{ijil}$  and  $d_{ijkj}$ . Indeed, if  $i \neq k$  and simultaneously  $j \neq l$ , then it follows from (4) and (5) that  $[d_{ijkl}, m_{rs}] = [d_{ijkl}, m_{rs}^*] = 0$ . Hence, using the Jacobi identity, we have also  $[d_{ijkl}, d_{klkl}] = 0$ . On the other hand, again using the Jacobi identity, we get

$$0 = [d_{ijkl}, d_{klkl}] = [[m_{ij}, m_{kl}^*], d_{klkl}] = [m_{ij}, 2m_{kl}^*] = 2d_{ijkl}.$$

The number of independent  $d_{ijil}, j \neq l$ , is limited by equalities  $d_{ijil} = d_{rjrl}$  (it is sufficient to consider the case  $i \neq r$ ): We have

$$[d_{ijil}, d_{rlrl}] = [[m_{ij}, m_{il}^*], d_{rlrl}] = [m_{ij}, m_{il}^*] = d_{ijil}.$$

On the other hand,

$$[d_{ijil}, d_{rlrl}] = [d_{ijil}, [m_{rl}, m_{rl}^*]] = [m_{rj}, m_{rl}^*] = d_{rjrl}.$$

Analogously, one shows that  $d_{ijkj} = d_{isks}, i \neq k$ .

It remains to consider elements  $h_{ij} := d_{ijij} = [m_{ij}, m_{ij}^*]$ . They are not independent. Indeed, for  $i \neq r$ , we have

$$[d_{rjrs}, d_{isij}] = [d_{rjrs}, [m_{is}, m_{ij}^*]] = [m_{ij}, m_{ij}^*] - [m_{is}, m_{is}^*],$$

and, on the other hand,

$$[d_{rjrs}, d_{isij}] = [[m_{rj}, m_{rs}^*], d_{isij}] = [m_{rj}, m_{rj}^*] - [m_{rs}, m_{rs}^*].$$

Hence

$$h_{ij} + h_{rs} - h_{is} - h_{rj} = 0, \quad (6)$$

which also holds trivially for  $i = r$ .

Let  $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$  be the unique linear mapping such that

$$\phi(E_{i\bar{j}}) = m_{ij},$$

$$\phi(E_{\bar{j}i}) = m_{ij}^*,$$

$$\phi(E_{ik}) = d_{ijkj}, \quad i \neq k,$$

$$\phi(E_{\bar{j}l}) = -d_{ilij}, \quad j \neq l,$$

$$\phi(E_{ii} - E_{\bar{j}\bar{j}}) = h_{ij},$$

(using (6) one verifies the existence of such a mapping). We prove that  $\phi$  is a Lie algebra homomorphism:

$$\phi([x, y]) = [\phi(x), \phi(y)]. \quad (7)$$

Since  $E_{i\bar{j}}, E_{\bar{j}i}$ , and their Lie brackets, span  $\mathfrak{g}$  (in the same way as  $m_{ij}, m_{ij}^*$ , and their Lie brackets, span  $\mathfrak{h}$ ), the proof reduces to the case  $x \in \{E_{i\bar{j}}, E_{\bar{j}i}\}$ . Indeed, assuming that (7) holds for this case, we obtain it also for the more general case  $x = [x_1, x_2]$ :

$$\begin{aligned} \phi([x_1, x_2], y) &= -\phi([x_2, [x_1, y]]) + \phi([x_1, [x_2, y]]) \\ &= -[\phi(x_2), \phi([x_1, y])] + [\phi(x_1), \phi([x_2, y])] \\ &= -[\phi(x_2), [\phi(x_1), \phi(y)]] + [\phi(x_1), [\phi(x_2), \phi(y)]] \\ &= [[\phi(x_1), \phi(x_2)], \phi(y)] = [\phi([x_1, x_2]), \phi(y)]. \end{aligned}$$

For the case  $x, y \in \{E_{i\bar{j}}, E_{\bar{j}i}\}$ , (7) is implied by the definition of  $\phi$ , because

$$[E_{i\bar{j}}, E_{l\bar{k}}] = \delta_{jl}E_{ik} - \delta_{ik}E_{l\bar{j}}, \quad (8)$$

$$[E_{i\bar{j}}, E_{k\bar{l}}] = 0, \quad (9)$$

$$[E_{\bar{j}i}, E_{l\bar{k}}] = 0. \quad (10)$$

For the remaining case  $y = [y_1, y_2]$ ,  $x, y_1, y_2 \in \{E_{i\bar{j}}, E_{\bar{j}i}\}$ , it is implied by the relations

$$[E_{i\bar{j}}, [E_{l\bar{k}}, E_{r\bar{s}}]] = \delta_{ki}\delta_{ls}E_{r\bar{j}} + \delta_{kr}\delta_{lj}E_{i\bar{s}}, \quad (11)$$

$$[E_{\bar{j}i}, [E_{k\bar{l}}, E_{\bar{s}r}]] = \delta_{ki}\delta_{ls}E_{\bar{j}r} + \delta_{kr}\delta_{lj}E_{\bar{s}i}. \quad (12)$$

The definition of  $\mathfrak{h}$  yields surjectivity of  $\phi$ . Since  $\mathfrak{g}$  is simple,  $\phi$  must be zero or injective. This proves the first part of the theorem.

To show the second part, one takes  $m_{ij} = E_{i\bar{j}}$ ,  $m_{ij}^* = E_{\bar{j}i}$ ; then (2)–(5) simply coincide with Equations (9)–(12).  $\square$

### 3. The Nilpotency Conditions

We denote the Lie algebra corresponding to an associative algebra  $\mathcal{A}$  by  $\mathcal{A}_L$ .

**THEOREM 2.** *For elements  $m_{ij}, m_{ij}^*$  of an associative algebra  $\mathcal{O}$  with identity, satisfying conditions of Theorem 1 (treated as elements of the Lie algebra  $\mathcal{O}_L$ ), the following properties are equivalent:*

- (1)  $m_{ij}m_{kl} = -m_{il}m_{kj}$ ,  $m_{ij}^*m_{kl}^* = -m_{il}^*m_{kj}^*$ ,
- (2)  $m_{ij}m_{il} = m_{ij}m_{kj} = m_{ij}^*m_{il}^* = m_{ij}^*m_{kj}^* = 0$ ,
- (3)  $m_{ij}^2 = m_{ij}^{*2} = 0$ ,
- (4)  $[m_{ij}, m_{ij}^*]^3 = [m_{ij}, m_{ij}^*]$ .

*Proof.* Implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) are trivial.

Assuming (3), identity  $[m_{ij}, [m_{ij}^*, m_{ij}]] = 2m_{ij}$  reduces to  $m_{ij}m_{ij}^*m_{ij} = m_{ij}$ , which allows us to calculate

$$\begin{aligned} [m_{ij}, m_{ij}^*]^3 &= (m_{ij}m_{ij}^* - m_{ij}^*m_{ij})^3 \\ &= (m_{ij}m_{ij}^*)^3 - (m_{ij}^*m_{ij})^3 = m_{ij}m_{ij}^* - m_{ij}^*m_{ij} = [m_{ij}, m_{ij}^*]. \end{aligned}$$

This shows that (3)  $\Rightarrow$  (4).

Let  $x \equiv [m_{ij}, m_{ij}^*]$  and  $m \equiv m_{ij}$ . Using  $[x, m] = 2m$  or, equivalently,  $(x - 2)m = mx$ , we get for any polynomial  $w(x)$  the relation  $w(x - 2) \cdot m = m \cdot w(x)$  or, equivalently,  $[w(x), m] = (w(x) - w(x - 2))m$ . In particular,

$$[x^3 - x, m] = 6(x - 1)^2m \quad \text{and} \quad [(x - 1)^2m, m] = 4(x - 2)m^2.$$

Now point 4 implies

$$x^3 - x = 0, \quad (x - 1)^2m = 0, \quad (x - 2)m^2 = 0.$$

From these equations, one easily gets  $m^2 = 0$ . Similarly, putting  $x = [m_{ij}^*, m_{ij}]$  and  $m = m_{ij}^*$ , one gets  $(m_{ij}^*)^2 = 0$ . This proves the implication (4)  $\Rightarrow$  (3).

Now, assume that (3) holds. Let  $r$  be arbitrary,  $j \neq l$ . From (4) we have  $m_{ij} = [m_{il}, [m_{rl}^*, m_{rj}]]$ . Using this and  $m_{il}^2 = 0$  gives

$$\begin{aligned} m_{ij}m_{il} &= [m_{il}, [m_{rl}^*, m_{rj}]]m_{il} = m_{il}[m_{rl}^*, m_{rj}]m_{il}, \\ m_{il}m_{ij} &= m_{il}[m_{il}, [m_{rl}^*, m_{rj}]] = -m_{il}[m_{rl}^*, m_{rj}]m_{il}, \end{aligned}$$

or  $2m_{ij}m_{il} = (m_{ij}m_{il} + m_{il}m_{ij}) + [m_{ij}, m_{il}] = 0$ . Analogously, one can derive the remaining equalities occurring under point (2). More precisely, it suffices to

notice the invariance of the considered relations under the substitution  $m_{ij} \rightarrow m_{ji}$ ,  $m_{ij}^* \rightarrow m_{ji}^*$ , and the substitution  $m_{ij} \rightarrow m_{ij}^*$ ,  $m_{ij}^* \rightarrow m_{ij}$ , respectively. Therefore, (3)  $\Rightarrow$  (2).

To prove the implication (2)  $\Rightarrow$  (1) we proceed as follows. For  $i \neq k$  and  $j \neq l$  we have

$$m_{ij} = [m_{kj}, [m_{kl}^*, m_{il}]] \quad \text{and} \quad m_{il} = [m_{kl}, [m_{kl}^*, m_{il}]],$$

yielding  $m_{ij}m_{kl} = m_{kj}[m_{kl}^*, m_{il}]m_{kl} = -m_{kj}m_{il}$ . The second part of point (1) is an immediate result of the substitution  $m_{ij} \rightarrow m_{ij}^*$ ,  $m_{ij}^* \rightarrow m_{ij}$ .  $\square$

**LEMMA 1.** *Relations  $[E_1, [E_2, E_1]] = E_1^2 = E_2^2 = 0$  imply  $[E_1, E_2]^2 = 0$ . In particular, if  $m_{ij}, m_{ij}^*$  satisfy the assumptions of Theorem 2 and have the properties formulated there, then  $[m_{ij}, m_{kl}^*]^2 = 0$ ,  $(i, j) \neq (k, l)$  (this generalizes property (3) of Theorem 2).*

*Proof.* Since

$$0 = [E_1, [E_2, E_1]] = 2E_1E_2E_1 - E_1E_1E_2 - E_2E_1E_1,$$

we have  $E_1E_2E_1 = 0$ , and, thus,  $[E_1, E_2]^2 = (E_1E_2 - E_2E_1)^2 = 0$ .

**DEFINITION 1.** Let  $\mathcal{O}_m$  denote the free algebra with identity, generated by  $m_{ij}, m_{ij}^*$  under relations:

$$[m_{ij}, m_{kl}] = 0, \tag{13}$$

$$[m_{ij}^*, m_{kl}^*] = 0, \tag{14}$$

$$[m_{ij}, [m_{kl}^*, m_{rs}]] = \delta_{ki}\delta_{ls}m_{rj} + \delta_{kr}\delta_{lj}m_{is}, \tag{15}$$

$$[m_{ij}^*, [m_{kl}, m_{rs}^*]] = \delta_{ki}\delta_{ls}m_{rj}^* + \delta_{kr}\delta_{lj}m_{is}^*, \tag{16}$$

$$m_{ij}^2 = m_{ij}^{*2} = 0. \tag{17}$$

Let us denote by  $U_{\mathfrak{g}}$  the enveloping algebra of the Lie algebra  $\mathfrak{g}$  and by  $I \subset U_{\mathfrak{g}}$  the ideal generated by  $\{E_{\alpha\beta}^2: \alpha \neq \beta\}$ .

**THEOREM 3.** *The algebra  $\mathcal{O}_m$  is isomorphic to the quotient algebra  $U_{\mathfrak{g}}/I$ . Moreover,  $\mathfrak{g} \subset U_{\mathfrak{g}}/I$ , i.e.  $\mathfrak{g} \cap I = \{0\}$ .*

*Proof.* By Theorem 1, there exists a Lie algebra homomorphism  $\phi_L: \mathfrak{g} \rightarrow (\mathcal{O}_m)_L$  such that  $\phi_L(E_{i\bar{j}}) = m_{ij}$ ,  $\phi_L(E_{\bar{j}i}) = m_{ij}^*$ . It lifts uniquely to a homomorphism  $\phi: U_{\mathfrak{g}} \rightarrow \mathcal{O}_m$ , and  $\ker \phi$  is the ideal generated by  $\{E_{i\bar{j}}^2, E_{\bar{j}i}^2\}$  (cf. relation (17)). Due to Lemma 1, this ideal coincides with  $I$ . Thus,  $\mathcal{O}_m$  is isomorphic to  $U_{\mathfrak{g}}/I$ .

Consider the homomorphism  $\tilde{R}: U_{\mathfrak{g}} \rightarrow M_{2N}(\mathbb{C})$  of associative algebras, uniquely defined by the condition  $\tilde{R}|_{\mathfrak{g}} = id_{\mathfrak{g}}$ . Obviously,  $\tilde{R}(I) = 0$ , because the square of the matrix  $E_{\alpha\beta} \in M_{2N}(\mathbb{C})$  vanishes for  $\alpha \neq \beta$ . Hence,  $\tilde{R}$  factorizes to a homomorphism  $R: U_{\mathfrak{g}}/I \rightarrow M_{2N}(\mathbb{C})$ . Of course,  $R([g]) = g$ , for  $g \in \mathfrak{g}$  and  $[g] := g + I \in U_{\mathfrak{g}}/I$ . Hence,  $\mathfrak{g} \subset U_{\mathfrak{g}}/I$ .  $\square$

The Weyl group of  $\mathfrak{sl}(2N, \mathbb{C})$  (i.e. the group of permutations of the  $2N$ -element set  $\mathcal{I}$ ), acting on the algebra  $U_{\mathfrak{g}}$ , leaves the set  $\{E_{\alpha\beta}^2: \alpha \neq \beta\}$  invariant. Thus, it leaves the ideal  $I$  invariant. This leads to a generalization of Theorem 2 (point (3) has no counterpart here, because we took it as the definition of  $I$ ).

**THEOREM 4.** *In the algebra  $U_{\mathfrak{g}}/I$  we have*

- (1)  $E_{\alpha\beta}E_{\gamma\delta} = -E_{\alpha\delta}E_{\gamma\beta}$ , for  $\alpha, \beta, \gamma, \delta$  mutually distinct,
- (2)  $E_{\alpha\beta}E_{\alpha\gamma} = 0 = E_{\beta\alpha}E_{\gamma\alpha}$ , for  $\alpha, \beta, \gamma$  mutually distinct,
- (3)  $H_{\alpha\beta}^3 = H_{\alpha\beta}$ , where  $H_{\alpha\beta} = E_{\alpha\alpha} - E_{\beta\beta}$ .

*Proof.* Let

$$r \in \mathbb{N}, X = \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathcal{I}^r: (i \neq j) \Rightarrow (\alpha_i \neq \alpha_j)\},$$

$$\sigma \cdot (\alpha_1, \alpha_2, \dots, \alpha_r) := (\sigma(\alpha_1), \sigma(\alpha_2), \dots, \sigma(\alpha_r)).$$

It is easy to verify that this defines a transitive action of the permutation group on  $X$ . Now, the statement follows from Theorems 3 and 2, and from transitivity of the action just defined, taking  $r = 4, 3, 2$ .  $\square$

**THEOREM 5.**  *$U_{\mathfrak{g}}/I$  is a finite-dimensional vector space.*

*Proof.* Take  $\{H_{\alpha, \alpha+1}\} \cup \{E_{\alpha\beta}: \alpha \neq \beta\}$  as an ordered basis in  $\mathfrak{g}$ . It follows from the Poincaré – Birkhoff – Witt theorem [1] that standard monomials built from elements of this basis form a basis in  $U_{\mathfrak{g}}$ . Due to point (3) of Theorem 4, dividing by the ideal  $I$  reduces the standard monomials built from these elements to a finite number.  $\square$

#### 4. Representations of $\mathcal{O}_m$

It is possible to describe all finite-dimensional representations of  $U_{\mathfrak{g}}/I$  (and, thus, also those of  $\mathcal{O}_m$ ) in terms of representations of the simple Lie algebra  $\mathfrak{g}$ .

Let there be given a representation  $\rho: U_{\mathfrak{g}}/I \rightarrow \text{End}(V)$ ,  $\dim V < \infty$ . The restriction of  $\rho$  to  $\mathfrak{g}$  is a representation  $\rho|_{\mathfrak{g}}$  of the Lie algebra  $\mathfrak{g}$ . Since  $\mathfrak{g}$  is simple, it decomposes into a direct sum of irreducible ones. Since  $\mathfrak{g} \subset U_{\mathfrak{g}}/I$  generates the algebra  $U_{\mathfrak{g}}/I$ , the decomposition of the representation  $\rho$  is the same. This shows that any finite-dimensional representation of the algebra  $U_{\mathfrak{g}}/I$  is completely

reducible, and each of its irreducible components is a unique extension to  $U_{\mathfrak{g}}/I$  of some irreducible representation of the Lie algebra  $\mathfrak{g}$ . Due to Theorems 2 and 4, the necessary and sufficient condition for the existence of such an extension is that  $\rho(H_{\alpha\beta})^3 = \rho(H_{\alpha\beta})$ , for any  $\alpha \neq \beta$ . The linear space  $V$  of any representation of  $\mathfrak{g}$  is a direct sum of weight spaces. Thus, the above condition is equivalent to the following one: for any weight  $m$  of the irreducible representation  $\rho$ , we have  $m(H_{\alpha\beta})^3 = m(H_{\alpha\beta})$ . This holds if the weights of  $\rho$  give only  $-1, 0$  or  $1$  when evaluated on the elements  $H_{\alpha\beta}$ . From the general theory ([11]), we know that the irreducible representations of  $\mathfrak{g}$  are in one-to-one correspondence with dominant integral weights (i.e. weights which on  $H_{\alpha\beta}$ ,  $\alpha < \beta$ , are nonnegative and integral). To determine a weight, one can give the sequence of its values on the elements  $H_{\alpha, \alpha+1}$ . If this sequence consists of exactly one unit and some zeros, the weight is called fundamental.

**COROLLARY 1.** *A finite-dimensional irreducible representation  $\rho$  of the algebra  $U_{\mathfrak{g}}/I$  is uniquely determined by the highest weight  $m$  of the Lie algebra representation  $\rho|_{\mathfrak{g}}$ ;  $m$  must be fundamental or zero.*

This statement completely characterizes the irreducible representations of  $U_{\mathfrak{g}}/I$ . Indeed, one easily constructs a representation for any fundamental weight and for the zero weight: Let  $V$  denote the free vector space spanned by  $\mathcal{P} = 2^{\mathcal{I}} = \{A: A \subset \mathcal{I}\}$ . For  $\alpha' = \alpha + 1$ , one defines  $\rho(E_{\alpha\alpha'}), \rho(E_{\alpha'\alpha}) \in \text{End}(V)$  as follows:

$$\rho(E_{\alpha\alpha'})(1 \cdot A) = \begin{cases} 1 \cdot (A \cup \{\alpha\} \setminus \{\alpha'\}), & \text{if } \alpha' \in A, \alpha \notin A, \\ 0, & \text{if } \alpha' \notin A \text{ or } \alpha \in A, \end{cases} \quad (18)$$

$$\rho(E_{\alpha'\alpha})(1 \cdot A) = \begin{cases} 1 \cdot (A \cup \{\alpha'\} \setminus \{\alpha\}), & \text{if } \alpha \in A, \alpha' \notin A, \\ 0, & \text{if } \alpha \notin A \text{ or } \alpha' \in A. \end{cases} \quad (19)$$

For a fixed  $Z \in \mathbb{Z}$ , let  $V_Z \subset V$  denote the linear subspace spanned by  $\mathcal{P}_Z := \{A \subset \mathcal{I}: |A| = Z\}$ . The subspace  $V_Z$  is  $\rho(U_{\mathfrak{g}}/I)$ -invariant. Let  $v_Z \in V_Z$  be  $1 \cdot A_Z$ , where  $A_Z \in \mathcal{P}_Z$  consists of the first  $Z$  elements in  $\mathcal{I}$ . The weights of the representation  $\rho$  are such that one concludes  $v_Z$  is a highest weight vector of the irreducible representation  $(V_Z, \rho|_{V_Z})$ . For  $Z = 1, 2, \dots, 2N - 1$  this gives the fundamental highest weights; for  $Z = 0$  or  $Z = 2N$ , the zero weight.

Now, let  $*$ :  $x \mapsto x^*$  be the unique antilinear anti-automorphism of the algebra  $U_{\mathfrak{g}}/I$  such that  $E_{\alpha\beta}^* = E_{\beta\alpha}$ , (equivalently, such that  $(m_{ij})^* = m_{ij}^*, (m_{ij}^*)^* = m_{ij}$ ). With this operation,  $U_{\mathfrak{g}}/I$  becomes a  $*$ -algebra.

We endow  $V_Z$  with a Hilbert space structure, taking the scalar product  $g_Z$  such that  $\{1 \cdot A: A \in \mathcal{P}_Z\}$  is an orthonormal set. The irreducible representation  $(H_Z, \rho|_{H_Z})$ , where  $H_Z = (V_Z, g_Z)$ , is a  $*$ -representation. This representation coincides with the canonical  $Z$ -representation defined in [3]. It was shown in [3] that  $H_Z$  is an eigenspace of the total charge operator  $\hat{Q}$ , with eigenvalue  $Q = e(Z - N)$ .

**COROLLARY 2.** *The representation  $\bigoplus_{Z=0}^{2N-1} (H_Z, \rho|_{H_Z})$  is faithful and defines an isomorphism of  $*$ -algebras*

$$U_{\mathfrak{g}}/I \cong \bigoplus_{Z=0}^{2N-1} B(H_Z), \quad \dim(H_Z) = \binom{2N}{Z}. \quad (20)$$

This is in accordance with the well-known fact that any  $*$ -algebra of linear operators on a finite-dimensional Hilbert space is a direct sum of factors. The center  $\mathcal{C}$  of  $U_{\mathfrak{g}}/I$  has the form  $\mathcal{C} = \bigoplus_i \mathcal{C}_i$ , with  $\mathcal{C}_i = \mathbb{C}\mathbb{1}_i$ . It is generated by the total charge operator  $\hat{Q}$  (see [3]), with the components in this direct sum corresponding to the spectral decomposition of  $\hat{Q}$ . Observe, however, that – strictly speaking –  $\hat{Q}$  cannot be fully reconstructed from (18), because one one-dimensional representation is lacking. On the other hand, we stress that  $\hat{Q}^2$  is a linear function of the Casimir operator and can be easily expressed in terms of the generators  $\{E_{\alpha\beta}\}$ .

**THEOREM 6.** *Any finite dimensional irreducible  $*$ -representation of the algebra  $U_{\mathfrak{g}}/I$  is unitarily equivalent to  $(H_Z, \rho|_{H_Z})$ , for some  $Z$ .*

*Proof.* Let  $(H, \pi)$  be a finite-dimensional irreducible  $*$ -representation of  $U_{\mathfrak{g}}/I$ . Of course,  $(H, \pi)$  is algebraically equivalent to  $(V_Z, \rho|_{V_Z})$ , for some  $Z$ . Thus,  $(H, \pi)$  is unitarily equivalent to the canonical  $Z$ -representation  $(H_Z, \rho|_{H_Z})$  – due to the following well-known fact [12], (point 2.9.6): If two irreducible representations of a  $C^*$ -algebra are algebraically equivalent, then they are also equivalent as  $*$ -representations.  $\square$

#### 4. Concluding Remarks

A comparison of our results with results obtained within the (continuum) algebraic quantum field theory approach is only partially possible.

(1) In Section 10 of [3], a possible strategy to construct the continuum theory as a limit of its lattice approximations was discussed. In particular, such a construction must include charge renormalization. It turns out that the algebra  $\mathcal{O}_m$  may be generated by operators  $\hat{\mathcal{M}}_{\gamma}$  corresponding to either trivial or 1-link paths  $\gamma$ . In the continuum limit, these generators would correspond to operators  $W(x) := \hat{\psi}^*(x)\hat{\psi}(x)$  and  $V_k(x) := \hat{\psi}^*(x)D_k\hat{\psi}(x)$ . Of course, such products are highly singular in the conventional formulation of QED. Our results suggest that, maybe, we must look for a continuum theory of the gauge-invariant fields  $W$  and  $V_k$ , which, together with the electromagnetic fields  $B$  and  $E$  should be treated as fundamental fields, instead of nonphysical objects  $\hat{\psi}(x)$ .

(2) It is probably too difficult to rigorously construct the full continuum limit addressed in the previous comment. But, due to the algebraic structure given by Theorem 3, it seems to be possible to construct at least the thermodynamic limit (infinite lattice) using methods developed in recent years, see, e.g., [13] and further references therein.

(3) In [9], the concept of charge classes was developed. It was shown that within every charge class there exist inequivalent representations corresponding to different asymptotic flux distributions which, however, are unitarily equivalent on lightcones  $V^+ + a$ , resp.  $V^- + a$ , (with apex  $a$ ) in position space. This way a criterion for distinguishing between the electric charge (carried by massive particles) and superselection sectors corresponding to inequivalent asymptotic infrared clouds of photons was found. In our case of a *finite* lattice, it turns out (see Section 8 in [3]) that for any fixed total charge sector representations corresponding to different flux distributions on the boundary are unitarily equivalent. However, after performing the thermodynamic limit, we hope to see the above charge classes.

(4) In [7], so-called charged morphisms, which had to be used to create charged states out of the physical vacuum, were introduced. They were obtained as weak limits of charge transfer cocycles. Heuristically, these cocycles correspond to ‘charged field bundles’, which formally were already invented in [8]. Using our wave function representation of physical states, from formula (8.1) in [3] we read off the finite lattice counterparts of these charged field bundles. On the other hand, due to the structure of  $\mathcal{O}_m$  given by Corollary 2, there are no nontrivial  $*$ -morphisms of  $\mathcal{O}_m$  intertwining between different sectors. But again, after performing the thermodynamic limit, we hope to construct charge transfer cocycles and, consequently, also rigorously charged morphisms.

## References

1. Kijowski, J., Rudolph, G. and Rudolph, M.: Functional integral of QED in terms of gauge invariant quantities, *Lett. Math. Phys.* **33** (1995), 139–146.
2. Kijowski, J., Rudolph, G. and Rudolph, M.: Effective bosonic degrees of freedom for one-flavour chromodynamics, *Ann. Inst. H. Poincaré* (in print).
3. Kijowski, J., Rudolph, G. and Thielman, A.: The algebra of observables and charge superselection sectors for QED on the lattice, *Comm. Math. Phys.* **188** (1997), 535–564.
4. Strocchi, F. and Wightman, A.: *J. Math. Phys.* **15** (1974), 2198.
5. Strocchi, F.: *Comm. Math. Phys.* **56** (1977), 57.
6. Strocchi, F.: *Phys. Rev. D* **17** (1978), 2010.
7. Fröhlich, J.: *Comm. Math. Phys.* **66** (1979), 223.
8. Fröhlich, J., Morchio, G. and Strocchi, F.: *Ann. of Phys.* **119** (1979), 241.
9. Buchholz, D.: *Comm. Math. Phys.* **85** (1982), 49; *Phys. Lett. B* **174** (1986), 331.
10. Fredenhagen, K. and Marcu, M.: *Comm. Math. Phys.* **92** (1983), 81.
11. Jacobson, N.: *Lie Algebras*, Dover, New York, 1979.
12. Dixmier, P. J.: *Les  $C^*$ -algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.
13. Palev, T. D.: *Rep. Math. Phys.* **31**(3) (1992), 241.