LETTER TO THE EDITOR

Magnetic solitons and elastic kink-like excitations in compressible Heisenberg chain

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Abstract. We show that the compressible Heisenberg chain remains, in the continuum limit, an example of a completely integrable system. The soliton excitations of the magnetic degrees of freedom lead to the kink-like excitations of the elastic degrees of freedom.

The dynamical properties of the continuum Heisenberg chain have been examined recently in view of the connection with modern field-theoretical concepts like solitons, complete integrability etc (Lakshmanan et al 1976, Lakshmanan 1977, Takhtajan 1977, Turski 1979). Possible extension of that kind of analysis to the quantum theory was discussed recently by Fogedby (1980) and the present authors (Cieplak and Turski 1980). In this Letter we would like to present results concerning properties of a one-dimensional compressible Heisenberg chain.

The compressible magnetic systems, i.e. those systems where the coupling between magnetic and elastic degrees of freedom is taken into account, have previously been studied in various contexts. Salinas (1973) gave extensive discussion of the compressible Ising model. Barma (1975) has discussed phonon-induced phase transitions in the classical Heisenberg chain.

We shall consider Heisenberg chain with free boundary conditions, i.e. we allow for changes in the length of the chain. Assuming only nearest-neighbour coupling between magnetic ions and using a harmonic approximation for the elastic degrees of freedom, we write the Hamiltonian for the system as

\[ H = -J \sum_i S_i \cdot S_{i+1} + \sum_i \left[ (u_i^2/2m) + (k/2)(u_{i+1} - u_i)^2 \right] \\
- \gamma \sum_i (u_{i+1} - u_i) S_i \cdot S_{i+1} \]

where \( u_i \) is the displacement of the magnetic ion from its equilibrium (without magnetic interactions) position, \( k \) is the spring constant and \( \gamma \) is the exchange striction coefficient (\( \gamma = (\partial J/\partial R) \)).

Following Barma, we perform the canonical transformation

\[ (\{p_i\}, \{u_i\}) \rightarrow (\{P_i\}, \{\xi_i\}) \]

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where the generator of the transformation \( F(\{\xi_i\}, \{p_i\}) \) is given as
\[
F(\{\xi_i\}, \{p_i\}) = -\sum_j p_j [\xi_j + (\gamma/k) \sum_{i \neq j} S_i \cdot S_{i+1}].
\]
(2)

Thus
\[
u_i = -\partial F/\partial p_i = \xi_i + (\gamma/k) \sum_{i \neq j} S_i \cdot S_{i+1}
\]
(3)

\[
p_i = -\partial F/\partial \xi_i = p_i.
\]
(4)

Using the above equations we obtain the new form of the Hamiltonian
\[
H' = -J \sum_i S_i \cdot S_{i+1} - (\gamma^2/2k) \sum_i (S_i \cdot S_{i+1})^2
\]
\[
+ \sum_i \left[ \left( p_i^2 / 2m \right) + (k/2) \left( \xi_{i+1} - \xi_i \right)^2 \right].
\]
(5)

Note, that in equation (5) the new degrees of freedom (\( \{\xi_i\}, \{p_i\} \)) are decoupled from the magnetic ones.

Using conventional Poisson brackets for classical spin vectors we obtain from equation (5) the following equations of motion for spin \( S_i \):
\[
\dot{\xi}_i = JS_i \times S_{i+1} \left[ 1 + (\gamma^2/Jk) S_i \cdot S_{i+1} \right]
\]
\[
+ JS_i \times S_{i-1} \left[ 1 + (\gamma^2/Jk) S_i \cdot S_{i-1} \right].
\]
(6)

We shall now look at the continuum limit of equation (6). That is, we expand \( S_{t+1} \equiv S(R_i + a) \) into the power series of \( a\partial/\partial R \equiv a\partial \), where \( a \) is the lattice spacing.

We obtain from equation (6)
\[
\dot{\xi}_i S(R_i, i) = a^2 J \left[ 1 + (\gamma^2 S^2/Jk) \right] S \times \partial^2 S.
\]
(7)

Equation (7) differs from the continuum limit of the rigid chain equation of motion in the value of the coefficient on its right-hand side:
\[
a^2 J \to a^2 J \left[ 1 + (\gamma^2 S^2/Jk) \right].
\]
(8)

If follows then that equation (7) has all the properties of the rigid chain model and therefore the continuous compressible Heisenberg chain is an example of a completely integrable system. Equation (7) has soliton solutions and finite-amplitude spin wave solutions.

The analysis of equation (7) is now possible via the Lakshmanan curvature–torsion variables. This leads to the explicit expressions for the magnetic soliton solutions of equation (7). We shall consider here only the special case of one soliton moving with velocity \( V \). For such a soliton the spin energy density \( e(R, t) = J a(\partial S)^2/2 \) is given by
\[
e(R, t) = J a^2 S \beta \tan \beta [\beta (R - R_0 - V t)/2]
\]
(9)

where the inverse width of the soliton \( \beta \) is related to its velocity \( V \) by the equation
\[
\beta = V/J a^2 S \left[ 1 + (\gamma^2 S^2/kJ) \right].
\]
(10)

Recalling now that \( Ja^2 S = \alpha \) is the magnon stiffness constant, and that the model contains two characteristic lengths besides the lattice spacing \( l_s = \gamma S^2/k \) is the character-
istic exchange striction length and $l = J/r^2$ is the length over which the exchange coupling changes appreciably), we can write equation (10) as

$$\beta^{-1} = (a/V) (1 + \delta) = \beta^{-1}_{r=0}(1 + \delta).$$

(11)

Thus the soliton width is larger in the compressible chain than in the rigid one.

We can now answer the question of what happens with the lattice deformation during the passage of the magnetic soliton through the lattice. From equation (3) we have

$$u_{i+1} - u_i = \xi_{i+1} - \xi_i + (\gamma/k) S_i S_{i+1}$$

(12)

and in the continuum limit

$$\epsilon(R, t) = \epsilon'(R, t) + (l_s/a) - 2l_s a \beta \partial \tanh[\beta(R - R_0 - Vt)/2]$$

(13)

where $\epsilon$ is the local lattice deformation.

From the Hamiltonian equation (5) we conclude that $\epsilon'(R, t)$ corresponds to local lattice deformation due to the harmonic oscillations, $(l_s/a)$ is the constant deformation due to the exchange striction and the last term in equation (13) describes magnetic soliton-induced deformation. Integrating equation (13), one easily sees that the magnetic soliton-induced lattice displacement field is of the form

$$\Delta u(R, t) = -2l_s a \beta \{1 + \tanh[\beta(R - R_0 - Vt)/2]\}$$

(14)

and therefore describes kink-like excitation of the lattice.

For $V > s_0$, where $s_0$ is the sound velocity equal to $a(k/m)^{1/2}$, this kink becomes a shock wave. The width of this shock exceeds the lattice spacing provided the characteristic magnon frequency for the compressible chain $SJ(1 + \delta)$ is larger than the lattice vibration frequency $(k/m)^{1/2}$. The relation between the magnetic solitons and elastic kinks in the compressible chain bears some similarity to the soliton–shock wave relation in the displacement form of the KdV equation (Kunin 1975).

In conclusion, we have shown that the compressible Heisenberg chain in the continuum limit supports magnetic solitons which differ from the rigid chain solitons only in the width–amplitude scaling. The magnetic soliton induces a kink-like displacement field for the lattice, which might become a shock wave.

It would be very interesting to see how much of this purely classical analysis remains true for the quantum case. Here one should start directly from the Hamiltonian (1) since the canonical transformation equation (2) is not unitary implementable for both quantum lattice and quantum spins. Our Schwinger bosons procedure (Cieplak and Turski 1980) can be used in that case, and subsequent analysis in terms of coherent states for both phonons and Schwinger bosons should lead to equations comparable to that used in this note.

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