THE VELOCITY OPERATOR FOR MANY-BOSON SYSTEMS

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Synopsis

Beginning with Landau's theory of superfluid systems various attempts were made to formulate the theory of many-boson systems in terms of hydrodynamical quantities, i.e., particle density and velocity fields. In the present paper a new approach is proposed to this question based on the application of Glauber's (or coherent-states) representation. The general form of the velocity operator is derived and special attention is paid to the properties of the velocity field in superfluid systems. The relation between the present approach and the one previously is discussed.

1. Introduction. The first satisfactory theory of superfluid systems was proposed by Landau\(^1\), and was called quantum hydrodynamics (QH). The meaning of the term QH ought to be understood as a synonym of the fluid-like picture of a quantum many-body system. In QH instead of working with non-hermitian operators \( \hat{\rho}(x, t) \) and \( \hat{J}(x, t) \) which create and destroy a particle at a given time-space point, we deal with hermitian operators \( \hat{\rho}(x, t) \) and \( \hat{J}(x, t) \) which describe the particle density and particle current, respectively. Both operators \( \hat{\rho}, \hat{J} \) correspond to measurable quantities which are of great importance in the description of the overall properties of the many-body systems. In particular various correlation functions, like van Hove's correlation function, are expressed in terms of expectation value of various products of operators \( \hat{\rho} \) and \( \hat{J} \).

The operators \( \hat{\rho} \) and \( \hat{J} \) are related to the field operators \( \hat{\rho} \) by standard formulas; commutation relations between \( \hat{\rho} \) and \( \hat{J} \) and various components of \( \hat{J} \) themselves could be derived either from the commutation relations between field operators, or from some general assumptions like locality, galilean invariance etc. Surprisingly, these commutation relations are the same for both kinds of particles, i.e., for bosons and fermions. In this paper we will be primarily interested in the properties of a many-boson system, the latter being the most interesting application of QH.

 Probably one of the most interesting questions which arise in QH is whether the velocity operator for a many-boson system may be defined. It
was shown by Landau\(^1\)) how such an operator may be defined through a properly symmetrized division of the current operator by the density operator. Using this definition, Landau was able to derive the commutation relations between \(\hat{\rho}\) and the velocity operator \(\hat{\mathbf{P}}\), as well as those between various components of the velocity themselves.

QH, formulated in terms of \(\hat{\rho}\) and \(\hat{\mathbf{P}}\) was recently criticized by Fröhlich\(^2\)). The essential point in Fröhlich’s criticism was that the division by the operator \(\hat{\rho}\) is not well defined, and, therefore, the velocity operator does not exist. While it is certainly true that the particle-density operator \(\hat{\rho}\) has no inverse in Fock space, the division by it is not necessary, and we intend to show that the velocity operator may be defined when the proper representation is adopted.

One of the characteristic features of the superfluid state is that it is the one with well-defined velocity and density patterns which are directly related to the microscopic properties of the system. In the superfluid system velocity and density fields are both fine-grained variables, rather than coarse-grained, as in the normal fluids.

2. Hydrodynamic current algebra. Let us consider system of identical, neutral, quantum particles, each of mass \(m = 1\). The field operators denoted by \(\hat{\mathbf{P}}(x)\) and \(\hat{\mathbf{P}}^\dagger(x)\) satisfy the commutation relations

\[
[\hat{\mathbf{P}}(x), \hat{\mathbf{P}}^\dagger(y)] = \delta(x - y); \quad [\hat{\mathbf{P}}(x), \hat{\mathbf{P}}(y)] = 0,
\]

where \([,]\) denotes a commutator.

The particle-density and particle-current operators are defined as usual:

\[
\hat{\rho}(x) = \hat{\mathbf{P}}^\dagger(x) \hat{\mathbf{P}}(x); \quad \hat{\mathbf{J}}(x) = \frac{\hbar}{2i} \hat{\mathbf{P}}^\dagger(x) \hat{\mathbf{V}} \hat{\mathbf{P}}(x),
\]

where

\[
[a(x) \hat{\mathbf{V}} \beta(x) = a(x) \hat{\mathbf{V}} \beta(x) - (x \alpha(x)) \beta(x).
\]

Using relations (2.1), we can derive the commutation relations between \(\hat{\rho}\) and \(\hat{\mathbf{J}}\)

\[
[\hat{\rho}(x), \hat{\rho}(y)] = 0; \quad [\hat{\rho}(x), \hat{\mathbf{J}}(y)] = \frac{\hbar}{i} \mathbf{V} \delta(x - y) \hat{\rho}(x),
\]

\[
[\hat{\mathbf{J}}^a(x), \hat{\mathbf{J}}^b(y)] = \frac{\hbar}{i} \frac{\partial}{\partial x_b} \{\delta(x - y) \hat{\mathbf{J}}^a(x)\} - \frac{\hbar}{i} \frac{\partial}{\partial y_b} \{\delta(x - y) \hat{\mathbf{J}}^b(y)\}.
\]

Now, if we recall that the operators \(\hat{\rho}(x)\) and \(\hat{\mathbf{J}}(x)\) satisfy the exact continuity equation (for each number of particle-conserving hamiltonians)

\[
\partial_\alpha \rho + \mathbf{V} \cdot \hat{\mathbf{J}} = 0,
\]
we may observe that the operators \( \hat{\rho} \) and \( \hat{J} \) form an example of a current algebra, known from the hadron dynamics. This fact was observed by Dashen and Sharp\(^3\).

It is worthwhile to notice that the generalized "charge" in this theory is the total number of particles (or the total mass). The very interesting properties of the systems violating the superselection rule attached to this particular generalized charge were discussed by Sarfatt\(^4\).

The hydrodynamic current algebra (HCA) spanned by the operators \( \hat{\rho}(x) \) and \( \hat{J}(x) \) has a classical limit in the form of the functions \( \rho(x) \) and \( J(x) \), defined as:

\[
\rho(x) = \sum_{i=1}^{N} \delta(x - q_i), \quad J(x) = \sum_{i=1}^{N} p_i \delta(x - q_i),
\]

(2.5)

where \( q_i \) and \( p_i \) are canonical positions and momenta, respectively.

This kind of function algebra was studied in the collective variables theory. A very important result in this area belongs to Kalman\(^6\), who shows that there exists a canonical transformation from the variables \( (q_i, p_i) \) to the new ones \( (\rho_k, \varphi_k) \), where \( \rho_k \) are Fourier components of the density \( \rho(x) \), such that

\[
J_k = i \sum_{\kappa'} \rho_{-\kappa'} \rho_{\kappa'},
\]

\[
J(x) = -i \rho(x) \mathcal{V} \varphi(x).
\]

(2.6)

Eqs (2.6) express the fact that for the classical system the microscopic velocity field \( \mathcal{V}(x) = J(x)/\rho(x) \) is always a potential field. This is a very important point in our considerations. In sec. 3 we introduce the concept of the coherent states; it is known that the coherent states are "as classical as possible" states of a quantum system. We show that the velocity field for the system appearing to be in a coherent state is also a potential field.

The next step in the HCA approach to the many-body problem is to express the hamiltonian in terms of \( \hat{\rho} \) and \( \hat{J} \). After some calculations we obtain

\[
\hat{H} = \frac{1}{\hbar} \int d^3x \left\{ \hbar \mathcal{V} \hat{\rho} - 2i \hat{J} \right\} \frac{1}{\hat{\rho}} \left\{ \hbar \mathcal{V} \hat{\rho} + 2i \hat{J} \right\}
\]

\[
+ \frac{1}{\hbar} \int d^3x \int d^3y \hat{\rho}(x) \ U(x - y) \hat{\rho}(y),
\]

(2.7)

where we dropped the constant term proportional to \( U(0) \) [\( U(x - y) \) being the interparticle interaction potential]. The hamiltonian (2.7) together with the algebraic relations between \( \hat{\rho} \) and \( \hat{J} \), could be used as a starting point for various applications.

In this paper we restrict ourselves to the problem of the velocity operator for a many-boson system.
Landau's definition of the velocity operator

\[ \hat{\mathbf{J}}(x) = \frac{i}{\hbar} \{ \hat{\mathbf{P}}(x) \cdot \hat{\mathbf{J}}(x) + \hat{\mathbf{J}}(x) \cdot \hat{\mathbf{P}}(x) \} \tag{2.8} \]

is implicit and we prefer to the conventional definition, also due to Landau, namely

\[ \hat{\mathbf{P}}(x) = \frac{i}{\hbar} \left\{ \frac{1}{\hat{\rho}(x)} \hat{\mathbf{J}}(x) + \hat{\mathbf{J}}(x) \frac{1}{\hat{\rho}(x)} \right\}. \tag{2.9} \]

The definition (2.9) contains the division by the particle-density operator \( \hat{\rho}(x) \) and is therefore subject to the Fröhlich criticism.

Using the commutation relations (2.3) and the standard procedure of dealing with commutators, we may obtain the following list of commutation relations between the operators \( \hat{\rho} \) and \( \hat{\mathbf{P}} \):

\[ [\hat{\rho}(x), \hat{\rho}(y)] = 0, \quad \tag{2.10a} \]

\[ [\hat{\rho}(x), \hat{P}^a(y)] = \frac{\hbar}{i} \frac{\partial}{\partial x^a} \delta(x - y), \quad \tag{2.10b} \]

\[ \hat{\rho}(x)[\hat{P}^a(x), \hat{P}^b(y)] = \frac{\hbar}{i} \delta(x - y) \left\{ \frac{\partial \hat{P}^b}{\partial x^a} - \frac{\partial \hat{P}^a}{\partial x^b} \right\}. \tag{2.10c} \]

In eq. (2.10c) the order of the operators \( \hat{\rho} \) and \( [\hat{P}^a(x), \hat{P}^b(y)] \) is arbitrary because these operators commute. It was assumed in the literature that there exists an operator \( \hat{\Phi}(x) \), such that

\[ \hat{\mathbf{P}}(x) = \hbar \hat{\mathbf{J}}(x). \tag{2.11} \]

In this case the right-hand side of (2.10c) vanishes, and the commutation relations simplify considerably; we have then:

\[ [\hat{\rho}(x), \hat{\rho}(y)] = 0, \quad [\hat{\Phi}(x), \hat{\Phi}(y)] = 0, \]

\[ [\hat{\rho}(x), \hat{\Phi}(y)] = -i \delta(x - y). \tag{2.12} \]

The above results were usually “derived” by means of polar decomposition of the field operator

\[ \hat{\Phi}(x) = \exp(i \hat{\Phi}(x)) \hat{\mathbf{J}}(x). \tag{2.13} \]

Unfortunately, the decomposition (2.13) is incorrect cf. Carruthers and Nieto. To the best of author’s knowledge, there is no proof in the literature, that the velocity operator \( \hat{\mathbf{P}} \) is of the form (2.11). In sec. 4 we show that the velocity operator indeed has that form.

3. Coherent states. The application of the coherent states to the theory of superfluid systems originated from the realization that the off-diagonal long-range order (ODLRO) in the many-boson system closely resembles the first-order coherence property of the electromagnetic field, known from Glauber’s theory of coherence. Indeed, the ODLRO theory states that at
the temperature $T < T_A \approx 2.17$ K for $^4$He, the first reduced density matrix
\( \Omega(x, y) \) factorizes into
\[
\Omega(x, y) = \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle = \psi^\dagger(x) \psi(y) + \bar{\Omega}(x, y),
\] (3.1)
where the term \( \bar{\Omega} \) vanishes as \( |x - y| \to \infty \). The factorized term, the
condensate wave function, is defined as the mean value of the field operator
\[
\psi(x) = \langle \hat{\Psi}(x) \rangle = \text{Tr}[\hat{R}\hat{\Psi}(x)],
\] (3.2)
where \( \hat{R} \) is the von Neuman statistical operator (properly normalized). Using
the properties of the trace operation and the commutation relation between \( \hat{\Psi}(x) \) and the number of particles operator, \( \hat{N} \), we observe that the
expectation value in (3.2) does not vanish if and only if \( \hat{R} \) does not commute
with \( \hat{N} \), \( \text{i.e.,} \) the quantum ensemble described by \( \hat{R} \) is a mixture of states
which are not the eigenstates of the operator \( \hat{N} \). Various ensembles possess
this property were proposed in the literature. According to the London idea
that the superfluid component of helium at $T < T_A$ is in the single pure
quantum state, Cummings and Johnston\(^8\) proposed the hypothesis that
the superfluid component of the helium is in the coherent state\(^7\). The
statistical operator \( \hat{R} \) for the system consists then of two parts; one of them
is the coherent part and the second is the "noise" part in the quantum-
electronic language. For detailed discussion of this subject and the details
of the construction of the operator \( \hat{R} \) as well as a most thorough explanation
of the ideas which lie behind the coherent-states approach to the many-boson
system we refer the reader to the paper of Anderson\(^9\) (cf. Langer\(^10\)).

Since the superfluid component may be regarded as the coherent field,
there should be a set of field equations, similar to Maxwell's equations in
optics, describing the motion of the condensate. In the sequel we show that
this set of equations turns out to be analogous to that of Gross's "Classical
Bose Field Theory"\(^11\).

After arguing that the theory of the Bose condensate may be formulated
in terms of coherent states, let us, very briefly repeat some facts from the
coherent-states theory; for the details the reader should consult the paper
of Glauber\(^12\).

Let us consider a many-boson system, in a box of volume $L^3 = 1$. The
field operators may be decomposed according to
\[
\hat{\Psi}(x) = \sum_k \hat{a}_k \exp(ik \cdot x). \] (3.3)

\(^7\) In real helium the condensation of particles into the state with zero momentum
is only fractional ($\approx 10\%$). The coherent state need not be monochromatic, \textit{v}ia,, it
may be considered as a combination of states with different wave vectors and thus
there is no contradiction between the assumption that the superfluid component is in
the coherent state and the fact that the condensation in zero-momentum mode is only
fractional.
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The coherent states $|\alpha_k\rangle$ are defined as the eigenstates of the annihilation operators $a_k$

$$a_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle.$$  \hspace{1cm} (3.4)

The tensor product of the states $|\alpha_k\rangle$

$$|\varphi\rangle = \bigotimes_k |\alpha_k\rangle,$$  \hspace{1cm} (3.5)

is the simultaneous eigenstate for all the annihilation operators $a_k$ and thus is the eigenstate for the field operator

$$\hat{\Psi}(\mathbf{x}) |\varphi\rangle = \sum_k \alpha_k e^{i k \cdot \mathbf{x}} |\alpha_k\rangle \equiv \alpha(\mathbf{x}) |\varphi\rangle.$$  \hspace{1cm} (3.6)

The eigenvalue of $\hat{\Psi}$, $\alpha(\mathbf{x})$ is a complex-valued function which may be identified with the condensate wave function. Indeed, if we calculate the mean value of the operator $\hat{\Psi}^{\dagger}(\mathbf{x}) \hat{\Psi}(\mathbf{y})$ in the state $|\alpha\rangle$ we obtain

$$\langle\alpha | \hat{\Psi}^{\dagger}(\mathbf{x}) \hat{\Psi}(\mathbf{y}) |\alpha\rangle = \alpha^*(\mathbf{x}) \alpha(\mathbf{y}).$$  \hspace{1cm} (3.7)

This equation should be compared with the definition of ODLRO.

The states $|\alpha\rangle$ form a non-orthogonal and over-complete set of vectors in Hilbert space. Indeed, the scalar product of two vectors $|\alpha\rangle$ never vanishes

$$\langle \alpha^1 | \alpha^2 \rangle = \exp[i (\langle \alpha^1 | \alpha^2 \rangle - \frac{1}{2} \|\alpha^1\|^2 - \frac{1}{2} \|\alpha^2\|^2)],$$  \hspace{1cm} (3.8)

where

$$(\alpha | \beta) = \sum_k \alpha^*_k \beta_k, \|\alpha\|^2 = (\alpha | \alpha),$$

and every state $|\alpha\rangle$ could be written as a linear combination of all states including itself. The last decomposition is, however, not unique. Nevertheless, there exists the resolution of the unit operator given by

$$\int \prod_k \frac{d^3 \alpha_k}{\pi} |\alpha_k\rangle \langle \alpha_k| = \int D(\alpha) |\alpha\rangle \langle \alpha| = 1.$$  \hspace{1cm} (3.9)

In the coherent-states representation we have the following forms of matrix elements of the particle density and particle current, respectively:

$$\langle \alpha | \rho(\mathbf{x}) | \beta \rangle = \langle \alpha | \beta \rangle \alpha^*(\mathbf{x}) \beta(\mathbf{x}),$$

$$\langle \alpha | \vec{J}(\mathbf{x}) | \beta \rangle = (\hbar / 2i) \langle \alpha | \beta \rangle \alpha^*(\mathbf{x}) \vec{V} \beta(\mathbf{x}).$$  \hspace{1cm} (3.10)

Using (3.10) and the continuity equation (2.4), we obtain the conservation law for diagonal elements of $\hat{\rho}$ and $\hat{J}$

$$\partial_t \rho_\alpha(\mathbf{x}) + \nabla \cdot J_\alpha(\mathbf{x}) = 0,$$  \hspace{1cm} (3.11)

where

$$\rho_\alpha(\mathbf{x}) = |\alpha(\mathbf{x})|^2.$$
\[ J_a(x) = (\hbar/2i) \alpha^*(x) \vec{V}_a(x). \]

Decomposing now the condensate wave function \( \alpha(x) \) into
\[ \alpha(x) = f_\alpha(x) \exp[\imath \varphi_\alpha(x)], \]  
we observe that \( f_\alpha^2(x) \) is the mean value of \( \bar{\beta} \) and that the gradient of the phase \( \varphi_\alpha \) is related to the current density by the formula
\[ \langle \alpha | \hat{J}(x) | \alpha \rangle = \hbar \langle \alpha | \bar{\beta}(x) | \alpha \rangle \vec{V}_{\varphi_\alpha} = \hbar \rho_\alpha \vec{V}_{\varphi_\alpha}. \]  

Using eq. (3.13) (3.13) we may define the velocity field as
\[ \vec{V}_a(x) = \langle \alpha | \hat{J}(x) | \alpha \rangle / \langle \alpha | \bar{\beta}(x) | \alpha \rangle = \hbar \vec{V}_{\varphi_\alpha}(x). \]  

It follows from eq. (3.14) that the velocity field for the system in the coherent state is given by the gradient of the phase of the condensate wave function \( \alpha(x) \). This fact was already used in the classical Bose field theory, and should be compared with Kalman’s result (2.6). Since the coherent states are the most nearly classical of all the quantum states, the result (3.14) is not unexpected.

Eq. (3.14) tells us nothing about the existence of the velocity operator, for which the diagonal matrix elements, in the Glauber representation, would have the form
\[ \langle \alpha | \hat{V}(x) | \alpha \rangle = \hbar \vec{V}_{\varphi_\alpha}(x). \]

The very existence of the velocity operator was questioned, mainly due to the fact that the matrix elements of such an operator should be an erratic function in the configuration-space representation. As we shall see, this difficulty may be overcome by adopting the Glauber representation.

4. The velocity operator. Let us consider an arbitrary operator \( \hat{O} \). Using the resolution of unity (3.9), we may write the following identity
\[ \hat{O} = \int D(\alpha) D(\beta) \exp(-\frac{1}{2} \lVert \alpha \rVert^2 - \frac{1}{2} \lVert \beta \rVert^2) O(\alpha^*, \beta) | \alpha \rangle \langle \beta |, \]  
where the function \( O(\alpha^*, \beta) \) is related to the matrix element of \( \hat{O} \) by
\[ O(\alpha^*, \beta) = \langle \alpha | \hat{O} | \beta \rangle \exp(\frac{1}{2} \lVert \alpha \rVert^2 + \frac{1}{2} \lVert \beta \rVert^2). \]  

The functions \( O \) corresponding to the operators \( \hat{\beta} \) and \( \hat{J} \) have the following forms:
\[ \hat{\beta} \rightarrow \hat{O}(\alpha^*, \beta; x) = \alpha^*(x) \beta(x) \exp[(\alpha | \beta)], \]
\[ \hat{J} \rightarrow \hat{O}(\alpha^*, \beta; x) = (\hbar/2i) \alpha^*(x) \vec{V}_\beta(x) \exp[(\alpha | \beta)]. \]  

The following theorem, due to Glauber, holds for the operator \( \hat{O} = \hat{O}_1 \hat{O}_2 \), if \( O, O_1, O_2 \), are their corresponding functions in the sense of (4.2), then
\[ O(\alpha^*, \beta) = \int O_1(\alpha^*, \gamma) O_2(\gamma^*, \beta) \exp(-\lVert \gamma \rVert^2) D(\gamma). \]  

Eq. (4.4) determines the function \( \theta_2 \) if the functions \( \theta_1 \) are known. The solution of (4.4), together with (4.1), gives us the form of unknown operator \( \hat{\theta}_2 \). This is exactly our problem. We have the implicit definition of the operator \( \hat{\mathcal{V}}(x) \) through (2.8), we know the functions \( \mathcal{R} \) and \( \mathcal{J} \) corresponding to the operators \( \hat{\rho} \) and \( \hat{\mathcal{J}} \) and we look for the function \( \mathcal{V}^\ast \) which satisfies the equation of the form (4.4), namely:

\[
\mathcal{J}(x^\ast, \beta; x) = \frac{1}{2} \int \mathcal{R}(x^\ast, x; x) \mathcal{V}(y^\ast, \beta; x) \exp(- ||y||^2) D(y) + \int \mathcal{V}(x^\ast, y; x) \mathcal{R}(y^\ast, \beta; x) \exp(- ||y||^2) D(y).
\]

(4.5)

In order to solve this equation and in order to simplify various calculations we change the definition (2.8) into:

\[
\mathcal{J}(x) = \frac{\hbar}{2i} \lim_{\epsilon \to 0} [\hat{\rho}(x) \hat{\mathcal{V}}(x + \epsilon) + \hat{\mathcal{V}}(x + \epsilon) \hat{\rho}(x)].
\]

(4.6)

Applying now the commutation relations (2.10) we are able to write (4.5) in the form

\[
\mathcal{V}(x^\ast, \beta; x) = \lim_{\epsilon \to 0} \int D(y) \mathcal{R}(x^\ast, x; x) \mathcal{V}(y^\ast, \beta; x + \epsilon) + \frac{\hbar}{2i} \exp(\langle \alpha | \beta \rangle) \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \delta(\epsilon).
\]

(4.7)

Inserting now the explicit forms of the functions \( \mathcal{R} \) and \( \mathcal{J} \) into (4.7) and using the integral identities

\[
\int \pi^{-1} d^2 \alpha_k \exp(\alpha_k^\ast \gamma_k - |\gamma_k|^2 \gamma_k)^n F(\gamma_k) = \left( \frac{\partial}{\partial \alpha_k} \right)^n F(\alpha_k),
\]

\[
\int \pi^{-1} d^2 \gamma_k \exp(\gamma_k^\ast \alpha_k - |\gamma_k|^2 \gamma_k)^n F(\gamma_k) = \left( \frac{\partial}{\partial \gamma_k} \right)^n F(\alpha_k),
\]

(4.8)

we obtain

\[
\frac{\hbar}{2i} \alpha^\ast(x) \mathcal{V}(x^\ast) \exp(\langle \alpha | \beta \rangle) = \frac{\hbar}{2i} \exp(\langle \alpha | \beta \rangle) \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \delta(\epsilon)
\]

\[
+ \lim_{\epsilon \to 0} \sum_k \alpha_k^\ast e^{ikx} \frac{\partial}{\partial \alpha_k} \mathcal{V}(\alpha_k^\ast, \beta_k; x + \epsilon)
\]

(4.9)

\[\dagger\] It is easily verified that the operator \( \hat{\mathcal{V}} \) defined by (4.6) has the same list of commutation relations as the one defined in (2.8). The limiting procedure in (4.6) is the trick used in order to remove the infinities of the form \( \Sigma_k 1 \) from our calculations. The infinities of this type occur very often in the calculations involving coherent-state techniques (cf. Langer\(^{10}\)).
Using the following properties of the delta function
\[
e^a \frac{\partial}{\partial e^b} \delta(e) = -\delta^a_b \delta(e), \quad e^a \delta(e) = 0,
\]
one can verify that the function $\mathcal{V}^\ast(\alpha^\ast, \beta; x)$, given below in (4.11), is the solution of (4.7).
\[
\mathcal{V}^\ast(\alpha^\ast, \beta; x) = \frac{\hbar}{2i} \exp[(\alpha | \beta)] \left[ \frac{1}{\beta(x)} \mathcal{V} \beta(x) - \frac{1}{\alpha^\ast(x)} \mathcal{V} \alpha^\ast(x) \right]. \tag{4.11}
\]

It is more convenient to rewrite (4.11) applying the polar decomposition of $\alpha(x)$ [cf. (3.12)]. We obtain then.
\[
\mathcal{V}^\ast(\alpha^\ast, \beta; x) = \frac{\hbar}{2i} \exp[(\alpha | \beta)] \left[ \mathcal{V} \alpha(x) + \alpha^\ast \right] + i \mathcal{V} \ln (f_\alpha | f_\beta). \tag{4.12}
\]

Comparing now eqs. (4.12) and (4.2) we observe that the operator $\hat{V}(x)$ defined according to (4.1), as
\[
\langle \alpha | \hat{V}(x) | \alpha \rangle = \hbar \mathcal{V} \alpha(x), \tag{4.13}
\]
has the matrix element between the states $| \alpha \rangle$ of the simple form
\[
\langle \alpha | \hat{V}(x) | \alpha \rangle = \hbar \mathcal{V} \alpha(x). \tag{4.14}
\]

As we may see from (4.14), the velocity operator $\hat{V}$ given by (4.13) has the property already imposed on the velocity operator for a many-boson system. The next important property of that particular form of velocity operator is that it is not only the reduced density matrix, which factorizes in the coherent-states representation, but also the current density
\[
J_\alpha(x) = \langle \alpha | \beta(x) | \alpha \rangle \langle \alpha | \hat{V}(x) | \alpha \rangle.
\]

This expresses the physical fact that for superfluid systems, not only the density field, but also the velocity field are fine-grained variables.

The integration in (4.13) may be carried out, and we obtain
\[
\hat{V} = (\hbar/2i) \int D(\alpha) D(\beta) \langle \alpha | \beta \rangle \left( \beta^{-1} \mathcal{V} \beta - \alpha^{-1} \mathcal{V} \alpha^\ast \right) \langle \alpha | \beta \rangle
\]
\[
= (\hbar/2i) \int D(\alpha) D(\beta) \langle \alpha | \beta \rangle \left( \mathcal{V} \ln \beta - \mathcal{V} \ln \alpha^\ast \right) \langle \alpha | \beta \rangle
\]
\[
= (\hbar/2i) \int D(x) \langle \alpha | \mathcal{V} \ln x \rangle \langle \alpha | \mathcal{V} \ln x^\ast \rangle
\]
\[
= (\hbar/2i) \int D(x) \langle \alpha | \mathcal{V} \ln x \rangle \langle \alpha | \mathcal{V} \ln x^\ast \rangle. \tag{4.15}
\]

Applying again the polar decomposition (3.12) we obtain
\[
\hat{V} = (\hbar/2i) \int D(x) \langle \alpha | \mathcal{V} \ln e^{i \phi_x} \rangle = \hbar \int D(x) \langle \alpha | \mathcal{V} \phi_x. \tag{4.16}
\]

Eq. (4.16) shows that the velocity operator for a many-boson system, as defined in (2.6) or (4.6), has the form
\[
\hat{V}(x) = \hbar \mathcal{V} \phi(x), \tag{4.17}
\]
where the velocity potential operator $\hat{\Phi}(x)$ is given as

$$\hat{\Phi}(x) = \int D(\alpha) \langle \alpha | \varphi_\alpha(x) \rangle.$$  \hspace{1cm} (4.18)

We should again compare that result with Kalman's, \textit{i.e.} (2.6).

Eq. (4.17) has exactly the same form as (2.11); however, the operator $\hat{\Phi}$ in (4.18) differs from the non-existing phase of the field operator [\textit{cf.} (2.13)]. In a sense our problem is similar to the famous problem of existence of the phase operator canonically conjugate to the number operator for a simple harmonic oscillator. It was shown (\textit{cf.} ref. 7) that such an operator does not exist. However, recently, Garrison and Wong\textsuperscript{18} reconsidered this problem once again. The difficulties that present the definition of the phase operator were eliminated by distinguishing between the Weyl and Heisenberg forms of the canonical commutation relations. The construction of the operator conjugate to the number operator in the Heisenberg sense, as proposed in Ref. 13, is somehow similar to our construction of the operator $\hat{\Phi}$.

At the end of this section we shall recapitulate its main results. Using the Glauber representation and the convolution theorem (4.4), together with the implicit definition of the velocity operator (2.8) or (4.6) and some other properties of HCA, we derive the general expression for the velocity operator of the many-boson system. The velocity operator is a potential operator in the sense of (4.17), but the velocity potential differs from that previously postulated in the literature. The form of the velocity operator (4.17) shows that various components of the velocity commute [\textit{cf.} (2.10)]. It means that we are able to measure various components of the velocity simultaneously. It is also suggested that the velocity correlation function may be of some use in the theory of superfluid systems.

5. \textit{Miscellanea}. The formalism presented in secs. 3 and 4 should be compared with the formal considerations frequently used in quantum hydrodynamics. For example, using (4.11), we may write

$$\langle \alpha | \hat{\mathcal{P}} | \beta \rangle = -\langle \hbar/2i \rangle \langle \alpha | \alpha^{-1} \hat{\mathcal{P}} \alpha - \beta^{-1} \hat{\mathcal{P}} \beta | \beta \rangle.$$  \hspace{1cm} (5.1)

If the states $|\alpha\rangle$, $|\beta\rangle$ are different from vacuum and if we agree to the definition of the inverse field operator

$$\hat{\Phi}^{-1}(x) |\alpha\rangle = \alpha^{-1}(x) |\alpha\rangle,$$  \hspace{1cm} (5.2)

then we may rewrite (5.1) as

$$\langle \alpha | \hat{\mathcal{P}} | \beta \rangle = -\langle \hbar/2i \rangle \langle \alpha | (\hat{\mathcal{P}} \hat{\Phi})(\hat{\Phi}^{-1})^\dagger \cdots \hat{\Phi}^{-1} \hat{\Phi} \beta | \beta \rangle.$$  \hspace{1cm} (5.3)

Eq. (5.3) may serve as the definition of the velocity operator in terms of non-hermitian operators $\hat{\mathcal{A}}(x)$ and $\hat{\mathcal{A}}^\dagger(x)$

$$\hat{\mathcal{P}}(x) = \langle \hbar/2i \rangle [\hat{\mathcal{A}}(x) - \hat{\mathcal{A}}^\dagger(x)],$$  \hspace{1cm} (5.4)
where
\[ \hat{Z} = \hat{P}^{-1} \hat{V} \hat{P} = (1/2\beta)(\hat{V}^2 + 2\hat{J}). \]

The operators \( \hat{Z} \) are evidently not defined in the usual occupation-number representation, and the above formulas are meaningful only through their connection with the coherent-states approach.

It is interesting, however, to look more closely at some properties of the operators \( \hat{Z} \). It is possible to prove that the operators \( G(x, y) \) defined as
\[ G(x, y) = \langle \hat{P}(x) \hat{P}(y) \rangle, \]
may be expressed in terms of \( \hat{p}(x) \) and \( \hat{A}(x) \). Indeed, the operator \( G(x, y) \) has the form
\[ G(x, y) = \hat{A}(x) \exp \left( \int_{x \rightarrow y} \hat{A}(s) \mathrm{d}s \right), \]
where the integration is carried along any curve \( P(x, y) \) joining the points \( x \) and \( y \) (cf. ref. 14). (It seems that the algebra of the operators \( G(x, y) \) is a more convenient tool for the description of a many-body system than the HGA (\( \rho \) ref. 2).)

At the end we shall show how the Gross equations of the classical Bose field theory could be derived from the coherent-states representation. In order to do this, let us write down the hamiltonian for a boson system in the second-quantization form
\[
\mathcal{H} = (\hbar^2/2) \int \left( \mathbf{\hat{P}} \mathbf{\hat{P}}^\dagger + \mathbf{\hat{V}} \mathbf{\hat{V}}^\dagger \right) + \int \int d^3x d^3y U(x - y) \mathbf{\hat{P}}^\dagger(x) \mathbf{\hat{P}}^\dagger(y) \mathbf{\hat{P}}(y) \mathbf{\hat{P}}(x), \tag{5.7}
\]

Evaluation of the matrix element \( \langle \alpha | \mathcal{H} | \alpha \rangle \) is straightforward and yields
\[
\langle \alpha | \mathcal{H} | \alpha \rangle = \mathcal{H}(\alpha, \alpha) = (\hbar^2/2) \int d^3x d^3y \mathbf{\hat{V}}(x) \mathbf{\hat{V}}^\dagger(y) \mathbf{\hat{P}}^\dagger(x) \mathbf{\hat{P}}(y) \mathbf{\hat{P}}(y) \mathbf{\hat{P}}(x), \tag{5.8}
\]

or, using the polar decomposition of \( \mathbf{\hat{P}}(x) \)
\[
\mathcal{H}(\alpha, \alpha) = \mathcal{H}(\rho_{\alpha}, \rho_{\alpha}) = (\hbar^2/2) \int d^3x d^3y \mathbf{\hat{V}}(x) \mathbf{\hat{V}}^\dagger(y) \rho_{\alpha}(x) \rho_{\alpha}(y) + \int \int d^3x d^3y U(x - y) \rho_{\alpha}(x) \rho_{\alpha}(y), \tag{5.9}
\]

The hamiltonian (5.9) is the hamiltonian used in the Gross theory. As is known, it leads to the proper form of the excitation spectrum and is a very convenient starting point for the investigation of quantized vortices.

The equations of motion for the fields \( \rho_{\alpha}(x) \) and \( \rho_{\alpha}(y) \) follow from (5.9) by simple functional differentiation with respect to \( \rho_{\alpha} \) and \( \rho_{\alpha} \). We obtain then:
\[
\partial_t \rho_{\alpha} + \mathbf{\hat{V}} \cdot \mathbf{\hat{V}} \rho_{\alpha} = 0, \tag{5.10}
\]
\[
\partial_{t \alpha} \rho_{\alpha} + (\hbar^2/2) \mathbf{\hat{V}} \rho_{\alpha}^2 - \int U(x - y) \rho_{\alpha}(y) (d^3y - (\hbar^2/2) \rho_{\alpha} \mathbf{\hat{V}} \rho_{\alpha} \mathbf{\hat{V}}) \tag{5.11}
\]
Eqs. (5.10) and (5.11) are the continuity equation and Bernoulli equation for the many-boson system in the state $|a\rangle$. The last term on the right-hand side of (5.11) is the so-called quantum pressure term.

In the fluid picture for a classical many-body system we obtain, in the notation of eq. (2.5), equations of motion in the form similar to (5.10) and (5.11). The continuity equations remain unchanged and in the Bernoulli equation we shall drop the quantum pressure term.

Eqs. (5.10) and (5.11) are equivalent to Hartree-like equations derived from (5.8), provided the velocity field $V_\alpha = \hbar V_\alpha$ satisfies the quantization of circulation condition, \textit{i.e.},

$$\int V_\alpha \cdot dx = 2\pi \hbar n,$$

$n = 0, 1, 2, ...$

We shall end our considerations with some comments on the application of the above presented theory and a possible further development of the theory. In our opinion, the coherent-states approach is convenient for the following problems: the more exact investigation of the excitation spectrum for the superfluid system, the problem of two coupled superfluid systems. We plan to generalize our theory for the case of a many-charge boson system.

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