The stability properties of the finite amplitude spin wave solutions of the equations of motion for continuous Heisenberg chains are examined and related to the Benjamin—Fair instability of the corresponding non-linear Schrödinger equation.

Recent experimental, theoretical and molecular dynamical work [1, 2] has indicated that there exists a wealth of interesting physical phenomena occurring in one-dimensional magnetic systems. At the same time it has been argued that the continuum limit of the one-dimensional ferromagnetic Heisenberg chain (CHC) model may serve as a convenient probing ground for various modern concepts like solitons, complete integrability etc.

Lakshmanan [3] and Takhtajan [4] have proven that the \( d = 1 \) CHC is an example of a completely integrable system. This means, as Lakshmanan has pointed out, that the solutions of the equations of motion for the CHC are equivalent to those of a certain non-linear Schrödinger equation. One obtains this equivalence by identification of the amplitude and the phase of the Schrödinger wave function with, respectively, the curvature and the line integral of the torsion of a space curve \( C \) onto which the CHC may be mapped via the use of the Frenet equation.

The non-linear Schrödinger equation obtained by Lakshmanan [3] belongs to the class of equations exhibiting the Benjamin—Fair instability [5]. That is, the exact time-dependent solutions of the form

\[
\psi_h(x, t) = a \exp \left[ -i (a^2 - K^2) t \right] \exp (-iKx),
\]

where \( a \) and \( K \) are constants, are unstable with respect to plane wave modulations with wave vectors \( q \) smaller than \( a \).

The Benjamin—Fair instability can be easily understood by following Gross's analysis of the Bogolubov spectrum for a weakly interacting Bose system. One notices that for purely attractive interactions the Bogolubov spectrum becomes unstable for wavelengths larger than the critical wavelength determined by the strength of the attractive interactions.

In this note we shall investigate the stability properties of a certain class of exact solutions for the CHC model, namely the finite amplitude spin waves, and we shall prove that these waves are unstable. This instability when investigated by means of the Lakshmanan equations is equivalent to the Benjamin—Fair instability. This, we feel, is an interesting result in view of the close relation between the Benjamin—Fair instability and the Fermi—Pasta—Ulam recurrence. Such a relation was recently noticed in a numerical experiment by Yuen and Ferguson [7].

The equations of motion for the CHC can be written in the form

\[
\frac{\partial}{\partial t} \mathbf{t}(x, t) = \mathbf{t}(x, t) \times \frac{\partial^2}{\partial x^2} \mathbf{t}(x, t),
\]

where we have normalized the spin vectors so that \( \mathbf{t} \cdot \mathbf{t} = 1 \), and all constants like exchange integral, lattice
spacing, spin value have been incorporated into the unit of time. The normalization condition allows an interpretation of the spin vector \( t \) as a unit vector tangent to a space curve \( C \) with \( x \) playing the role of the arc length. Using the Frenet equations, familiar from geometry, one rewrites eq. (1) in the form

\[
\partial_t t(x,t) = -\kappa \tau n(x,t) + b(x,t) \partial_x \kappa.
\]

(2)

Here \( \kappa \) and \( \tau \) are the curvature and torsion of the curve \( C \) and \( n, b \) are normal and binormal vectors, respectively. Eq. (2) together with the Frenet equations form the basis of our analysis.

Finite amplitude spin waves are exact solutions of eq. (2) which depend on space and time via \( \xi = x - vt \) only. It follows from eq. (2) that the curvature for such a curve \( C \) is constant and the torsion is equal to the constant frame velocity \( v \). It follows also that the Darboux vector \( D = \tau t + \kappa b \) is a constant of the motion and its length \( D \) is equal to \((v/\cos \theta)\), where \( \theta \) is the angle between the vector \( D \) and the spin direction. An explicit solution of eq. (2) reads

\[
t(t) = e_1(v/D) + (1 - v^2/D^2)^{1/2} [e_2 \cos (D\xi) + e_3 \sin (D\xi)],
\]

where the \( \{e_i\} \) form a right-handed triad with \( e_1 \) in the direction of \( D \). Returning to the coordinates \( x, t \) one sees that solution (3) represents a spin wave with wave vector \( q \) equal to \( D \) and frequency \( \omega = q^2 \cos \theta \). Notice that solution (3) does not belong to the class of solutions admitted in the work of Takhtajan [4].

The geometrical interpretation of solution (3) is that of uniform rotation of the Frenet triad around the fixed direction of \( D \).

The stability analysis of the spin wave solutions can be performed easily by means of the Lakshmanan equations [3] for \( \kappa \) and \( \tau \):

\[
\partial_t (\kappa^2/2) = - \partial_x (\kappa^2 \tau),
\]

(4)

\[
\partial_t \tau = \partial_x (-\tau^2 + \kappa^2/2) + \partial_x [(\partial_x^2 \kappa)/\kappa].
\]

(5)

It is instructive to notice that these equations are just the quantum hydrodynamical equations used by Gross [6]. Eq. (4) is the continuity equation and eq. (5) is the Euler equation with the last term in its r.h.s. playing the role of the quantum pressure term.

We look first at the stationary solutions of eqs. (4), (5). One sees that the current \( J = \kappa^2 \tau \) is now constant and the curvature \( \kappa \) obeys second-order differential equations of the form

\[
\partial^2 \kappa/\partial x^2 = - \partial V(\kappa)/\partial \kappa,
\]

(6)

where the potential \( V(\kappa) \) is equal to

\[
V(\kappa) = -\frac{1}{2}c\kappa^2 + \frac{1}{8}\kappa^4 + J^2/2\kappa^2.
\]

(7)

Here \( c \) is a constant of integration from eq. (5).

All the solutions of eq. (6) can be interpreted as those of the equation of motion for a soft Duffing oscillator in a "centrifugal" force field. The spin wave solutions are the spatially homogeneous solutions of eq. (6). The constant \( c \) fixes the value of the angle \( \theta \) and for example \( c = 0 \) corresponds to \( \theta \approx 63.4^\circ \).

Having found all the spin wave solutions we may assess their stability with respect to plane wave modulations of the form \( \kappa = \kappa_0 + \delta \kappa, \tau = \tau_0 + \delta \tau \), where \( \delta \kappa, \delta \tau \) are proportional to \( \exp (i\omega t - iqx) \). Linearizing eqs. (4,5) we obtain the following dispersion relation:

\[
(\omega - 2q\tau_0)^2 = q^2(q^2 - \kappa_0^2),
\]

(8)

which clearly exhibits instability for \( 0 < q < \kappa_0 \). The most unstable mode is that with wave vector \( \kappa_0/\sqrt{2} \).

By comparing the above result with the dispersion relation following from the non-linear Schrödinger equation one concludes that eq. (8) describes the Benjamin–Fair instability. We claim therefore that the Benjamin–Fair instability of the spatially periodic solutions of the non-linear Schrödinger equation appears in the dynamics of the CHC as the instability of the finite amplitude spin wave solutions.

The stability properties of other, spatially inhomogeneous solutions of eq. (6) can be investigated by a method originally invented for Benard-like problems [8]. Similarly one can investigate the stability of the running wave solutions of eqs. (4,5) which in the original spin variables represent soliton-like solutions.

We want to close with the following comments.

The finite amplitude spin wave (3) is a solution of eq. (1) with very special initial conditions. Such conditions have low probability of occurrence in \( d = 1 \) systems in the absence of an external magnetic field. Even if these conditions are met, the spin wave will exhibit the Benjamin–Fair instability. Both these facts seem to indicate that the finite amplitude spin waves play a rather minor role in the dynamics of real systems. However, in a recent numerical experiment on the non-linear Schrödinger equation, Yuen and
Ferguson [7] have shown that the spectrum of unstable modulations with wave vectors \( \frac{1}{2} \kappa_0 < q < \kappa_0 \) leads to the Fermi—Pasta—Ulam recurrence. It will be very interesting to see, on the computer, how such a recurrence will look in the original spin variables. Finally one should mention that the short-wavelength propagating spin density modes which are known to exist in one-dimensional magnets are “small” amplitude excitations essentially different from those investigated in this letter.

I have benefitted from discussions held about the CHC, over the last three years, with Professors Alf Sjölander, George Reiter, Bill Kerr and Dr. Marek Cieplak. I thank Bill and George for bringing to my attention the work of Lakshmanan and Takhtajan.

References