ON THE DYNAMICS OF LOW-TEMPERATURE MODELS FOR PHASE TRANSITIONS WITH CONSERVED ORDER PARAMETER

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Starting from the time-dependent Ginzburg-Landau model, we derive dynamic versions of a non-linear \(\sigma\)-model and a drumhead model, both with conserved order parameters. In both cases there appears a non-ordering field that adiabatically follows the order parameter. In this way a constraint is imposed on the dynamics which guarantees consistency of conservation of the order parameter and the symmetries of the models.

1. Introduction

When a continuous symmetry is broken by a phase transition then the low-temperature behaviour of the system is controlled by Goldstone modes. An example of such modes are the spin waves in an isotropic \(n\)-component spin system where the \(O(n)\) symmetry is broken in the ferromagnetic phase. Another example are the capillary waves of an interface in a one-component spin system in \(d\) spatial dimensions. In this case the Euclidean symmetry \(E(d)\) is broken by the interface in the two-phase region.

Effective Hamiltonians for Goldstone modes have been established, with a convenient representation of these modes by field variables. The models defined in this way are the non-linear \(\sigma\)-model for the \(n\)-component spin system, and the drumhead model for an interface in a one-component spin system.

In both models there occurs a second-order phase transition at a critical temperature \(T_c \sim \epsilon + O(\epsilon^3)\) where \(\epsilon = d - 2\) in the non-linear \(\sigma\)-model and \(\epsilon = d - 1\) in the drumhead model. The asymptotic critical behaviour of these transitions has been discussed within an \(\epsilon\)-expansion. Furthermore the critical dynamics has been considered for systems that undergo relaxation towards an equilibrium state described by the above static models. This has been done for the non-linear \(\sigma\)-model with non-conserved order parameter as well as conserved order parameter. A similar discussion has been given for the drumhead model.
with a non-conserved bulk order parameter\textsuperscript{7}. The case of a conserved order parameter has been considered mainly in connection with the first-order transition at temperature $T < T_c$.\textsuperscript{8} In general models with conserved (B-type) and models with non-conserved (A-type) dynamics belong to different dynamic universality classes though they lead to the same static critical behaviour.\textsuperscript{9}

It is our impression that the case of A-type dynamics of a non-linear $\sigma$-model has been treated far more lucidly in the literature than the B-type dynamics. This is due to a difficulty in B-type dynamics which has been pointed out by de Dominicis, Ma and Peliti in their discussion of the dynamic non-linear $\sigma$-model\textsuperscript{3}. The difficulty is an apparent conflict between the conditions of fixed spin length and the condition of conservation of total magnetisation. One purpose of this work is to show that this conflict only appears if a hydrodynamic mode is ignored which exists apart from the Goldstone modes. Although this mode is fast as compared to the Goldstone modes it is coupled to these and therefore should be treated in an adiabatic approximation. As shown in the present paper this approximation changes the hydrodynamic mode into a constraint field that removes the abovementioned conflict.

A similar situation occurs in the dynamics of an interface with a conserved bulk order parameter. A second intention of our work is to formulate the derivation of a (dynamic) drumhead model from a bulk model as parallel as possible to the derivation of a non-linear $\sigma$-model for Goldstone modes in a spin system. We will see that the parallelism is suggestive but not perfect at the present time. It remains an interesting question whether drumhead models can be derived as non-linear $\sigma$-models for the Goldstone modes of the broken Euclidean symmetry by introducing a convenient coset space parametrisation of $E(d)/E(d - 1)$ in analogy with the $O(n)/O(n - 1)$ parametrisation used for an $n$-component spin model.

2. The Non-Linear $\sigma$-Model

A convenient method to derive the non-linear $\sigma$-model for spin-waves from a Ginzburg-Landau Hamiltonian is the use of collective coordinates\textsuperscript{10} in a functional integral formulation. This can be done systematically for the partition sum of spin waves\textsuperscript{11}. To treat the dynamics in a similar way we start from a representation of the time-dependent Ginzburg-Landau model by an action $K$ that enters the generating functional

$$Z[\tilde{j}, j] = \int D[\tilde{\phi}, \phi] \exp \left( - K[\tilde{\phi}, \phi]/T + \int dx dt \{ \tilde{j}(x, t) \tilde{\phi}(x, t) + j(x, t) \phi(x, t) \} \right)$$

(2.1)

of correlation functions of the order parameter $\phi$ and its Martin-Siggia-Rose response field $\tilde{\phi}$\textsuperscript{12}. This representation has been used frequently in the theory of critical dynamics and in the method of stochastic quantization. We refer
the reader to Refs. 12, 13 for details about its use and its relation to other formulations as, e.g., the description of dynamics by Langevin equations.

For the Ginzburg-Landau model $K$ reads explicitly

$$K = \int d^d x d t \left[ - \dot{\phi} \Gamma \phi + \phi [ \partial_t \phi + \Gamma \delta H(\phi) / \delta \phi ] \right]$$

(2.2)

with an operator

$$\Gamma = \gamma ( - \nabla^2 )^a$$

(2.3)

where $a = 0$ for the non-conserved and $a = 1$ for the conserved case, and the Hamiltonian

$$H = \int d^d x \left[ \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right].$$

(2.4)

In $K$ the coefficient of the linear $\dot{\phi}$-term is a Langevin force which is Gaussian distributed as can be seen from the term quadratic in $\dot{\phi}$. From (2.1) and (2.2) it is easily seen that correlation functions involving $\dot{\phi}$ are response functions to an external field $\vec{f}$ that adds to the Langevin force.

In the present section, $\phi$ and $\dot{\phi}$ are assumed to be $n$-component fields, and products of these fields, as e.g. $\dot{\phi} \phi$, denote scalar products. Furthermore the local potential $V$ in the Hamiltonian is assumed to depend only on the modulus of $\phi$ so that the whole model defined by (2.2)–(2.4) is $O(n)$ symmetric. The detailed structure of $V$ is irrelevant for the low-temperature behaviour of the system. For this it is only important that $V$ has a sharply peaked minimum at some value say $|\phi| \approx 1$, where

$$V'(1) = 0, \quad V''(1) = m^2.$$  

(2.5)

In the Ginzburg-Landau Hamiltonian $m^2$ is a measure of the distance from the critical temperature. Since we are interested in a situation well below the critical temperature we consider the limit $m^2 \to \infty$ whereby fluctuations of the modulus of $\phi$ are completely suppressed. Let us therefore put $V = m^2 U$ and assume that $U$ is a smooth and regular function in the limit $m^2 \to \infty$. To perform this limit it is suggestive to use the following parametrisation of $\phi$:

$$\phi_{\alpha}(x, t) = [1 + m^{-2} \rho(x, t)] s_{\alpha}(x, t), \quad \alpha = 1, \ldots, n,$$  

(2.6)

$$s^2(x, t) = 1.$$  

(2.7)

The field $s(x, t)$ represents the Goldstone modes due to the broken $O(n)$
symmetry and \( \rho(x, t) \) describes radial fluctuations of the magnetization vector.

Since (2.7) implies \( s \partial \partial_s = 0 \) this also suggests the following decomposition of \( \bar{\psi} \):

\[
\bar{\psi}_s(x, t) = \bar{s}_s(x, t) + \bar{\rho}_s(x, t) s_s(x, t),
\]

\[
\bar{s}(x, t) s(x, t) = 0.
\]

For the present case the transformation to the new variables \( \bar{s}, s, \bar{\rho}, \rho \) constitutes the essence of the method of collective coordinates^{10}. A convenient formal way to transform to the new variables is to insert into (2.1) the identity

\[
\int D[\bar{s}, s, \bar{\rho}, \rho] R[\bar{s}, s, \bar{\rho}, \rho] \delta[\bar{s} - (1 + m^{-2}\rho)s] \delta[s^2 - 1] \delta[\bar{\rho} - \bar{s} - \bar{\rho}s] \delta[\bar{s}s]
\]

which also defines the Jacobian I. All the \( \delta \)-symbols are to be understood as \( \delta \)-functionals.

This leads to a generating functional for the Goldstone modes:

\[
Y[\bar{h}, h] = \int D[\bar{s}, s, \bar{\rho}, \rho] \delta[s^2 - 1] \delta[\bar{s}s] \exp \left[ -J[\bar{s}, s, \bar{\rho}, \rho] / T \right]
\]
\[
+ \int dxd t \left[ \bar{h}(x, t) \bar{s}(x, t) + h(x, t) s(x, t) \right].
\]

To get an explicit expression for the action \( J \) we now use the following expansions in (2.2):

\[
\partial_t \bar{\psi} = \partial_t s + m^{-2}\rho \partial_t \rho + O(m^{-2} \partial_t s)
\]

(2.12)

\[
\nabla^2 \bar{\psi} = \nabla^2 s + O(m^{-2} \nabla^2 s; m^{-2} (\nabla \rho) \nabla s; m^{-2} \nabla \rho).
\]

(2.13)

\[
\partial \nabla / \partial \bar{\psi} = \rho s + O(m^{-2}).
\]

(2.14)

Note that we kept the term \( m^{-2} s \partial_t \rho \) in (2.12). As there is no term of order \( m^0 \) which contains the time derivative of \( \rho \), the \( O(m^{-2}) \) term is necessary to ensure the causality of the theory. This will become more obvious if we now give the result of the expansion (2.12)–(2.14) in (2.2). Since due to (2.6) the logarithm of the Jacobian does not contribute to order \( m^0 \) we obtain the leading order result:

\[
J = \int d^4 x dt \left[ - (\bar{s} + \bar{\rho}s) \Gamma(\bar{s} + \bar{\rho}s) + \bar{s} \partial_t s + m^{-2} \bar{\rho} \partial_t \rho + (\bar{s} + \bar{\rho}s) \Gamma(-\nabla^2 + \rho)s \right].
\]

(2.15)

Equations (2.11) and (2.15) are the main results of the present section. From (2.15) it can be seen, that in perturbation theory the term \( O(m^{-2}) \) in \( J \) only affects the bare response propagator of the \( \rho \)-field:
\begin{equation}
\langle \rho(q', t), \rho(q, 0) \rangle_0 = \int D[\tilde{\rho}, \rho] \rho(q', t) \tilde{\rho}(q, 0) \exp(-J_0[\tilde{\rho}, \rho]) ,
\end{equation}

\begin{equation}
J_0 = \int \frac{d^d q dt}{(2\pi)^d} \left[ \gamma q^{2a} |\tilde{\rho}(q, t)|^2 + \tilde{\rho}(q, t)[m^{-2} \partial_t + \gamma q^{2a}] \rho(-q, t) \right].
\end{equation}

We find explicitly

\begin{equation}
\langle \rho(q', t), \rho(q, 0) \rangle_0 = (2\pi)^d \delta(q + q') R(q, t)
\end{equation}

where

\begin{equation}
\lim_{m \to \infty} R(q, t) = \lim_{m \to \infty} \frac{\partial(t) m^2 \exp(-\gamma q^{2a} m^2 t)}{\gamma q^{2a}} = \delta_+(t)/\gamma q^{2a}.
\end{equation}

Thus the effect of performing the limit $m^2 \to \infty$ at the end of the calculation is to generate a $\delta$-function in (2.18) that is peaked at $t = +0$ and therefore respects causality.

In the case of a non-conserved order parameter where $\Gamma$ is a constant the integral over $\rho$ in (2.11) generates a $\delta$-functional $\sim \delta[\tilde{\rho}]$ that also makes the integration over $\tilde{\rho}$ trivial. The result is a representation of the model that has been proposed previously\textsuperscript{6}, and that has been used more recently in the stochastic quantization of the non-linear $\sigma$-model\textsuperscript{14}. In the case of a conserved order parameter it seems to be more convenient not to integrate over $\rho$ and $\tilde{\rho}$ since this would generate hydrodynamic singularities in the action.

Let us now elucidate our remarks about the role of the additional field $\rho$ which we made in the introduction. A simple discussion of this point uses the equations of motion which correspond to (2.15):

\begin{equation}
\partial_t s = \gamma P \nabla^{2a}(\nabla^2 + \rho)s + \xi,
\end{equation}

\begin{equation}
m^{-2} \partial_t \rho = \gamma s \nabla^{2a}(\nabla^2 + \rho)s + \xi_\rho.
\end{equation}

$P$ is the projection operator

\begin{equation}
P_{\alpha\beta}(x, t) = \delta_{\alpha\beta} - s_\alpha(x, t) s_\beta(x, t).
\end{equation}

Note that the Langevin forces $\xi$ and $\xi_\rho$ have cross correlations according to the first term in (2.15).

The conflict pointed out in the introduction can already be seen in the deterministic part of Eq. (2.20). The projection operator $P$ originating from the fixed spin length and the Laplace operator in $\Gamma$ originating from the conservation of magnetization do not commute. Therefore the equations for $s$ and $\rho$ become
non-trivially coupled. (Note that in the case of a non-conserved order parameter \( \Gamma = \text{const.} \), and \( P \rho s = 0 \) so that \( s \) obeys a simple diffusion equation.) On the other hand, in the adiabatic limit \( m^2 \to \infty \) Eq. (2.21) states that the second term of the projection operator (2.22) has no effect in (2.20) and therefore may be ignored. In this sense \( \rho(x, t) \) plays the role of a constraint field in the adiabatic approximation.

### 3. The Drumhead Model

The dynamics of the drumhead model will also be derived from the time-dependent Ginzburg-Landau model using the method of collective coordinates. This method has already been applied to the statics of the system\(^{15}\). We will adopt here a version of this procedure that has been proposed by Zia.\(^{16}\)

Again our starting point is the model described by Eqs. (2.1)–(2.4). Now we consider one-component fields \( \phi \) and \( \tilde{\phi} \) and a potential \( V = m^2 U \) that has two degenerate minima at \( \phi = \pm 1 \).

In the two-phase region we obtain a static saddle-point solution \( \phi_c(z) \) describing a planar interface located at \( z = 0 \) from the equation

\[
-d_x^2 \phi_c + V'(\phi_c) = 0
\]

with boundary conditions \( \phi_c(\pm \infty) = \pm 1 \). The interface becomes sharp in the limit

\[
V''(1) = m^2 \to \infty .
\]

Equations (3.1) and (3.2) correspond to Eq. (2.5) of the previous section. The Goldstone modes in the present case correspond to long wavelength deviations of the interface from planar. In an expansion around the saddle-point they appear as soft eigen-modes, i.e.

\[
(-\nabla^2 + V'(\phi_c)) [\partial_x \phi_c(z) \exp(iqy)] = q^2 [\partial_z \phi_c(z) \exp(iqy)].
\]

This follows from (3.1) with \((d - 1)\)-dimensional vectors \( q \) and \( y \) in the plane \( z = 0 \). All other eigen-modes of the operator in (3.3) are assumed to be hard modes, having eigenvalues of the order of \( m^2 \).

At this stage it is convenient to introduce a parametrisation of non-planar configurations \( z = f(y) \) of the interface by the normal coordinate \( u \) and a \((d - 1)\)-dimensional set of curvilinear coordinates \( v \) within the interface\(^{16}\), so that

\[
x = \xi + un .
\]

Here the components of

\[
\xi(v) = (v, f(v))
\]
are Cartesian coordinates of some point at the interface and

$$n = \frac{1}{\sqrt{g}} (- \partial f(v), 1)$$  \hspace{1cm} (3.6)

with

$$g = 1 + (\partial f)^2$$  \hspace{1cm} (3.7)

is the normal unit vector at this point.

Now we decompose $\varphi(x)$ in the form

$$\varphi(x) = \varphi_c(u) + M^{-2} \eta(x)$$  \hspace{1cm} (3.8)

and require

$$\langle \partial_u \varphi_c; \eta \rangle = \int du \eta(u, v) \partial_u \varphi_c(u) = 0 .$$  \hspace{1cm} (3.9)

The Goldstone modes are included in $\varphi_c(u)$ and by analogy with (2.6), $M^{-2}$ is the inverse of the operator of Eq. (3.3) in the hard-mode subspace. Since

$$\partial_s \varphi_c = [\partial_u \varphi_c] \partial_s u$$

and from (3.4) $\partial_s u = -\partial_s f/\sqrt{g}$ it becomes apparent that $\partial_s \varphi_c$ factorizes into $\partial_u \varphi_c$ and a function of $v$ only. This suggests to write

$$\tilde{\varphi}(x) = -\tilde{f}(v) \partial_u \varphi_c(u) + \tilde{\eta}(u, v)$$  \hspace{1cm} (3.10)

and to require

$$\langle \partial_u \varphi_c; \tilde{\eta} \rangle = 0 .$$  \hspace{1cm} (3.11)

By a trick similar to (2.10) we can transform to the new variables $\tilde{f}, f$, $\tilde{\eta}, \eta$ and obtain a generating functional

$$Y[\tilde{h}, h] = \int D[\tilde{f}, f, \tilde{\eta}, \eta] \exp[-J[\tilde{f}, f, \tilde{\eta}, \eta, \xi]/T$$

$$+ \int \partial^{d-1} vdt[h(v, t)f(v, t) + \tilde{h}(v, t)\tilde{f}(v, t)].$$  \hspace{1cm} (3.12)

An explicit expression for $J$ is obtained from the expansions

$$\partial_s \varphi = -[\partial_u \varphi_c] \partial_s f/\sqrt{g} + M^{-2} \partial_s \eta + O(m^{-2}; \partial_s f),$$  \hspace{1cm} (3.13)

$$-\nabla^2 \varphi + \partial \nabla / \partial \varphi = [\partial_u \varphi_c] \partial (\partial f/\sqrt{g}) + \eta + O(m^{-2}; \partial^4 f),$$  \hspace{1cm} (3.14)

$$\partial^4 x = \partial^{d-1} vdu(\sqrt{g} + O(\partial^2 f)).$$  \hspace{1cm} (3.15)
Equation (3.14) follows from

\[-\nabla^2 \varphi_c + V^\varphi_c = -(\partial_u \varphi_c) \nabla^2 u,
\]

which is a consequence of (3.1) and of the identity \( \nabla \cdot u = n \) implied by (3.4) (see also Ref. 16).

Since again the Jacobian of the transformations (3.8)–(3.11) does not contribute in order \( m^0 \) (as in Ref. 15) we obtain from (2.2), (3.8)–(3.11) and (3.13)–(3.15)

\[
J = \int d^{d-1}v dt \sqrt{g} \left\{ -\langle \tilde{f} \partial_v \varphi_c + \tilde{\eta} | \Gamma \rangle - \tilde{f} \partial_u \varphi_c + \tilde{\eta} \rangle + \frac{1}{\sqrt{g}} \langle \partial_u \varphi_c \rangle \tilde{f} \partial_v \right. \\
+ \langle \tilde{\eta} | M^{-2} \partial_t \tilde{\eta} \rangle + \langle -\tilde{f} \partial_u \varphi_c + \tilde{\eta} | \Gamma \rangle - (\partial_u \varphi_c) \partial (\partial f/\sqrt{g}) + \tilde{\eta} \rangle \right\}. 
\]

Equation (3.17) defines the dynamic drumhead model and is the main result of this section. Comparing it to (2.15) it can easily be seen that most of our remarks which concerned the non-linear \( \sigma \)-model carry over to the case of the drumhead model.

If the kinetic coefficient \( \Gamma \) is just a constant the \( \eta \)-integration produces a functional \( \sim \delta[\tilde{\eta}] \) and we recover the well known drumhead model for a non-conserved order parameter. On the other hand if we consider a conserved order parameter so that \( \Gamma = -\rho \nabla^2 \), the diffusive \( \rho \)-modes are coupled to the interface field and the integration over \( \eta \) leads to long-ranged hydrodynamic interactions. The deterministic part of the resulting dynamics is then identical to the equation of motion derived by Langer and Turski from phenomenological arguments. It should be noted, however, that the Laplace operator in \( \Gamma \) generates a leading order term (due to \( \nabla \cdot u = n \)) as well as several higher order terms in curvature which consequently should be ignored in (3.17).

The role of the \( \eta \)-field in (3.17) is essentially the same as that of the \( \rho \)-field in (2.15). Its time derivative appears with a prefactor \( O(m^2) \) and for \( m^2 \to \infty \) it becomes a constraint field. The analogy to (2.20) and (2.21) is closest if one considers the equation of motion for \( \varphi_c(u) \). The deterministic part of this equation is proportional to the projector \( P \sim |\partial_u \varphi_c \times \partial_u \varphi_c| \) and the constraint, which appears in the adiabatic limit allows to neglect \( P \) just as for the non-linear \( \sigma \)-model.

4. Conclusions

We have derived models which describe the low-temperature dynamics of systems with a conserved order parameter. Our analysis is quite similar for both spin waves of an \( n \)-component magnet and for the capillary waves of an interface in a one-component model. The important differences in the argument indicate
that we were unable to use the Euclidean symmetry structure as effectively as that of an internal continuous symmetry. Indeed, we needed to use explicitly the saddle-point solution $\varphi_c$ describing the interface, though only global properties of $\varphi_c$ enter the actual calculations.

We think that these different states of knowledge become important if we want to study (2.15) and (3.17) by perturbation theory. For both models the linear eigen-mode spectra can be found straightforwardly\textsuperscript{5,7,8}. One finds $\omega \sim iq^2$ (A type) and $\omega \sim iq^4$ (B type) for the non-linear $\sigma$-model and $\omega \sim iq^2$ (A type) and $\omega \sim iq^3$ (B type) for the drumhead model. The easiest way to obtain the latter result is to solve the linearized equation of motion for $\eta$ in terms of $f$ and insert this solution into (3.9). If we now want to study the effects of interactions between the Goldstone modes it becomes decisive to make use of the symmetry in order to construct a renormalizable perturbation expansion. The dynamic functional (2.15) seems to be renormalizable. For A-type dynamics renormalizability can be proved using the $O(n)$ symmetry and the constraint (2.17)\textsuperscript{7}. In the case of a conserved order parameter (B-type) the theory remains $O(n)$ symmetric. The new fields $\rho$ and $\bar{\rho}$ are non-ordering fields with canonical dimensions $d_\rho = d_{\bar{\rho}} = 2$ (which is also the dimension of $\bar{\delta}$). Therefore the theory seems to stay renormalizable by symmetry and power counting. For the drumhead model, on the other hand, the question of renormalizability is in general much harder. For A-type dynamics the model is known to be renormalizable\textsuperscript{7} but for B-type dynamics the situation is unclear. This is due to the fact that the additional field $\eta$ is a bulk mode which is coupled to the interface. Therefore results for critical exponents cannot be obtained from (3.15) for model B at present. It is known however that the dynamic critical exponent $\gamma = 4 - \eta$ for B-type dynamics in all orders of an $\varepsilon = 4 - d$ expansion of the bulk model\textsuperscript{13}, where $\eta$ denotes the static exponent governing the decay of order parameter correlations.

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