The Impulse of Quantized Vortex

by

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§ 1. In an earlier paper [1] we made an attempt to formulate the theory of motion for a set of quantized vortices using the theory of discrete defects borrowed from the theory of continuous media.

After [1] appeared in print a paper of Mamaladze [2] was brought to our attention. The model of vortex used in [2] is very similar to that used in [1]; the main difference consists in that Mamaladze employs an essentially two-dimensional model and, taking advantage of it, uses the mechanism of the Laplace equation in the complex variable form.

It was shown in [2] that when the dynamical properties of vortices are of importance, the concept of “impulse” first introduced by Kelvin should be used rather than ordinary momentum, the latter being very sensitive to boundary conditions and vanishing in the case of stationary boundaries.

In [1] we also argue that generalized momentum of vortex, as defined in [1], is related to Kelvin or Lin impulses but the argumentation was not sufficiently clear. In the present note we intend to show explicitly that the momentum of vortex as defined implicitly through equation (11) in [1] is exactly the Kelvin impulse. We also show that our procedure can be generalized to the case when the vortices are studied within the scope of a suitably modified Gross model [3], i.e. when the “quantum pressure” term is present.

§ 2. Throughout this paper we follow the notation of [1], but for the sake of completeness we repeat here certain facts from [1].

We consider the nonviscous, incompressible, irrotational fluid described by velocity potential $\phi$. The vortices are defined as cylindrical surfaces each of radius $a$ which can move as a whole and across which the velocity potential is discontinuous. The jump of $\phi$, denoted as $[\phi]$ (evaluated on the element of a cylinder's perimeter) is given by

\[ [\phi] = (\pi a/ma) \theta, \]

where $a$ is the radius of the cylinder, $\theta$ — the polar angle, and $n$ — an integer.
The jump of $\phi$ is then a function of the point on the surface. The position of a point on the surface $S_A$ is denoted by radius vector $\zeta_A$.

Due to linearity of the Laplace equation we may write

\begin{equation}
\phi = \phi_0 + \sum_A \phi_A,
\end{equation}

where $\phi_0$ is the velocity potential for the fluid free of singularity, and $\phi_A$ denotes a part of potential $\phi$ generated by the surface $S_A$.

The action functional can then be written in the form

\begin{equation}
\mathcal{L} = \sum_A \int_{S_A} dt \int_{S_A} dS \frac{1}{2} \rho_0 \left\{ L_A + L_{AO} + \sum_{B \leq A} L_{AB} \right\},
\end{equation}

where

\begin{align*}
L_A &= [\phi_A n \cdot \nabla \phi_A], \\
L_{AO} &= [\phi_A n \cdot \nabla \phi_0], \\
L_{AB} &= [\phi_A n \cdot \nabla \phi_B].
\end{align*}

The formal procedure [4] and more or less intuitive argumentation in [1] permit us to define the time derivative of the momentum density of the surface $S_A$ as

\begin{equation}
\dot{P} = \frac{\delta}{\delta \phi_A} \int_{S_A} dS L_A.
\end{equation}

The variational derivatives of all other terms in the action $\mathcal{L}$ are identified with the density forces acting on the vortex $A$ due to the presence of other vortices and free fluid flow. The term $L_A$ is a singular function because it involves double integration of the Green function for the Laplace equation, over the same domain. Applying the "breaking of bars" method

\begin{equation}
[\phi \psi] = [\phi] \langle \psi \rangle + \langle \phi \rangle [\psi],
\end{equation}

where $\langle \psi \rangle$ denotes the mean value on the surface, we obtain the following expression for $L_A$

\begin{equation}
L_A = [\phi_A] \langle n \cdot \nabla \phi_A \rangle.
\end{equation}

The normal velocity of the fluid is continuous across the surface and the mean value of $n \cdot \nabla \phi_A$ is equal to $n \cdot \zeta_A$, where $\zeta_A$ is the velocity of the surface point.

Now we are able to derive

\begin{equation}
\dot{P}_A = - \frac{\partial}{\partial t} \frac{\partial}{\partial \zeta_A} \left( \frac{1}{2} \rho_0 \dot{\zeta}_A \cdot \frac{\partial}{\partial \zeta_A} \right) L_A = - \zeta_A \times \dot{\zeta}_A,
\end{equation}

where $\mathbf{x} = -\rho_0 e^{\alpha \rho} \frac{\partial}{\partial \zeta_A} \left[ \phi_A \right]$ is the "vorticity" vector pointing along the vortex axis; $\zeta_A = \frac{\partial}{\partial \eta} \zeta_A$ are tangent vectors to the coordinate lines on the surface $S_A$ and where we use the relation

\begin{equation}
n_A = \frac{1}{2} e^{R_\sigma} \epsilon_{\zeta_A \zeta_A \sigma} \zeta_A \zeta_A; \quad e^{R_\sigma} = g^{-1/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g = \det \left( \zeta_A \cdot \zeta_A \right).
\end{equation}
The form of equation of motion is then

\[ \dot{\mathbf{P}} = Mf - \sum_{\text{B}} \rho_0 [\phi_A] \nabla (n \cdot u) + \rho_0 [\phi_A] \nabla (n \cdot u), \]

where \( Mf \) is the Magnus force and \( u = -\nabla \phi_B; \) \( u = -\nabla \phi_0. \)

Using the boundary conditions satisfied by the normal component of velocity on the surface \( S_A, \) we obtain the final form of equation of motion

\[ \dot{\zeta} \times v = Mf. \]

The left-hand side of Eq. (7) is the time derivative of Kelvin impulse density. In order to show this, let us consider a single vortex filament stretched along the \( z \) axis. The jump of \( \phi \) is then given as previously and the vector \( \mathbf{v} \) is given by \( \mathbf{v} = (\rho_0 / m) e_z, \) where \( m \) is the mass of helium atom and \( n \) is an integer.

The Kelvin impulse may be written as \[ (8) \]

\[ \mathbf{J}^k = \rho_0 \int_{S_A} [\phi_A] n dS. \]

Using the method of potential theory for the Laplace equation, one can show that

\[ (8') \quad \mathbf{J}^k = \rho_0 \mathbf{R} \times \mathbf{\Gamma}, \quad \text{(per element of the length of vortex filament)}, \]

where \( \mathbf{R} \) is the radius vector of the center of vortex and \( \mathbf{\Gamma} = \frac{n \hbar}{m} e_z. \)

Now, integrating (7) over the perimeter of the vortex filament we obtain exactly (8').

Indeed, decomposing the velocity \( \dot{\zeta} \) into the translational and rotational parts (see Figure)

\[ (9) \quad \dot{\zeta} = \dot{\zeta}^{\text{tr}} + \dot{\zeta}^{\text{rot}} = \dot{\mathbf{R}} + \omega e_z \times \mathbf{u}; \]
inserting then this expression into (7), and performing the integration, we obtain (8').

Eq. (7) has then the form: time derivative of the density of the Kelvin impulse is equal to the density of the total force exerted on the vortex. The above statement is also the result of Maimaladze.

We should like to stress that the above form of the equation of motion cannot justify using the impulse concept in other calculations such as estimation of critical velocity for vortices system to be created. This follows from the fact that our equations were derived under special boundary conditions as we pointed out in [1].

§ 3. Let us consider for a moment the Lagrangian for the case of Gross model [3]. We have

\[ L = \int d^3 x \left\{ m \rho \dot{\varphi} - \frac{1}{2} m \rho \nabla \varphi \cdot \nabla \varphi - \frac{\hbar^2}{2m} \nabla \rho^{1/2} \cdot \nabla \rho^{1/2} \right\} - \int d^3 x d^3 y (\rho (x) - \rho_0) V(x - y) (\rho (y) - \rho_0). \]

Assume now that the vortices are defined as before. Due to the nonlinearity of field equations corresponding to (10), the superposition principle (i) is not valid any longer. The qualitative consideration of Fetter shows that one can assume the velocity potential to satisfy (1), if on the boundary of vortex core the density satisfies the conditions which follow from solution of [5] and which may be written as

\[ [\rho]_{S_A} = 0; \quad [n \cdot \nabla \rho]_{S_A} = 0; \quad \rho|_{S_A} = \rho_0 \]

and if the separation between vortices is large compared to the radius of the vortex core. It is easy to check that performing the same integrations by parts procedure as in [1] we obtain exactly the same Lagrangian for vortices system as in [1]. The only difference is that the terms independent of the position of $S_A$ are different and that the relation between $\varphi_A$ and discontinuity of $\varphi$ on $S_A$ is no longer given by the potential-theory formula [1, eq. (6)].

The equations of motion which follow from (10) are then equivalent to those given in § 2. This can be explained on the grounds that the assumption that $\varphi$ is given as superpositions of $\varphi_A$ together with the boundary conditions for $\rho$ corresponds in some way to a kind of linearization of the Gross model.

On the other hand, our procedure is not completely satisfactory, because one can expect that the inclusion of the "quantum pressure" term should change the equation of motion in a significant way, particularly when the array of vortices is dense. Unfortunately, the theory of discrete defects which is the very basis of our consideration is, as it is now, formulated for linear fields only. Although there are some recent attempts to generalize it, particularly in connection with the non-linear elasticity, there is little hope, in our opinion, that it will be useful in our problem.

In conclusion, the concepts of Magnus force and Kelvin impulse will not be so useful a tool of description for liquid helium vortices as they were for vortices in
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\end{equation}

Assume now that the vortices are defined as before. Due to the nonlinearity of field equations corresponding to (10), the superposition principle \((1)\) is not valid any longer. The qualitative consideration of Fetter shows that one can assume the velocity potential to satisfy \((1)\), if on the boundary of vortex core the density satisfies the conditions which follow from solution of \([5]\) and which may be written as

\begin{equation}
|\rho|_{S_\varphi} = 0; \quad |\mathbf{n} \cdot \nabla \rho|_{S_\varphi} = 0; \quad \rho|_{S_\varphi} = \rho_0
\end{equation}

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In conclusion, the concepts of Magnus force and Kelvin impulse will not be so useful a tool of description for liquid helium vortices as they were for vortices in
ideal fluid. The vortex in liquid helium may be connected with singularities in the flow other than those used in the theory of ideal, incompressible, nonviscous fluid [6, p. 120].

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REFERENCES