

## GAUGE-INDEPENDENT CANONICAL FORMULATION OF RELATIVISTIC PLASMA THEORY

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Following an earlier work we derive a gauge-independent canonical structure for a fully relativistic multicomponent plasma theory. The Klimontovich form of the distribution function is used to derive the basic Poisson bracket relations for the canonical variables  $\hat{f}_a$ ,  $\mathbf{B}$ , and  $\mathbf{E}$ . The Poisson bracket relations provide an explicit canonical realization of the Lie algebra of the Poincaré group and they lead to the correct transformation properties for the canonical variables. We stress the importance of a canonical realization of the full symmetry group of the evolution equations. The covariance of the theory under the symmetry group can be used as a criterion to discriminate among different canonical structures for the evolution equations.

### 1. Introduction

The purpose of this paper is to present a straightforward derivation, based on particle dynamics, of a canonical structure for the equations that describe the interaction of a relativistic multicomponent plasma with the electromagnetic field.

Since the concept of a canonical or Hamiltonian structure for a system of evolution equations has undergone considerable changes over the past decade, we shall begin with a few remarks on our usage of the term *canonical formulation*. Some authors reserve this term exclusively for those situations in which, by a proper choice of variables, the evolution of the system can be described in terms of the standard Poisson bracket relations (Kronecker deltas for the discrete case or Dirac delta-functions for the continuous case). We feel

that this requirement is too restrictive because it often necessitates the introduction of various types of potentials–variables that are not in one-to-one correspondence with the states of the system. Although the introduction of potentials is difficult to avoid in quantum theories (they play a vital role in finding explicit realizations of operator algebras), in classical theories one can easily do without them. The elimination of potentials in classical theories merely requires a natural extension of the concept of canonical formulation.

Abstracting from our previous experience<sup>1,2)</sup> with gauge-independent canonical formulations, we assert that usage of the term *canonical* to describe the structure of a system of evolution equations is fully justified for a theory in which the following elements are present:

i) The state of the system at each instant of time is fully described by a set of fundamental dynamical variables – canonical variables. The rate of change (time derivative) of each canonical variable at a given instant of time is determined entirely by the state of the system at that time. Thus, the time evolution equations for the canonical variables are first-order differential equations with respect to time.

ii) Poisson bracket relations, which satisfy all the standard properties of linearity, antisymmetry, product rule, and Jacobi identity, are defined on the space of functions of the canonical variables.

iii) The group of canonical transformations (transformations of the canonical variables that leave the Poisson bracket relations invariant) contains as a subgroup all the symmetry transformations of the theory, that is, all transformations that leave the evolution equations unchanged. The generators of time translations, space translations, and space rotations are identified respectively with the energy, momentum, and angular momentum of the system. In particular, the time evolution equations can be expressed as Poisson bracket relations

$$\partial_t \chi^i = \{\chi^i, H\} \quad (1)$$

between a set of fundamental dynamical variables  $\chi^i$  and the Hamiltonian  $H$  (energy) of the system.

A canonical formulation, in the above sense, for the relativistic dynamics of charged particles and for the relativistic dynamics of charged fluids has been given by Bialynicki-Birula and Iwinski<sup>2)</sup>. The other branch of physics where the canonical formulation of the relativistic dynamics is clearly needed is the statistical plasma theory. Balescu and Poulain<sup>3)</sup> have used the formulation<sup>2)</sup> to develop a Liouvillean description of relativistic plasma physics. This is however not the only way one may proceed. Indeed, in several problems of non-relativistic plasma physics as well as in the theory of weakly relativistic

(Breit–Darwin) plasma the formulation based upon the use of the Klimontovich exact one-particle distribution functions happens to be more useful<sup>4,5</sup>). We believe this is also the case in the fully relativistic theory.

In the present paper, following earlier work of Iwinski and Turski<sup>6</sup>) we derive a gauge-independent canonical formulation of relativistic multicomponent plasma theory. We show that all three requirements (i)–(iii) can be met and that the full Poincaré group can be realized as a subgroup of the group of canonical transformations. The Poincaré invariance of the canonical structure derived for the relativistic plasma theory puts the theory on an equal footing with other relativistic field theories. Our method can also be applied to a plasma theory based on nonrelativistic particle dynamics; however, the group of symmetry transformations is smaller in this case because it does not contain Lorentz transformations (Galilean transformations do not leave the Maxwell equations invariant), and this makes the theory less appealing.

As canonical variables (fundamental dynamical variables) describing the state of a system composed of a multicomponent plasma and the electromagnetic field, we choose the magnetic field vector  $\mathbf{B}(\mathbf{r}, t)$ , the electric field vector  $\mathbf{E}(\mathbf{r}, t)$ , and a set of distribution functions  $\hat{f}_a(\mathbf{r}, \mathbf{p}, t)$ , one for each component of the plasma. The electromagnetic potentials will not be used in our formulation; our approach is manifestly gauge-independent. Using the Klimontovich<sup>4</sup>) formulation of relativistic plasma theory, we derive the basic Poisson bracket relations<sup>6</sup>) for the fundamental dynamical variables from the canonical formulation of relativistic charged-particle dynamics given in ref. 2.

Morrison<sup>7</sup>) has derived a canonical structure for the nonrelativistic Vlasov–Maxwell equations. Recently, Marsden and Weinstein<sup>8</sup>), using infinite-dimensional symplectic manifold theory, derived a different canonical structure for the nonrelativistic equations. In the nonrelativistic case our canonical structure is equivalent to the canonical structure derived by Marsden and Weinstein and by Morrison and Weinstein<sup>8</sup>). We show that the canonical structure proposed by Morrison is inconsistent with the symmetries that are inherent in the evolution equations.

## 2. Derivation of the Poisson bracket relations

As was shown by Born and Infeld<sup>9</sup>) in 1935, Poisson bracket relations for the electromagnetic field can be defined in the form

$$\{B_i(\mathbf{r}), D_j(\mathbf{r}')\} = \varepsilon_{ijk} \partial_k \delta(\mathbf{r} - \mathbf{r}'), \quad (2)$$

even for the general case in which the constitutive equations relating the  $\mathbf{E}$  and  $\mathbf{B}$  vectors with the  $\mathbf{D}$  and  $\mathbf{H}$  vectors are nonlinear. In the linear theory we can

identify  $\mathbf{D}$  with  $\mathbf{E}$ , which we shall do below. As all Poisson bracket relations used in this paper are defined at equal times, we shall not explicitly indicate the time arguments.

In order to derive the Poisson bracket relations involving the distribution functions  $f_\alpha$ , we use the representation introduced by Klimontovich<sup>4</sup>). This representation expresses each distribution function

$$\hat{f}_\alpha(\mathbf{r}, \mathbf{p}, t) = \sum_{A \in S_\alpha} \delta(\mathbf{r} - \boldsymbol{\xi}_A(t)) \delta(\mathbf{p} - \boldsymbol{\pi}_A(t)) \quad (3)$$

as a sum of contributions from isolated point particles. In eq. (3),  $\boldsymbol{\xi}_A$  and  $\boldsymbol{\pi}_A$  are the position and *kinetic momentum* vectors of the  $A$ th particle and  $S_\alpha$  denotes the set of particles of type  $\alpha$ . In our formulation we do not use the canonical momenta of the charged particles. The canonical momenta are dependent on a choice of gauge for the electromagnetic potentials; therefore they are not uniquely determined by the state of the system. On the other hand, the kinetic momenta—which are functions solely of the particle velocities—are manifestly gauge-independent variables. The argument  $\mathbf{p}$  of the distribution function  $f_\alpha$  is therefore to be interpreted as

$$\mathbf{p} = \frac{m_\alpha \mathbf{v}}{\sqrt{1 - v^2/c^2}}, \quad (4)$$

where  $m_\alpha$  is the rest mass of a particle of type  $\alpha$ .

The description of a many body system by means of the Klimontovich function bears some similarity to the second quantization, with  $\hat{f}_\alpha$  playing the role of the field operator. On the other hand  $\hat{f}_\alpha$ 's are closely related to the quantum Wigner distribution functions. In classical, nonrelativistic statistical mechanics it is the  $\hat{f}$ -function method which is particularly suited for developing the fully renormalized kinetic theory<sup>10</sup>).

The relativistic dynamics of charged particles interacting with the electromagnetic field is described by the Maxwell–Lorentz equations:

$$\frac{d\boldsymbol{\xi}_A(t)}{dt} = \mathbf{v}_A(t), \quad (5a)$$

$$\frac{d\boldsymbol{\pi}_A(t)}{dt} = e_A(\mathbf{E}(\boldsymbol{\xi}_A(t), t) + \mathbf{v}_A(t) \times \mathbf{B}(\boldsymbol{\xi}_A(t), t)), \quad (5b)$$

$$\partial_t \mathbf{B}(\mathbf{r}, t) = -\nabla \times \mathbf{E}(\mathbf{r}, t), \quad (5c)$$

$$\partial_t \mathbf{E}(\mathbf{r}, t) = \nabla \times \mathbf{B}(\mathbf{r}, t) - \sum_A e_A \mathbf{v}_A(t) \delta(\mathbf{r} - \boldsymbol{\xi}_A(t)), \quad (5d)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \sum_A e_A \delta(\mathbf{r} - \boldsymbol{\xi}_A(t)), \quad (5e)$$

where we have used a rationalized system of units and have set  $c = 1$ . Eqs. (5e), which do not involve the time derivatives, are treated as constraints on the initial data. These constraints are preserved by the time evolution. Eqs. (5a)–(5e) have only a formal meaning because the fields are singular at the positions of the particles, but they are used by us just as a heuristic tool.

The Maxwell–Lorentz equations (see ref. 2) can be expressed in the canonical form (1) by using the following set of Poisson bracket relations for the canonical variables  $\xi_A$ ,  $\pi_A$ ,  $\mathbf{B}$  and  $\mathbf{E}$ :

$$\{\xi_A^i, \pi_B^j\} = \delta_{AB} \delta_{ij}, \quad (6a)$$

$$\{\pi_A^i, \pi_B^j\} = \delta_{AB} e_A \varepsilon_{ijk} B^k(\xi_A), \quad (6b)$$

$$\{\pi_A^i, E^j(\mathbf{r})\} = e_A \delta_{ij} \delta(\mathbf{r} - \xi_A), \quad (6c)$$

$$\{B^i(\mathbf{r}), E^j(\mathbf{r}')\} = \varepsilon_{ijk} \partial_k \delta(\mathbf{r} - \mathbf{r}'). \quad (6d)$$

All the remaining Poisson brackets vanish. These Poisson brackets are consistent with the constraints (5e).

With the above choice of Poisson bracket relations for the canonical variables, the full Poincaré group<sup>2)</sup> is realized as a subgroup of the group of canonical transformations. To show that this is also true for the relativistic plasma theory, we need to express the generators of the Poincaré group in terms of the distribution functions  $\hat{f}_\alpha$ . All the generators can be expressed as appropriate space integrals of the components of the energy–momentum tensor  $T^{\mu\nu}$ . For a system composed of charged particles and the electromagnetic field, the components of the symmetric energy–momentum tensor are

$$T^{00} = \sum_A \sqrt{m_A^2 + \pi_A^2} \delta(\mathbf{r} - \xi_A) + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad (7a)$$

$$T^{0i} = \sum_A \pi_A^i \delta(\mathbf{r} - \xi_A) + (\mathbf{E} \times \mathbf{B})^i, \quad (7b)$$

$$T^{ij} = \sum_A \pi_A^i v_B^j \delta(\mathbf{r} - \xi_A) - E^i E^j - B^i B^j + \frac{1}{2} \delta_{ij} (\mathbf{E}^2 + \mathbf{B}^2). \quad (7c)$$

From eqs. (5a)–(5e) one can formally obtain the continuity relation

$$\partial_\mu T^{\mu\nu} = 0 \quad (8)$$

for the energy–momentum tensor. Using the distribution functions  $\hat{f}_\alpha$ , we can express the energy–momentum tensor in the form

$$T^{00}(\mathbf{r}, t) = \sum_\alpha \int d^3p E^\alpha(p) \hat{f}_\alpha(\mathbf{r}, \mathbf{p}, t) + T_{\text{em}}^{00}(\mathbf{r}, t), \quad (9a)$$

$$T^{0i}(\mathbf{r}, t) = \sum_\alpha \int d^3p p^i \hat{f}_\alpha(\mathbf{r}, \mathbf{p}, t) + T_{\text{em}}^{0i}(\mathbf{r}, t), \quad (9b)$$

$$T^{ij}(\mathbf{r}, t) = \sum_{\alpha} \int \frac{d^3p}{E^{\alpha}(p)} p^i p^j \hat{f}_{\alpha}(\mathbf{r}, \mathbf{p}, t) + T_{\text{em}}^{ij}(\mathbf{r}, t), \quad (9c)$$

where  $E^{\alpha}(p) = \sqrt{m_{\alpha}^2 + p^2}$  and  $T_{\text{em}}^{\mu\nu}$  is the energy-momentum tensor of the electromagnetic field. The continuity equation for the energy-momentum tensor can also be derived directly from the Klimontovich-Maxwell equations,

$$[\partial_t + \mathbf{v}_{\alpha} \cdot \nabla + e_{\alpha}(\mathbf{E}(\mathbf{r}, t) + \mathbf{v}_{\alpha} \times \mathbf{B}(\mathbf{r}, t)) \cdot \partial] \hat{f}_{\alpha}(\mathbf{r}, \mathbf{p}, t) = 0, \quad (10a)$$

$$\partial_t \mathbf{B}(\mathbf{r}, t) = -\nabla \times \mathbf{E}(\mathbf{r}, t), \quad (10b)$$

$$\partial_t \mathbf{E}(\mathbf{r}, t) = \nabla \times \mathbf{B}(\mathbf{r}, t) - \sum_{\alpha} e_{\alpha} \int d^3p \mathbf{v}_{\alpha} \hat{f}_{\alpha}(\mathbf{r}, \mathbf{p}, t), \quad (10c)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad \nabla \cdot \mathbf{E}(\mathbf{r}, t) = \sum_{\alpha} e_{\alpha} \int d^3p \hat{f}_{\alpha}(\mathbf{r}, \mathbf{p}, t), \quad (10d)$$

where  $\mathbf{v}_{\alpha} = \mathbf{p} \sqrt{m_{\alpha}^2 + p^2}$  and  $\nabla = \partial/\partial \mathbf{r}$ ,  $\partial = \partial/\partial \mathbf{p}$ . On the right-hand sides of eqs. (10c,d) one recognizes the particle density  $n_{\alpha}(\mathbf{r}, t)$  and particle current  $\mathbf{j}_{\alpha}(\mathbf{r}, t)$  expressed in terms of the Klimontovich distribution  $\hat{f}_{\alpha}$ , that is  $n_{\alpha} = \int d^3p \hat{f}_{\alpha}$  and  $\mathbf{j}_{\alpha} = \int d^3p \mathbf{v}_{\alpha} \hat{f}_{\alpha}$ . Since  $\hat{f}_{\alpha}$  transforms as a scalar under the Lorentz transformations the fields  $n_{\alpha}$  and  $\mathbf{j}_{\alpha}$  form a four-vector  $j_{\alpha}^{\mu}$  which is conserved by virtue of eqs. (10).

We now turn to the evaluation of the Poisson bracket relations in relativistic plasma theory. Treating the Klimontovich distribution functions (3) as functions of the dynamical variables  $\xi_A$  and  $\pi_A$  (with parametric dependence on  $\mathbf{r}$  and  $\mathbf{p}$ ), we can calculate the Poisson bracket relations with the help of the following general rule:

$$\{F, G\} = \sum_{i,j} \frac{\partial F}{\partial \chi^i} \{\chi^i, \chi^j\} \frac{\partial G}{\partial \chi^j}, \quad (11)$$

where  $F$  and  $G$  are arbitrary functions of the canonical variables  $\chi^i$ . For continuous systems, partial derivatives are replaced by functional derivatives and sums are replaced by integrals. From (3), (6), and (11) we obtain the basic Poisson bracket relations for the canonical variables  $\hat{f}_{\alpha}$ ,  $\mathbf{B}$ , and  $\mathbf{E}$ :

$$\{\hat{f}_{\alpha}(x), \hat{f}_{\beta}(x')\} = \delta_{\alpha\beta} [(\hat{f}_{\alpha}(\mathbf{r}, \mathbf{p}') - \hat{f}_{\alpha}(\mathbf{r}', \mathbf{p})) \nabla \cdot \partial + e_{\alpha} \mathbf{B}(\mathbf{r}) \cdot (\partial \hat{f}_{\alpha}(x) \times \partial)] \delta(x - x'), \quad (12a)$$

$$\{\hat{f}_{\alpha}(x), \mathbf{E}(\mathbf{r}')\} = -e_{\alpha} \partial \hat{f}_{\alpha}(x) \delta(\mathbf{r} - \mathbf{r}'), \quad (12b)$$

$$\{\hat{f}_{\alpha}(x), \mathbf{B}(\mathbf{r}')\} = 0. \quad (12c)$$

Eq. (12a) can also be written in an equivalent form,

$$\{\hat{f}_{\alpha}(x), \hat{f}_{\beta}(x')\} = \delta_{\alpha\beta} [\nabla \hat{f}_{\alpha}(x') \cdot \partial \delta(x - x') - \partial \hat{f}_{\alpha}(x) \cdot \nabla \delta(x - x') + e_{\alpha} \mathbf{B}(\mathbf{r}) \cdot (\partial \hat{f}_{\alpha}(x) \times \partial \delta(x - x'))]. \quad (12a')$$

Here  $x = (\mathbf{r}, \mathbf{p})$  denotes a point in the 6-dimensional parameter space of positions and kinetic momenta. The Poisson bracket relations between  $\mathbf{E}$  and  $\mathbf{B}$  retain the same form (6d). Using (12), (6), and (11), we get the following general form of the Poisson bracket for arbitrary functionals of  $\hat{f}_\alpha$ ,  $\mathbf{B}$ , and  $\mathbf{E}$ :

$$\begin{aligned} \{F, G\} = & \sum_\alpha \int d^6x f_\alpha \left[ \nabla \frac{\delta F}{\delta \hat{f}_\alpha} \cdot \partial \frac{\delta G}{\delta \hat{f}_\alpha} - \nabla \frac{\delta G}{\delta \hat{f}_\alpha} \cdot \partial \frac{\delta F}{\delta \hat{f}_\alpha} + e_\alpha \mathbf{B} \cdot \left( \partial \frac{\delta F}{\delta \hat{f}_\alpha} \times \partial \frac{\delta G}{\delta \hat{f}_\alpha} \right) \right. \\ & \left. + e_\alpha \left( \partial \frac{\delta F}{\delta \hat{f}_\alpha} \cdot \frac{\delta G}{\delta \mathbf{E}} - \partial \frac{\delta G}{\delta \hat{f}_\alpha} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) \right] \\ & + \int d^3r \left[ \frac{\delta F}{\delta \mathbf{E}} \cdot \left( \nabla \times \frac{\delta G}{\delta \mathbf{B}} \right) - \frac{\delta G}{\delta \mathbf{E}} \cdot \left( \nabla \times \frac{\delta F}{\delta \mathbf{B}} \right) \right]. \end{aligned} \quad (13)$$

These Poisson bracket relations are equivalent to those derived by Marsden and Weinstein<sup>8</sup>).

### 3. Hamiltonian and other generators of the Poincaré group

Now, we disregard the heuristic origin of formula (13) and check by direct calculations that it constitutes a consistent basis for a canonical formulation of relativistic plasma theory when the plasma is treated as a continuous phase space fluid (Vlasov approximation). The singular distribution functions (3) are replaced by smooth distribution functions, and the evolution equations (5a)–(5e) are replaced by the Vlasov–Maxwell equations, which have the same form as the Klimontovich–Maxwell equations. We shall denote the smooth distribution functions by  $f_\alpha$ .

In the complete statistical mechanics approach one should be able to derive the smooth Vlasov equation from the exact set of eqs. (10). In the non-relativistic limit that can be done either at the level of the B.B.G.K.Y. hierarchy or, more exactly, by showing that the Vlasov equation becomes an exact dynamical equation in the limit  $e^2 \rightarrow 0$ ,  $N \rightarrow \infty$  with  $e^2 N = \text{const.}$ <sup>11</sup>). Unfortunately, relativistic statistical mechanics is not yet developed to the extent that would allow for a repetition of such an analysis.

Since the Poisson brackets are defined for a fixed time, the Poincaré invariance of the canonical formulation is not explicit. We can, however, demonstrate by an explicit calculation that the Poincaré group is realized as a subgroup of the group of canonical transformations defined with respect to the Poisson bracket (13). Following the approach of refs. 2 and 12, we first prove that the Dirac–Schwinger<sup>13</sup>) conditions

$$\{T^{00}(\mathbf{r}), T^{00}(\mathbf{r}')\} = -(T^{0k}(\mathbf{r}) + T^{0k}(\mathbf{r}'))\partial_k\delta(\mathbf{r} - \mathbf{r}'), \quad (14a)$$

$$\{T^{00}(\mathbf{r}), T^{0k}(\mathbf{r}')\} = -(T^{ki}(\mathbf{r}) + T^{00}(\mathbf{r}')\delta_{ki})\partial_i\delta(\mathbf{r} - \mathbf{r}'), \quad (14b)$$

$$\{T^{0k}(\mathbf{r}), T^{0l}(\mathbf{r}')\} = -(T^{0l}(\mathbf{r})\partial_k + T^{0k}(\mathbf{r}')\partial_l)\delta(\mathbf{r} - \mathbf{r}') \quad (14c)$$

hold for the components of the energy–momentum tensor (9). By an explicit but lengthy calculation using the Poisson bracket relations (13), we verified that the Dirac–Schwinger conditions are satisfied for the relativistic plasma theory. These results remain true<sup>12)</sup> even if the linear theory of the electromagnetic field is replaced by a general nonlinear theory, for example, of the Born–Infeld type.

The generators of the Poincaré transformations expressed in terms of the distribution functions  $f_\alpha$  have the form

$$H = \sum_\alpha \int d^6q E^\alpha(p) f_\alpha(q) + \frac{1}{2} \int d^3r (\mathbf{E}^2 + \mathbf{B}^2), \quad (15a)$$

$$\mathbf{P} = \sum_\alpha \int d^6q \mathbf{p} f_\alpha(q) + \int d^3r (\mathbf{E} \times \mathbf{B}), \quad (15b)$$

$$\mathbf{M} = \sum_\alpha \int d^6q (\mathbf{r} \times \mathbf{p}) f_\alpha(q) + \int d^3r \mathbf{r} \times (\mathbf{E} \times \mathbf{B}), \quad (15c)$$

$$\mathbf{N} = \sum_\alpha \int d^6q \mathbf{r} E^\alpha(p) f_\alpha(q) + \frac{1}{2} \int d^3r \mathbf{r} (\mathbf{E}^2 + \mathbf{B}^2) - t\mathbf{P}. \quad (15d)$$

As a result of (14), these generators form a realization of the Lie algebra of the Poincaré group.

The generators (15) of the Poincaré group are the generators that one would normally construct for a noninteracting system. The Hamiltonian (15a) does not contain the electric charges  $e_\alpha$ , the coupling constants of the interaction. In our approach the interaction between the plasma and the electromagnetic field is introduced entirely through the Poisson bracket (13), which explicitly contains the electric charges  $e_\alpha$  and the magnetic field vector  $\mathbf{B}$ . The canonical formulation of relativistic plasma theory developed in this paper may be viewed as a realization of the Souriau–Sternberg<sup>14)</sup> approach (although these authors consider only motion in external fields) of introducing the interaction as a modification of the Poisson bracket relations (symplectic structure). We emphasize that we treat as a canonical theory the full interacting system, not just the motion of the charged particles in an external electromagnetic field.

The second step of our proof of relativistic invariance is to show that the generators of the Poincaré group act on the canonical variables  $f_\alpha$ ,  $\mathbf{B}$ , and  $\mathbf{E}$  in a manner that is consistent with their known transformation properties. These transformation properties are determined by the physical interpretation of the canonical variables.

Using the Hamiltonian  $H$  as the generator of time translations, we find that the evolution equations can be expressed in the canonical form

$$\partial_t f_\alpha = \{f_\alpha, H\}, \quad \partial_t \mathbf{B} = \{\mathbf{B}, H\}, \quad \partial_t \mathbf{E} = \{\mathbf{E}, H\}, \quad (16)$$

because

$$\{f_\alpha, H\} = -[\mathbf{v}_\alpha \cdot \nabla + e_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) \cdot \partial] f_\alpha, \quad (17a)$$

$$\{\mathbf{B}, H\} = -\nabla \times \mathbf{E}, \quad (17b)$$

$$\{\mathbf{E}, H\} = \nabla \times \mathbf{B} - \sum_\alpha e_\alpha \int d^3p \mathbf{v}_\alpha f_\alpha. \quad (17c)$$

The components of the momentum vector  $\mathbf{P}$  generate the following infinitesimal transformations:

$$\{f_\alpha, P^k\} = -\partial_k f_\alpha, \quad (18a)$$

$$\{B^i, P^k\} = \delta_{ik} (\nabla \cdot \mathbf{B}) - \partial_k B^i, \quad (18b)$$

$$\{E^i, P^k\} = \delta_{ik} \left( \nabla \cdot \mathbf{E} - \sum_\alpha e_\alpha \int d^3p f_\alpha \right) - \partial_k E^i. \quad (18c)$$

Imposing the constraints, we find that the momentum vector  $\mathbf{P}$  is the generator of space translations. One can also check that our Poisson bracket relations (13) lead to the correct changes of  $f_\alpha$ ,  $\mathbf{B}$ , and  $\mathbf{E}$  under infinitesimal rotations when the components of the angular momentum vector  $\mathbf{M}$  are used as generators. The Poisson bracket relations proposed by Morrison<sup>7)</sup> for the nonrelativistic Vlasov–Maxwell equations do not lead to the correct changes in the canonical variables  $f_\alpha$ ,  $\mathbf{B}$ , and  $\mathbf{E}$  under infinitesimal transformations generated by the momentum vector  $\mathbf{P}$  or by the angular momentum vector  $\mathbf{M}$ .

Finally, we determine the behavior of  $f_\alpha$ ,  $\mathbf{B}$ , and  $\mathbf{E}$  under infinitesimal Lorentz transformations generated by the first moment of energy  $\mathbf{N}$ . Invariance of the theory under Lorentz transformations requires that the distribution functions  $f_\alpha$  transform as Lorentz scalars and that  $\mathbf{B}$  and  $\mathbf{E}$  transform as the components of an antisymmetric tensor of rank two. From (13) and (15d) we obtain

$$\{f_\alpha, N\} = (\mathbf{r} \partial_t + t \nabla + E^\alpha(p) \partial) f_\alpha, \quad (19a)$$

$$\{B^i, N^k\} = (x^k \partial_t + t \partial_k) B^i - \varepsilon_{ijk} E^j, \quad (19b)$$

$$\{E^i, N^k\} = (X^k \partial_t + t \partial_k) E^i + \varepsilon_{ijk} B^j. \quad (19c)$$

These are the correct changes in the canonical variables under infinitesimal Lorentz transformations.

This concludes our proof that the equal-time Poisson bracket relations (13)

used to define the canonical structure of relativistic plasma theory are covariant under the action of the Poincaré group.

#### 4. Conclusions

The canonical theory of a multicomponent plasma and the electromagnetic field developed in this paper is a Poincaré invariant theory of interacting particles and fields. It can be used either to describe a given history of the system (given initial data) or as the basis of a statistical description by introducing a probability distribution on the space of the canonical variables at a given instant of time (space of initial data).

However, one should be aware of that such a procedure would require the knowledge of equilibrium correlation functions for both fields electromagnetic ( $\mathbf{E}, \mathbf{B}$ ) and particle ( $f_\alpha$ ) ones. So far no equilibrium relativistic statistical theory exists which would be able to provide that input.

The formal symplectic structure of the Klimontovich–Maxwell equations may also serve as a foundation for the derivation of relativistic kinetic equations, but then one would have to cope with the field singularities at the positions of the particles. In our approach we have avoided these difficulties by dealing only (apart from initial heuristic steps) with smooth distribution functions.

The formulation of the Vlasov dynamics we propose is completely free of unphysical problems occurring in the theories based on gauge-noninvariant formulation. There is no need, for example, in taking a “quantum mechanical de tour” as in<sup>15</sup>) in order to avoid confusion with two possible forms of the Vlasov equation.

The relativistic field theory that we have considered is different from the standard field theories studied in elementary particle physics because one of the fields  $f_\alpha(\mathbf{r}, \mathbf{p}, t)$  is defined on the phase space. The canonical formulation described in this paper may perhaps be a convenient starting point for the quantization of such unusual field theories.

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The idea of using only gauge-independent variables in a canonical formulation of electrodynamics can be traced back to W. Pauli. In the first edition of his classic *Handbuch der Physik* article on quantum mechanics, Pauli defined the quantum commutator version of the basic Poisson bracket relations (6) that we use in our canonical formulation of relativistic plasma theory. An English translation of this article, including the chapter on Quantum Electrodynamics from the original 1933 edition, is now available from Springer-Verlag: *General Principles of Quantum Mechanics* (Springer, New York, 1981).
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