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Examples of the Dirac approach to dynamics of systems with constraints

Sonnet Nguyen, Łukasz A. Turski*

Center for Theoretical Physics, College of Science, Polish Academy of Sciences. Al. Lotników 32/46, 02-668 Warszawa, Poland

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Abstract

The Dirac brackets approach to description of the dynamics of dynamical systems in presence of the phase-space constraints is illustrated here on few examples taken from classical and continuum mechanics course.

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1. Introduction

In classical analytic mechanics dynamical systems consists, typically, of many individual *objects* subjected to action of *forces* (external and internal). The motion of these objects is often restricted by externally applied *constraints*. Concepts familiar from textbooks of classical mechanics, the Lagrange equation of II-kind approach, and subsequent paradigm of least action principle [1,2] reside on the incorporation of the constraints into the construction of the generalized coordinates and momenta. Once the Lagrange equations are constructed the Hamiltonian formulation, and subsequent phase-space description, is presented as an application of the Legendre transformation [2]. In modern formulation of the analytic mechanics this is replaced by a more general approach, called *symplectic dynamics* more suitable for many applications, particularly in numerical analysis [3]. One of the most appealing features of the symplectic dynamics description is, that it can be easily extended for quantum mechanics, and it can

* Corresponding author. Fax: +48-22-843-1369.
E-mail addresses: sonnet@cft.edu.pl (S. Nguyen), laturski@cft.edu.pl (Ł.A. Turski).

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be generalized to include a broad spectrum of field theoretical applications, for example continuous media mechanics [4,5]. Symplectic dynamics can also be refashion to encompass dynamics of some dissipative systems. This later generalization is called *metriplectic* description [4–6].

Systems described by symplectic dynamics are also subject to the constraints. One of the elegant and powerful methods for handling these constraints is the Dirac brackets approach [3,7–9]. Unfortunately, very few examples of a "practical" use of that powerful technique are available in the literature. Recently we have shown how the Dirac brackets can be used to construct the symplectic dynamics of an incompressible fluid [10]. In this paper we would like to show that the Dirac brackets are very convenient and easy to use for studying many other problems in classical mechanics including problems of rigid-body dynamics in which the body is indeed treated as consisting of many individual objects – particles – subject to well-known constraints. To illustrate this we shall analyze the simple problem of *physical pendulum* treated as a collection of N-*individual pendulums* connected together by rigid bonds, as in rigid body. It is instructive to see how various quantities, such as moment of inertia, appears, quite naturally, in our description. Another example, more fundamental and potentially attractive for other application, is the demonstration that the non-canonical Poisson brackets for hydrodynamics [10–13] can be derived from some canonical Poisson brackets structure [14] using the Dirac brackets procedure.

In order to make reading of this paper self-contained we include in the following section a short primer in symplectic dynamics followed by brief exposition of the Dirac brackets approach.

2. Symplectic dynamics

In classical dynamic of complex systems one often follows the method developed for classical particles in Hamiltonian dynamics and describes the system dynamics in terms of properly chosen (generalized) positions and momenta spanning the even ($2K$) dimensional phase space $\mathscr{P}$. Denoting the collection of these coordinates and momenta as $z^A = (q^1, q^2, \ldots, q^K, p_1, p_2, \ldots, p_K)$ and making further assumption that the dynamics of the system is governed by a Hamilton like equations of motion we can write them as

$$\partial_t z^A = \{z^A, \mathscr{H}\} \tag{2.1}$$

where $\mathscr{H}$ is the system hamiltonian and $\{\cdot\,,\cdot\}$ denotes the *Poisson bracket*, a bilinear operation which satisfies three requirements:

(i) *Antisymmetry*: $\{\mathscr{F}, \mathscr{G}\} = -\{\mathscr{G}, \mathscr{F}\}$

(ii) *Leibniz rule*: $\{\mathscr{F}, \mathscr{G}\mathscr{E}\} = \mathscr{G}\{\mathscr{F}, \mathscr{E}\} + \{\mathscr{F}, \mathscr{G}\}\mathscr{E}$

(iii) *Jacobi identity*: *Alt* $\{\mathscr{F}, \{\mathscr{G}, \mathscr{E}\}\} \equiv \{\mathscr{F}, \{\mathscr{G}, \mathscr{E}\}\} + \{\mathscr{G}, \{\mathscr{E}, \mathscr{F}\}\} + \{\mathscr{E}, \{\mathscr{F}, \mathscr{G}\}\} = 0$.

A Poisson structure on N-dimensional manifold $\mathscr{P}$ consists of the space of smooth functions $\mathscr{F}$ defined on $\mathscr{P}$, i.e., $C^\infty(\mathscr{P})$, and a Poisson bracket on it. A smooth manifold equipped with a Poisson structure is called Poisson manifold. The Leibniz rule states that the Poisson bracket $\{\cdot,\cdot\}$ acts on each factor as a vector field, therefore it must be of the form $\{\mathscr{F},\mathscr{G}\} = \gamma(\mathrm{d}\mathscr{F},\mathrm{d}\mathscr{G})$ where γ is a field of bivectors on $\mathscr{P}$. If such a field defines a Poisson bracket, it is called a *Poisson tensor*. In the local coordinates (z^A), each Poisson bracket has the form

$$\{\mathscr{F},\mathscr{G}\} = \sum_{A,B=1}^{N} \gamma^{AB}\, \partial_A\mathscr{F}\, \partial_B\mathscr{G}, \quad A,B = 1,\ldots,N, \quad \partial_A \equiv \frac{\partial}{\partial z^A}\,, \tag{2.2}$$

where γ is the antisymmetric tensor such that

$$\sum_{D=1}^{N} \left[\gamma^{DA}\, \partial_D\gamma^{BC} + \gamma^{DB}\, \partial_D\gamma^{CA} + \gamma^{DC}\, \partial_D\gamma^{AB}\right] = 0\,. \tag{2.3}$$

Indeed, condition (2.3) is equivalent to the Jacobi identity for the Poisson bracket given by (2.2).

Given a Poisson structure γ on a manifold $\mathscr{P}$, according to the *splitting theorem* [15], locally (in the neighborhood of a point $z = 0$) there exists a coordinate system (z^A) such that the Poisson tensor γ is of the form

$$\gamma = \begin{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} & 0 \\ 0 & T \end{bmatrix}, \tag{2.4}$$

where $\mathbf{I}$ is $k\times k$ unit matrix and $2k = rank\ \gamma$ at the point $z = 0$, $\mathbf{T} = [T_{ij}]$ is a $(N-2k)\times(N-2k)$ matrix, T_{ij} are functions of $(z^{2k+1},\ldots,z^N)$ and $T_{ij}(0) = 0$ for $i,j = 2k+1,\ldots,N$. Iff γ has *constant rank* and $rank\ \gamma = 2k = N$ at every point, the Poisson tensor γ defines a symplectic structure and locally

$$\gamma = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{2.5}$$

A Poisson structure is called *canonical Poisson structure* iff its Poisson tensor is of the form (2.5).

As previously the equation of motion for any *observable* $\mathscr{F}$ is given as

$$\partial_t\mathscr{F} = \{\mathscr{F},\mathscr{H}\}\,, \tag{2.6}$$

where $\mathscr{H}$ is the system Hamiltonian.

3. Dirac brackets

We shall use the definition of the Dirac brackets which is a natural generalization of the original construction proposed by Dirac [7–9] and discussed in detail in Refs. [3–5].

When the physical system with phase space $\mathscr{P}$ (which we suppose to be a symplectic manifold) is subject to a set of even number of constraints $\{\Theta_a = 0, a = 1,\ldots,2n\}$ then its motion proceeds on a submanifold

$$\mathscr{P} \supset \mathscr{S} = \bigcap_{a=1}^{2n} \{z \in \mathscr{P} | \Theta_a(z) = 0\} . \tag{3.1}$$

Let us denote the Poisson bracket for two arbitrary (sufficiently smooth) phase-space functions $\mathscr{F}$ and $\mathscr{G}$ by $\{\mathscr{F}, \mathscr{G}\}$ and let us suppose that all constraints are of the *second class* in the Dirac classification, which means the matrix of Poisson brackets of the constraints, $\mathbf{W} = [W_{ab}]$,

$$W_{ab} = \{\Theta_a, \Theta_b\} \tag{3.2}$$

has the maximally rank and is invertible. The Dirac bracket $\sqsubseteq \mathscr{F}, \mathscr{G} \sqsupseteq$ for two smooth functions $\mathscr{F}$ and $\mathscr{G}$ is defined as

$$\sqsubseteq \mathscr{F}, \mathscr{G} \sqsupseteq = \{\mathscr{F}, \mathscr{G}\} - \sum_{a,\,b}^{2n} \{\mathscr{F}, \Theta_a\}\ M_{ab}\ \{\Theta_b, \mathscr{G}\} , \tag{3.3}$$

where $[M_{ab}]$ is the inverse of the constraints matrix $[W_{ab}]$. One important property of the Dirac bracket is that all the constraints Θ_a are *Casimirs*, i.e., for an arbitrary function $\mathscr{F}$,

$$\sqsubseteq \Theta_a, \mathscr{F} \sqsupseteq = 0 . \tag{3.4}$$

For finite-dimensional phase space $\mathscr{P}$, dim $\mathscr{P} = 2N$, and for $2n$ second class constraints Θ_a we can always find a *canonical transformation* such that the constraints, Θ_a, lie on the first $2n$ coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ of the phase space and the remaining degrees of freedom, $(Q_1, \ldots, Q_{N-n}, P_1, \ldots, P_{N-n})$ are unconstrained. The Dirac bracket in the whole phase space is equal to the canonical Poisson bracket in the space $(Q_1, \ldots, Q_{N-n}, P_1, \ldots, P_{N-n})$, namely the *reduced phase space* [7–9]. An explicit construction of such a canonical transformation is, in general, quite difficult. Formally, one can use the Dirac formalism in the Poisson context, however this formalism will be useless since the matrix of constraints $\mathbf{W}$ is no longer invertible.

Dirac brackets, given by Eq. (3.3) replace the original Poisson brackets in the equation of motion for the constrained system. Thus for a phase-space function $\mathscr{F}$ the time evolution on the submanifold $\mathscr{S}$ is governed by

$$\left(\frac{\partial \mathscr{F}}{\partial t}\right)_{\mathscr{S}} = \sqsubseteq \mathscr{F}, \mathscr{H} \sqsupseteq , \tag{3.5}$$

where $\mathscr{H}$ is system Hamiltonian.

4. Hamiltonian dynamics on the surfaces

In this section we shall illustrate Dirac bracket applications studying simple example of a particle moving on a surface $\mathscr{S}_{\text{config}} = \{\mathbf{x} \in \mathbf{R}^n | f(\mathbf{x}) = 0\}$ in the configuration space.

Let us consider $\mathscr{P} = \mathscr{R}^{2n}$ with a canonical Poisson (symplectic) structure and a $(2n-2)$-dimensional submanifold $\mathscr{S} \subset \mathscr{P}$ defined as follows:

$$\mathscr{S} = \{(\mathbf{x}, \mathbf{p}) \in \mathscr{P} \mid \Theta_1 = \Theta_2 = 0\} ,$$

where

$$\Theta_1 \equiv f(\mathbf{x}), \qquad \Theta_2 \equiv \mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{x}}. \tag{4.1}$$

The physical interpretation of (4.1) is simple. The first constraint $\Theta_1 = 0$ describes a given fixed algebraic surface in the configuration space $\mathscr{S}_{\text{config}} \subset \mathscr{R}^n$. Our main assumptions are that f is smooth and zero is a regular value of the function f. The second assumption ensures that $\mathscr{S} = f^{-1}(0)$ is a close regular $(n-1)$-dimensional differential submanifold in the configuration space $\mathscr{R}^n$ [16]. Moreover this assumption guarantees $\nabla f \neq 0$, so we can use the gradient to define a normal on $\mathscr{S}$. The second constraint $\Theta_2 = 0$ implies that the particle momentum is always tangent to that surface $\mathscr{S}_{\text{config}}$. It is clear that for a class of functions f with the regular value zero, the constraints Θ_i are of the second class in the Dirac classification. It is quite straightforward to generalize this example to the case of several $(2k)$ constraints of the type (4.1) and the case of constrained electric charges.

The matrix of constraints $\mathbf{W}$ has now the form

$$\mathbf{W} = \left|\frac{\partial f}{\partial \mathbf{x}}\right|^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and its inverse is } \mathbf{M} = \frac{1}{|\partial f/\partial \mathbf{x}|^2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.2}$$

Denoting the unit normal vector to the surface f at the point x by $\mathbf{n}(x)$, $\mathbf{n}(\mathbf{x}) = (1/\partial f/\partial)(\partial f/\partial \mathbf{x})$ and using the Dirac formula (3.3) we get

$$\sqsubseteq x_i, x_j \sqsupseteq = 0, \quad \sqsubseteq x_i, p_j \sqsupseteq = \delta_{ij} - \frac{1}{\partial f/\partial \mathbf{x}^2} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \delta_{ij} - n_i(\mathbf{x}) n_j(\mathbf{x}),$$

$$\begin{aligned} \sqsubseteq p_i, p_j \sqsupseteq &= \frac{1}{\partial f/\partial \mathbf{x}^2} \left\{ \frac{\partial f}{\partial x_j} \left[\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{x}}\right] \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_i} \left[\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{x}}\right] \frac{\partial f}{\partial x_j} \right\} \\ &= n_j(\mathbf{x}) \left[\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{x}}\right] n_i(\mathbf{x}) - n_i(\mathbf{x}) \left[\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{x}}\right] n_j(\mathbf{x}) . \end{aligned} \tag{4.3}$$

In the particular case $d = 3$ the bracket $\sqsubseteq p_i, p_j \sqsupseteq$ can be rewritten as follows:

$$\sqsubseteq p_i, p_j \sqsupseteq = - \varepsilon_{ijk} \left\{ \mathbf{n} \times \left[\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{x}}\right] \mathbf{n} \right\}_k ,$$

where ε_{ijk} is the Levi–Civitta symbol.

In the particular case when $\Theta_1 = x_n$ and $\Theta_2 = p_n$, the constraints can be easily eliminated by choosing a smaller number of phase variables, then the Dirac brackets (4.3) reduce to ordinary canonical brackets

$$\sqsubseteq x_i, x_j \sqsupseteq = \sqsubseteq p_i, p_j \sqsupseteq = 0, \qquad \sqsubseteq x_i, p_j \sqsupseteq = \delta_{ij}, \quad \text{here } i, j = 1, \ldots, n-1$$

and $\sqsubseteq x_n, \cdot \sqsupseteq = \sqsubseteq \cdot\, , p_n \sqsupseteq = 0$. The coordinates (x_n, p_n) should be omitted and this is a simplest possible example of symplectic reduction.

In order to make use of the non-canonical brackets (4.3) we consider a single classical particle of mass m moving in the potential field $V(x)$ in the presence of constraints (4.1). The Hamiltonian for that system is

$$\mathscr{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{4.4}$$

and we denote the potential force $-\partial V/\partial \mathbf{x}$ by $\mathbf{F}$. The Hamilton–Dirac equations of motion (3.5) follow

$$\begin{aligned}\dot{\mathbf{x}} &= \sqsubseteq \mathbf{x}, \mathscr{H} \sqsupseteq = \frac{1}{m}[\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\mathbf{n}] = \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= \sqsubseteq \mathbf{p}, \mathscr{H} \sqsupseteq = \mathbf{F} - \left[\mathbf{F}\cdot\mathbf{n} + \frac{1}{m}\mathbf{p}\cdot\left[\left(\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\mathbf{n}\right]\right]\mathbf{n}\,.\end{aligned} \tag{4.5}$$

One can rewrite Eqs. (4.5) in a “Newtonian” form

$$\begin{aligned} m\ddot{\mathbf{x}} &= \mathbf{F} - \frac{1}{\partial f/\partial \mathbf{x}^2}\left[\mathbf{F}\cdot\frac{\partial f}{\partial \mathbf{x}} + m\,\dot{\mathbf{x}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \mathbf{x}}\right)\right]\frac{\partial f}{\partial \mathbf{x}} \\ &= \mathbf{F} - \left[\mathbf{F}\cdot\mathbf{n} + m\,\dot{\mathbf{x}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}\right]\mathbf{n}\,. \end{aligned} \tag{4.6}$$

The r.h.s. of Eq. (4.6) describes the force acting on the constrained particle moving on the surface $\mathscr{S}$. This force consists of two parts: the potential force and the constraint reaction’s force, which is always orthogonal to the surface.

5. The physical pendulum

A rigid body, in classical non-relativistic mechanics, is defined as a constrained system of finite number of particles (atoms). For rigid body consisting of N particles one has $(3N - 6)$ configuration constraints. Can one handle this within the context of the Dirac constraints discussed in Section 3? On a first glance the use of the Dirac constraints for this purpose looks impractical for one has, seemingly, to handle a $(6N - 12) \times (6N - 12)$ constraints matrix. The general way around that difficulty will be presented elsewhere. Here we will show how a model, or the toy version, of the Euler equation for rigid-body dynamics can be derived using the Dirac constraints formulation. For that model we choose the physical pendulum consisting of N rigidly tide planar mathematical pendulums as shown in Fig. 1. The usual polar coordinates are used.

The relevant $2(N - 1)$ constraints are given by

$$\Theta_i = \varphi_{i+1} - \varphi_1, \qquad \Theta_{N-1+i} = \frac{p_{\varphi_{i+1}}}{m_{i+1}r_{i+1}^2} - \frac{p_{\varphi_1}}{m_1 r_1^2}, \quad i = 1, \ldots, N-1\,. \tag{5.1}$$

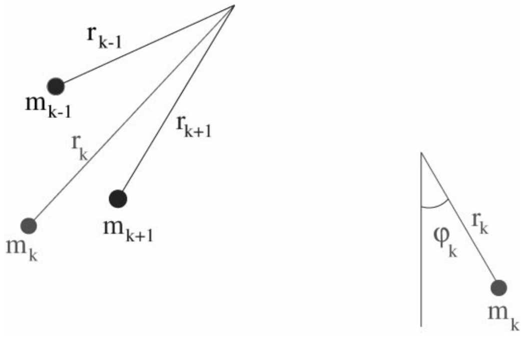

Fig. 1. Physical pendulum envisaged as consisting from N two-dimensional mathematical pendulums. The gravitational field is pointing vertical (along the z-axis).

It is convenient to introduce the $(N-1)\times(N-1)$ matrix $\mathbf{A}$,

$$\mathbf{A} = [A_{ij}] = \begin{bmatrix} A_1 & X & X & \dots & X & X \\ X & A_2 & X & \dots & X & X \\ X & X & A_3 & \dots & X & X \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X & X & X & \dots & X & A_{N-1} \end{bmatrix}, \tag{5.2}$$

where

$$X = \frac{1}{m_1 r_1^2}, \quad A_i = X + \frac{1}{m_{i+1} r_{i+1}^2}.$$

The $2(N-1)\times 2(N-1)$-matrix of constraints, $\mathbf{W}$ is then

$$\mathbf{W} = \begin{bmatrix} 0 & \mathbf{A} \\ -\mathrm{A} & 0 \end{bmatrix}. \tag{5.3}$$

Using the properties of determinants and the induction method one can prove that the inverse of the constraints matrix $\mathbf{M} = \mathbf{W}^{-1}$ has the form:

$$\mathbf{M} = \begin{bmatrix} 0 & -\mathbf{A}^{-1} \\ \mathbf{A}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{B} \\ \mathbf{B} & 0 \end{bmatrix},$$

where

$$B_{ij} = \frac{1}{I_N} \begin{cases} -m_{i+1} r_{i+1}^2 m_{j+1} r_{j+1}^2 & \text{if } i \neq j, \\ (I_N - m_{i+1} r_{i+1}^2) m_{i+1} r_{i+1}^2 & \text{if } i = j \end{cases} \tag{5.4}$$

and $I_N = \sum_{i=1}^{N} m_i r_i^2$ is a physical pendulum moment of inertia. The technique used for calculating the matrix $\mathbf{B} = \mathbf{A}^{-1}$ is presented in the Appendix. Having a explicit form of the matrix $\mathbf{M}$ we can evaluate the Dirac brackets using expression (3.3). We obtain

$$\sqsubseteq \varphi_i, \varphi_j \sqsupseteq = \sqsubseteq p_{\varphi_i}, p_{\varphi_j} \sqsupseteq = 0,$$

$$\sqsubseteq \varphi_i, p_{\varphi_j} \sqsupseteq = \frac{m_j r_j^2}{I_N} \quad \text{for } i, \ j = 1, 2, \dots, N. \tag{5.5}$$

We shall call the above Poisson structure the Dirac structure of the planar physical pendulum. By adding Dirac brackets of the remaining degrees of freedom (namely translation motion) to system (5.5), we shall obtain the full *Dirac structure for planar motion* of two-dimensional rigid bodies.

In order to illustrate how one can use the non-canonical brackets (5.5), we consider a motion of physical pendulum in the external (homogenous) gravitational field. The Hamiltonian of the system is obviously the sum of individual particles contributions:

$$\mathscr{H} = \sum_{i=1}^{N} \left[\frac{p_{\varphi_i}^2}{2m_i r_i^2} - g m_i r_i \cos\varphi_i \right] . \tag{5.6}$$

Using Dirac brackets (5.5) we get the equations of motion for individual pendulums:

$$\dot{\varphi}_i = ⊑ \varphi_i, \mathscr{H} ⊒ = \frac{1}{I_N} \sum_{j=1}^{N} p_{\varphi_j} ,$$

$$\dot{p}_{\varphi_i} = ⊑ p_{\varphi_i}, \mathscr{H} ⊒ = -\frac{g}{I_N} m_i r_i^2 \sum_{j=1}^{N} m_j r_j \sin\varphi_j . \tag{5.7}$$

Introducing now the center of mass coordinates (R, φ) defined as

$$MR\sin\varphi = \sum_{j=1}^{N} m_j r_j \sin\varphi_j , \qquad MR\cos\varphi = \sum_{j=1}^{N} m_j r_j \cos\varphi_j , \tag{5.8}$$

where $M = \sum_{j=1}^{N} m_j$, we can easily see that the equation of motion for φ is

$$\ddot{\varphi} = -\frac{gMR}{I_N} \sin\varphi . \tag{5.9}$$

The set of equations (5.7) is indeed equivalent to that of two-dimensional physical pendulum.

6. Dirac bracket for inviscid compressible fluid

In this section we show how the Poisson non-canonical brackets for ideal compressible fluid [11–13,17], and therefore the dynamic of a barotropic fluid, can be derived from that of a simple non-interacting dust subject to the Dirac constraints and following the Dirac construction of constrained systems dynamics [7–9,14]. Elsewhere we have shown how the use of the Dirac algorithm permits for a simple geometrical interpretations for some non-canonical Poisson brackets appearing in physics, for example description of ideal incompressible fluid [10].

One can make use of our description in different context solving mathematical problems. For example, one immediate consequence of our construction is that the Jacobi identity for the hydrodynamical brackets is automatically satisfied, when the direct proof of that identity is cumbersome.

6.1. Potential fluid

Consider now a phase space of a "dust" with field variables $\varrho(\mathbf{x}), \xi(\mathbf{x})$ and their canonical conjugate momentum $\Pi_\varrho(\mathbf{x}), \Pi_\xi(\mathbf{x})$. The physical interpretation of these variables is quite intuitive. $\boldsymbol{\xi}(\mathbf{x})$ is the radius vector of the dust particle at the point $\mathbf{x}$ and its canonical momentum $\Pi_\xi(x) = \varrho_0 m\,\mathbf{u}(\mathbf{x})$, where ϱ_0 is constant with dimension of density, $\mathbf{u}(\mathbf{x})$ is the particle velocity, m is the particle mass. One should imagine that each dust particle is labelled by its initial position and $\xi(\mathbf{x}, t)$ is the position of a particle labelled by a vector $\mathbf{x}$ at time t. $\varrho(\mathbf{x})$ is an additional field variable which is interpreted as the "ghost" particle number density, and $\Pi_\varrho(\mathbf{x})$ is its canonical conjugate momentum.

The canonical Poisson structure for our dust plus the ghost fields is specified by assumption that

$$\{\varrho(\mathbf{x}), \Pi_\varrho(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\,,$$
$$\{\xi^i(\mathbf{x}), \Pi^j_\xi(\mathbf{y})\} = \{\xi^i(\mathbf{x}), m\varrho_0 u^j(\mathbf{y})\} = \delta^{ij}\delta(\mathbf{x}-\mathbf{y}) \tag{6.1}$$

with all other Poisson brackets equal to zero.

To define the system dynamics we postulate the "dust" Hamiltonian as

$$\mathscr{H} = \mathscr{H}_k + \mathscr{H}_p = \int \frac{|\Pi_\xi(\mathbf{x})|^2}{2m\varrho_0}\,\mathrm{d}^3x + \int \mathscr{F}\left(\varrho(\mathbf{x}), \Pi_\varrho(\mathbf{x})\right)\mathrm{d}^3x$$
$$= \int \tfrac{1}{2}m\varrho_0|\mathbf{u}(\mathbf{x})|^2\mathrm{d}^3x + \int \mathscr{F}\left(\varrho(\mathbf{x}), \Pi_\varrho(\mathbf{x})\right)\,\mathrm{d}^3x\,. \tag{6.2}$$

The physical interpretation of expression (6.2) is as follows. The first term represents the dust kinetic energy and the second attaches some "energy" to the auxiliary density field. The equations of motion for the dust, derived from (6.2) and using the original Poisson brackets, (6.1), are indeed the dust equations of motion, i.e., each dust particle moves with constant velocity. Indeed,

$$\frac{\partial\varrho}{\partial t} = \{\varrho, \mathscr{H}\} = \frac{\partial\mathscr{F}}{\partial\Pi_\varrho}\,, \quad \frac{\partial\Pi_\varrho}{\partial t} = \{\Pi_\varrho, \mathscr{H}\} = -\frac{\partial\mathscr{F}}{\partial\varrho}\,,$$
$$\frac{\partial\xi^i}{\partial t} = \{\xi^i, \mathscr{H}\} = u^i, \quad \frac{\partial u^i}{\partial t} = \{u^i, \mathscr{H}\} = 0\,. \tag{6.3}$$

The ghost-field dynamics, specified entirely by choice of the "energy" $\mathscr{F}$, is of no importance.

Let us now subject our dust+ghost-field dynamics to the set of Dirac constraints. Introducing $\eta(\mathbf{x})$ as

$$\eta(\mathbf{x}) = \varrho_0[1 - \nabla\cdot\xi(\mathbf{x})]\,, \tag{6.4}$$

we write the constraints as

$$\Theta_1(\mathbf{x}) \equiv \Pi_\varrho(\mathbf{x}) = 0\,,$$
$$\Theta_2(\mathbf{x}) \equiv \varrho(\mathbf{x}) - \eta(\mathbf{x}) = \varrho(\mathbf{x}) - \varrho_0[1 - \nabla\cdot\xi(\mathbf{x})] = 0\,. \tag{6.5}$$

The constraints $\Theta_2 = 0$ has a geometrical interpretation similar to that from elasticity theory [18], namely that the divergence of the strain tensor equals to the volume change of the material. The result of that constraint is that the "dust" field ξ and ϱ become coupled.

The constraints matrix $\mathbf{W}_{ij}(\mathbf{x},\mathbf{y}) \equiv \{\Theta_i, \Theta_j\}$ has the following form:

$$\mathbf{W}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{6.6}$$

and its inverse matrix kernel $\mathbf{M} = \mathbf{W}^{-1} = -\mathbf{W}$. Applying now the Dirac formula (3.3), see also [3,7–9], we found

$$\begin{aligned} &\sqsubseteq \varrho(\mathbf{x}), m\varrho_0 u^i(\mathbf{y}) \sqsupseteq = -\,\varrho_0 \frac{\partial}{\partial x^i}[\delta(\mathbf{x}-\mathbf{y})]\,, \\ &\sqsubseteq \xi^i(\mathbf{x}), m\varrho_0 u^j(\mathbf{y}) \sqsupseteq = \delta^{ij}\delta(\mathbf{x}-\mathbf{y})\,, \end{aligned} \tag{6.7}$$

all the other Dirac brackets vanish.

The fact that $\sqsubseteq u^i(\mathbf{x}), u^j(\mathbf{y}) \sqsupseteq = 0$ is responsible for the fact that our formulation is restricted to potential flow dynamics. Indeed, the vorticity field $\omega = \nabla \times \mathbf{u}$ is a Casimir since $\sqsubseteq \varrho(\mathbf{x}), \omega^k(\mathbf{y}) \sqsupseteq = \sqsubseteq u^j(\mathbf{x}), \omega^k(\mathbf{y}) \sqsupseteq = 0$, i.e., $\sqsubseteq \omega^k(\mathbf{x}), \mathscr{K}(\varrho,\mathbf{u}) \sqsupseteq = 0$ for every smooth function $\mathscr{K}$.

Using constraints (6.5), and for $\nabla\cdot\xi \ll 1$ the kinetic part of the "dust" Hamiltonian (6.2) can be rewritten as

$$\begin{aligned} \mathscr{H}_k &= \frac{1}{2}\int m\varrho_0|\mathbf{u}(\mathbf{x})|^2\,\mathrm{d}^3x = \frac{1}{2}\int \frac{m\varrho(\mathbf{x})|\mathbf{u}(\mathbf{x})|^2}{[1-\nabla\cdot\xi(\mathbf{x})]}\,\mathrm{d}^3x \\ &\simeq \frac{1}{2}\int m\varrho(\mathbf{x})|\mathbf{u}(\mathbf{x})|^2\,\mathrm{d}^3x\,. \end{aligned} \tag{6.8}$$

The physical interpretation of the above approximation (6.8) is the following: the kinetic energy of the dust particles in the infinitesimal volume $\mathrm{d}V'$ is equal $\varrho_0 u^2\,dV' = \varrho u^2 \mathrm{d}V'/(1-\nabla\cdot\xi) \simeq \varrho u^2\,\mathrm{d}V$, here $\mathrm{d}V = \mathrm{d}V'/(1-\nabla\cdot\xi)$ is an infinitesimal volume which is obtained from $\mathrm{d}V'$ by a deformation $\mathbf{x}\to\mathbf{x}-\xi(\mathbf{x})$ [18].

Applying now the Dirac brackets to "dust" Hamiltonian (6.8) we obtain equations of motion for dust subject to constraints (6.5). The resulting Dirac brackets equations of motion are identical to these for ideal, barotropic, potential liquid flow equations:

$$\frac{\partial\varrho}{\partial t} = \sqsubseteq \varrho, H \sqsupseteq = -\sum_{k=1}^{d}\frac{\partial}{\partial x^k}(\varrho u^k) = -\nabla\cdot(\varrho\mathbf{u})\,, \tag{6.9}$$

$$\frac{\partial u^i}{\partial t} = \sqsubseteq u^i, \mathscr{H} \sqsupseteq = -\frac{\partial}{\partial x^i}\left[\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{m}\frac{\partial\mathscr{F}}{\partial\varrho}\right]\,, \tag{6.10}$$

$$\frac{\partial\Pi_\varrho}{\partial t} = \sqsubseteq \Pi_\varrho, \mathscr{H} \sqsupseteq = 0\,, \tag{6.11}$$

$$\frac{\partial\xi^i}{\partial t} = \sqsubseteq \xi^i, \mathscr{H} \sqsupseteq = \frac{\varrho}{\varrho_0}u^i\,. \tag{6.12}$$

Eq. (6.9) is the continuity equation and in order to show that the Eq. (6.10) is the Euler equation for potential flows one can use the identity

$$\frac{\partial}{\partial x^i}\left[\frac{1}{2}|\mathbf{u}|^2\right] = [\mathbf{u}\cdot\nabla]u^i + [\mathbf{u}\times(\nabla\times\mathbf{u})]^i\,. \tag{6.13}$$

The remaining two equations have interesting interpretation. First of them, $\Pi_\varrho=0$, is just the other formulation of the continuity equation. The second one, $\varrho_0\,\partial_t\xi(\mathbf{x})=\varrho(\mathbf{x})\,\mathbf{u}(\mathbf{x})$, provides the expressions of the particle current in Lagrangian and Eulerian picture of the fluid.

In conclusion, we have shown that the canonical formulation of a barotropic, potential fluid dynamics can be envisaged as that of a dust subjected to the Dirac constraints.

6.2. Ideal compressible fluid

We shall denote the fluid mass density by ϱ and its velocity by $\mathbf{u}$. To analyze non-potential flow of a barotropic fluid we decompose the fluid field velocity into the Clebsch potentials as follows:

$$\mathbf{u}(\mathbf{x},t) = \nabla_{\mathbf{x}}\phi(\mathbf{x},t) + \alpha(\mathbf{x},t)\nabla_{\mathbf{x}}\beta(\mathbf{x},t)\,. \tag{6.14}$$

To use the Dirac formulation, we envisaged ideal compressible fluid as a constrained system which is described by four pairs of canonical conjugate functional variables $\varrho,\Pi_\varrho,\phi,\Pi_\phi,\alpha,\Pi_\alpha,\beta,\Pi_\beta$ with four following constraints $\Theta_i=0$, $i=1,\ldots,4$

$$\Theta_1\equiv\Pi_\varrho,\qquad \Theta_2\equiv\varrho+\Pi_\phi,\qquad \Theta_3\equiv\Pi_\alpha,\qquad \Theta_4\equiv\varrho\alpha+\Pi_\beta. \tag{6.15}$$

Note that for a potential flow the system of four constraints (6.15) reduces to the system of two constraints (6.5) in the section (6.1)

$$\Theta_1=\Pi_\varrho,\qquad \Theta_2=\varrho+\Pi_\phi=\varrho-\nabla\cdot\Pi_{\mathbf{u}}. \tag{6.16}$$

The matrix of constraints $\mathbf{W}(\mathbf{x},\mathbf{y})$ has the following form:

$$\mathbf{W}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\begin{bmatrix} 0 & -1 & 0 & -\alpha \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varrho \\ \alpha & 0 & \varrho & 0 \end{bmatrix}. \tag{6.17}$$

The inverse matrix $\mathbf{M}=\mathbf{W}^{-1}$ is given by

$$\mathbf{M}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{\alpha}{\varrho} & 0 \\ 0 & -\frac{\alpha}{\varrho} & 0 & \frac{1}{\varrho} \\ 0 & 0 & -\frac{1}{\varrho} & 0 \end{bmatrix}. \tag{6.18}$$

Applying the Dirac formulation we obtain the following non-zero brackets:

$$\sqsubseteq \varrho(\mathbf{x}), \phi(\mathbf{y}) \sqsupseteq = \delta(\mathbf{x}-\mathbf{y}), \qquad \sqsubseteq \phi(\mathbf{x}), \alpha(\mathbf{y}) \sqsupseteq = \frac{\alpha}{\varrho}\delta(\mathbf{x}-\mathbf{y}),$$

$$\sqsubseteq \alpha(\mathbf{x}), \beta(\mathbf{y}) \sqsupseteq = \frac{1}{\varrho}\delta(\mathbf{x}-\mathbf{y}), \qquad \sqsubseteq \phi(\mathbf{x}), \Pi_\phi(\mathbf{y}) \sqsupseteq = \delta(\mathbf{x}-\mathbf{y}),$$
$$\sqsubseteq \beta(\mathbf{x}), \Pi_\beta(\mathbf{y}) \sqsupseteq = \delta(\mathbf{x}-\mathbf{y}), \tag{6.19}$$

all other brackets are zeros.

Introducing now variable $\lambda = \alpha\varrho$ and rewriting $\mathbf{u}$ in the form

$$\mathbf{u} = \nabla_{\mathbf{X}}\ \phi(\mathbf{x}, t) + \frac{\lambda}{\varrho}\nabla_{\mathbf{X}}\ \mu(\mathbf{x}, t), \tag{6.20}$$

we obtain well-known canonical Poisson brackets

$$\sqsubseteq \varrho(\mathbf{x}), \phi(\mathbf{y}) \sqsupseteq = \sqsubseteq \lambda(\mathbf{x}), \mu(\mathbf{y}) \sqsupseteq = \delta(\mathbf{x}-\mathbf{y}), \tag{6.21}$$

and all other Dirac brackets between $\phi, \mu, \lambda, \varrho$ vanish.

Writing the Hamiltonian (6.2) in the Clebsch representation

$$\mathscr{H} = \int \left\{ \frac{1}{2}\varrho\left[(\nabla\phi)^2 + 2\alpha(\nabla\phi\cdot\nabla\beta) + \alpha^2(\nabla\beta)^2\right] + \mathscr{F}(\varrho) \right\} \mathrm{d}^{\mathrm{d}}x. \tag{6.22}$$

We see that the equations of motion for non-constrained system are

$$\begin{aligned}
&\frac{\partial\varrho}{\partial t} = 0, \qquad \frac{\partial\phi}{\partial t} = 0, \qquad \frac{\partial\alpha}{\partial t} = 0, \qquad \frac{\partial\beta}{\partial t} = 0,\\
&\frac{\partial\Pi_\varrho}{\partial t} = -\left[\frac{|\mathbf{u}|^2}{2} + \frac{\partial\mathscr{F}}{\partial\varrho}\right],\\
&\frac{\partial\Pi_\phi}{\partial t} = \partial_i\left[\varrho(\partial_i\phi + \alpha\partial_i\beta)\right] = \nabla\cdot(\varrho\mathbf{u}),\\
&\frac{\partial\Pi_\alpha}{\partial t} = -\varrho(\partial_i\phi + \alpha\partial_i\beta)\partial_i\beta = -\varrho\mathbf{u}\cdot\nabla\beta,\\
&\frac{\partial\Pi_\beta}{\partial t} = \partial_i[\varrho\alpha(\partial_i\phi + \alpha\partial_i\beta)] = \nabla\cdot(\varrho\alpha\mathbf{u}).
\end{aligned} \tag{6.23}$$

Under the presence of constraints (6.15) the simple dynamics (6.23) modifies to the dynamics for constrained system which follows:

$$\begin{aligned}
&\frac{\partial\varrho}{\partial t} = -\nabla\left[\varrho(\nabla\phi + \alpha\nabla\beta)\right] = -\nabla[\varrho\mathbf{u}], \qquad \frac{\partial\Pi_\varrho}{\partial t} = 0,\\
&\frac{\partial\phi}{\partial t} = \alpha\mathbf{u}\cdot\nabla\beta - \left(\frac{|\mathbf{u}|^2}{2} + \frac{\partial\mathscr{F}}{\partial\varrho}\right), \qquad \frac{\partial\Pi_\phi}{\partial t} = \nabla\cdot(\varrho\mathbf{u}),\\
&\frac{\partial\alpha}{\partial t} = 0, \qquad \frac{\partial\Pi_\alpha}{\partial t} = 0,\\
&\frac{\partial\beta}{\partial t} = -\mathbf{u}\cdot\nabla\beta, \qquad \frac{\partial\Pi_\beta}{\partial t} = \nabla\cdot(\varrho\alpha\mathbf{u}).
\end{aligned} \tag{6.24}$$

Knowing brackets (6.19) one can easily calculate non-canonical Poisson brackets [11,12,18]:

$$\sqsubseteq \varrho(\mathbf{x}), \varrho(\mathbf{y}) \sqsupseteq = 0, \qquad \sqsubseteq \varrho(\mathbf{x}), u^i(\mathbf{y}) \sqsupseteq = \frac{\partial}{\partial y_i}\delta(\mathbf{x}-\mathbf{y}),$$

$$\sqsubseteq u^i(\mathbf{x}), u^j(\mathbf{y}) \sqsupseteq = \frac{1}{\varrho}\left(\partial_i\alpha\partial_j\beta - \partial_i\alpha\partial_j\beta\right)\delta(\mathbf{x}-\mathbf{y})$$

$$= \frac{1}{\varrho}\varepsilon^{ijk}[\nabla\times\mathbf{u}]_k\delta(\mathbf{x}-\mathbf{y}). \tag{6.25}$$

The Dirac bracket (6.25) which applies to flow with non-zero vorticity, generalizes bracket (6.7) for potential flow. It is worth to note that the Kelvin theorem follows Eq. (6.25).

Having then we can write the Dirac brackets for two arbitrary functionals F, G of the field $(\varrho, \mathbf{u})$ as

$$\sqsubseteq F, G \sqsupseteq (\varrho, \mathbf{u}) = \int \mathrm{d}\boldsymbol{x}\ \mathrm{d}\boldsymbol{y}\left(\frac{\delta F}{\delta\varrho(\mathbf{x})}\frac{\delta G}{\delta u^i(\mathbf{y})} - \frac{\delta G}{\delta\varrho(\mathbf{x})}\frac{\delta F}{\delta u^i(\mathbf{y})}\right) \sqsubseteq \varrho(\mathbf{x}), u^i(\mathbf{y}) \sqsupseteq$$

$$+\left(\frac{\delta F}{\delta u^i(\mathbf{x})}\frac{\delta G}{\delta u^j(\mathbf{y})} - \frac{\delta G}{\delta u^i(\mathbf{x})}\frac{\delta F}{\delta u^j(\mathbf{y})}\right) \sqsubseteq u^i(\mathbf{x}), u^j(\mathbf{y}) \sqsupseteq$$

$$= \int \mathrm{d}\boldsymbol{x}\left(-\frac{\delta F}{\delta\varrho}\left[\nabla\cdot\frac{\delta G}{\delta\mathbf{u}}\right] + \frac{\delta G}{\delta\varrho}\left[\nabla\cdot\frac{\delta F}{\delta\mathbf{u}}\right]\right)$$

$$+\left(\frac{\delta F}{\delta\mathbf{u}}\times\frac{\delta G}{\delta\mathbf{u}}\right)\cdot(\nabla\times\mathbf{u})\frac{1}{\varrho}. \tag{6.26}$$

It is easy to see that the Dirac bracket (6.26) is the same as in [13].

7. Conclusions

We have shown how some problems from classical mechanics of constrained systems can be handled within the framework of the Dirac formulation of constraints. The particularly interested example of physical pendulum illustrates the general approach to the constrained many particles (many objects) system. It also indicates that the Dirac brackets can be generalized for the case of symplectic description on continuous media mechanics. Elsewhere [10] we have shown how the Dirac constraints and the following Dirac brackets can be used to describe symplectic dynamics of incompressible fluid and the constrained dust dynamics [14]. The Dirac constraints can also be used to formulate the generalized (symmetric with respect to the electric and magnetic charges) electrodynamics, to determine the Poisson structures for elasticity theory, non-linear σ-model [19], etc. The Dirac approach was found to be useful in [20].

Appendix. Calculation of the matrix B

Following expression (5.2) let us denote

$$D_{N-1}(A_1, \dots, A_{N-1}; X) = \det \mathbf{A} . \tag{A.1}$$

One easily finds the following recurrent property:

$$D_N(A_1, \dots, A_N; X) = (A_N - X)D_{N-1}(A_1, \dots, A_{N-1}; X) + (A_{N-1} - X)(A_{N-2} - X)\dots(A_1 - X)X . \tag{A.2}$$

Let $N_{ij} = \{1, 2, \dots, N-1\} - \{i, j\}$ be a set of natural number less than N and does not contain element i, j, and let $\mathbf{B} = [B_{ij}] = \mathbf{A}^{-1}$. The elements of the matrix $\mathbf{B}$ are

$$B_{ij} = \frac{1}{D_{N-1}(A_1, \dots, A_{N-1}; X)} \begin{cases} -X \prod_{k \in N_{ij}} (A_k - X) & \text{if } i \neq j , \\ D_{N-2}(A_1, \dots, \hat{A}_i, \dots, A_{N-1}; X) & \text{if } i = j , \end{cases} \tag{A.3}$$

where $\hat{A}_k$ means that A_k is missing. Using the recurrent property (A.2), by induction principle, one easily finds that

$$D_{N-1}(A_1, \dots, A_{N-1}; X) = \frac{\sum_{k=1}^{N} m_k r_k^2}{\Pi_{k=1}^{N} m_k r_k^2} = \frac{I_N}{\Pi_{k=1}^{N} m_k r_k^2} . \tag{A.4}$$

Substituting Eqs. (A.4) to (A.3) we obtain Eq. (5.4).

Note that one can also use the theorem of Cayley–Hamilton to compute the matrix **B**.

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