HAMiltonian formulation of spherical shell’s dynamics from first principles

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The Lagrangian function of self-gravitating spherically symmetric matter shells composed of an elastic fluid is derived from variational principles in the General Relativistic context. The natural Hamiltonian of the system, representing the total energy for a distant observer, is then calculated in terms of canonical variables. Known results for the dust case can be recovered.

Keywords: list your keywords here.

1. Introduction

Thin shells were first introduced by Werner Israel [1] as a simple model to study gravitational collapse and since then have become an useful toy model to investigate a variety of issues in classical general relativity (see [2] and reference therein), and eventually quantum gravity [3]. A thin matter shell is a 3-dimensional time-like hyper-surface Σ that separates two smooth 4-dimensional manifolds. Tailoring of the two manifolds along the shell implies that the induced 3-geometry must be the same from both sides, while a jump of its first derivatives across the hyper-surface is related to the matter content of the shell. The dynamics of a spherically symmetric shell+gravity system is then described by the history of the two-dimensional surface that tailors the two portions of spherical space-times. Einstein equations split into a regular part outside the shell and a delta-like part concentrated on Σ and can be derived from an appropriate Lagrangian.

[1]
The Hamiltonian formulation of dynamics of thin shells was constructed in [4] and the variational approach from first principles was developed in [5] for the dust shell and in [6] for spherical shells with general equation of state. The Hamiltonian that was derived in this approach correctly reproduces the numerical value of the total energy of the system as seen by an observer at spatial infinity. The work is based on the analysis of boundary phenomena that occur when performing the variation of the Hilbert action of the system [7]. The implications of changes in the time gauge were thoroughly discussed in [8].

In the present paper we shall discuss how to extend the results obtained for one shell to a system composed of two or more shells.

2. Variational principle for a one-shell model

Consider a model consisting of two portions of spherically symmetric vacuum space-times tailored together across the history of a time-like hyper-surface. As was already shown in [9], the dynamics of the shell+gravity system can be derived from first principles performing the variation of an appropriate Hilbert action $A$ composed of a gravitational part, a matter part and a boundary part. The gravitational part of the action accounts for the Ricci scalar and the jump of the extrinsic curvature across the shell. It is, therefore, divided into a regular part outside of the shell and a singular part on the shell ($A_{\text{grav}} = A_{\text{reg}}^{\text{grav}} + A_{\text{sing}}^{\text{grav}}$). The matter part of the action describes energy density of the (possibly interacting) particles of the shell [10]. Finally, the boundary part is included to account for boundary phenomena [7]:

$$A = A_{\text{grav}} + A_{\text{matter}} + A_{\text{boundary}}.$$  \hspace{1cm} \text{(1)}

It was noted by many authors that the variation of the Hilbert action in the usual framework (without boundary terms) presents contributions on the boundary together with the standard volume parts (that give Euler-Lagrange equations). Therefore, in order to obtain a true variational principle for the action those boundary terms must be killed. Different control modes at the boundary lead to different Hamiltonian formulations. All of them imply the same equations of motion. However, only one of them has the property that the Hamiltonian function represents the total energy for the system as seen by a faraway observer (because corresponds to ‘adiabatic insulation’ at infinity, cf. [7]). The boundary part of the action, corresponding to the above ‘true Hamiltonian formulation’, takes the following form:

$$A_{\text{boundary}} = \frac{1}{16\pi} \int_{\partial T} (g_{AB} Q^{AB}),$$ \hspace{1cm} \text{(2)}

where $\partial T$ is the boundary of the world-tube $T$ containing the shell (which, at the end of the procedure, can be shifted to infinity, cf. [6]), $a, b = 0, 1, 2$ label coordinates on the boundary and $Q^{ab} := \sqrt{\det g}(Lg^{ab} - L^{ab})$, where by $\hat{g}_{ab}$ and $L_{ab}$ we denote the 3-geometry and the extrinsic curvature of $\partial T$. Moreover, $A, B = \vartheta, \varphi$ are angular coordinates, whereas $x^0$ is the time variable. The boundary contribution (eq. 2) acts as if
there was another shell of matter, whose radius is shifted at infinity, outside our system. Variation of the standard Hilbert action produces the boundary term $g_{ab} \delta Q^{ab}$ (cf. [7]) which vanishes only if we control the external curvature $Q^{ab}$ of the tube. In the spherically symmetric case this implies controlling the external Schwarzschild mass, i.e. controlling the energy (Hamiltonian) of the system. This way we are not able to obtain a true Hamilton-like variational principle but rather a Maupertuis-like principle.

Adding the boundary term (eq. 2) to the action converts the boundary contribution to the variation of the action into:

$$-g_{00} \delta Q^{00} + Q_{AB} \delta g^{AB},$$

i.e. changes the way we control the field at the boundary. This allows us to consider a broader family of initial field configurations. Namely, we may tailor an ‘interior’ Schwarzschild space-time

$$ds_1^2 = -\left(1 - \frac{2M_1}{r_1}\right)dt_1^2 + \left(1 - \frac{2M_1}{r_1}\right)^{-1} dr_1^2 + r_1^2 d\Omega^2$$

(3)

to an ‘exterior’ ‘Schwarzschild-like geometry’, with a time-depending mass parameter (cf. [6]):

$$ds_2^2 = -\left(1 - \frac{2M_2(t_2)}{r_2}\right)dt_2^2 + \left(1 - \frac{2M_2(t_2)}{r_2}\right)^{-1} dr_2^2 + r_2^2 d\Omega^2.$$  

(4)

This way we achieve a ‘true’ variational principle $\delta A = 0$ as a correct starting point for the Hamiltonian description of the shell dynamics, where the energy (mass) conservation: $M_2(t_2) = \text{const.}$ is not assumed a priori but is a consequence of the dynamics of the system (see again [6] for details).

Following [6] we now derive the Lagrangian picture for a single shell $\Sigma_2$. As the configuration variable of the system we choose the geometric radius $\psi_2$ of the shell. The Hilbert action for one shell separating the Schwarzschild interior (eq. 3) from the ‘Schwarzschild-like’ exterior (eq. 4) results as the integral (with respect to the exterior space-time $t_2$) of the total Lagrangian that, in general, depends upon the configuration variables $(\psi_2, \dot{\psi}_2)$, representing the mass parameter and its derivative with respect to $t_2$ (dotted quantities indicate derivatives with respect to the exterior time variable):

$$A = \int_{t_1}^{t_2} L(\psi_2, \dot{\psi}_2, M_2, \dot{M}_2) dt_2.$$

(5)

It turns out, however, that $L$ does not depend upon $\dot{M}_2$ and therefore, from Euler-Lagrange equation $\frac{dL}{dM_2} = 0$ it is possible to evaluate $M_2$ as a function depending upon the ‘true’ configuration variables $\psi_2, \dot{\psi}_2$:

$$M_2 = \frac{1}{2} \psi_2 \left\{1 - \left(1 - \frac{2M_1}{\psi_2} \cosh \mu_2 - \sqrt{\frac{m_2^2}{\psi_2^2} + \left(1 - \frac{2M_1}{\psi_2} \right) \sinh \mu_2^2}\right)^2\right\}. \quad (6)$$

Here, $m_2 = m_2(\psi)$ describes the total rest mass (including interaction energy of its particles, cf. [10]) of the shell and can be treated as its equation of state. Moreover,
by $\mu_2$ we denote the hyperbolic angle between the surfaces $t_1 = \text{const.}$ on the interior Schwarzschild side and the surfaces $t_2 = \text{const.}$ on the exterior side. The value of $\mu_2$ is given as an implicit function of $\psi_2, \dot{\psi}_2$ obtained from the junction conditions of the two metrics:

$$\frac{\sinh \mu_2}{\cosh \mu_2 - \sqrt{\frac{1 - 2M_2}{1 - 2M_1} \psi_2}} = \frac{\dot{\psi}_2}{1 - 2M_2 \psi_2} = 0.$$  \hspace{1cm} (7)

Once (eq. 6) is substituted into (eq. 5), the Lagrangian for the single shell system becomes [6]:

$$L = \psi_2 \dot{\psi}_2 \mu_2 - M_2,$$  \hspace{1cm} (8)

the momentum canonically conjugated to $\psi_2$ is $p_2 := \frac{\partial}{\partial \dot{\psi}_2} = \psi_2 \mu_2$ and, therefore, the Hamiltonian obtained from the Legendre transformation $H := p_2 \dot{\psi}_2 - L$ is equal to:

$$H = M_2.$$  \hspace{1cm} (9)

Equation $\dot{M}_2 = 0$ is, therefore, implied by the Hamiltonian dynamics of the system.

3. **Lagrangian and Hamiltonian for a system composed of two shells**

In order to consider a system composed of two gravitating shells (Figure 1) we shall follow a ‘constructive’ procedure. Namely, once the Lagrangian for a single shell $\Sigma_2$ is obtained, we add a second shell $\Sigma_1$ ‘inside’ the first one and derive the Lagrangian and Hamiltonian picture for the whole system. Our procedure extends in an obvious way to the system of $n$ gravitating shells.

The reason to follow this ‘constructive’ approach resides in the aforementioned analysis.

![Fig. 1: Two shells, $\Sigma_1$ and $\Sigma_2$, that separate three spherically symmetric space-times with different mass parameter ($M_0$, $M_1$ and $M_2$).](image)
of the variational principle. In fact, in order to derive the Hamiltonian function for one shell we must assume that the mass parameter of the ‘exterior’ space-time is another configuration variable depending on the time coordinate. Considering from the very beginning a system composed of two shells that separate Schwarzschild-like geometries with time-depending mass parameters, it will be in general impossible to construct regular Cauchy surfaces \( t_i = \text{const.} \) \((i = 0, 1, 2)\) over the whole manifold.

Adding the second shell \( \Sigma_1 \) to the system leaves the Lagrangian formulation for the external shell unchanged since the outermost portion of the space-time ‘doesn’t know’ what is happening in the interior part and sees the mass parameter \( M_1 \) as a constant. Consequently, the variational principle for the system composed of two shells is obtained in the footsteps of the first one considering the space-time (eq. 3) with \( M_1 = M_1(t_1) \) as the new Schwarzschild-like ‘exterior’ and a Schwarzschild space-time with \( M_0 = \text{const.} \) and coordinates \( \{t_0, r_0, \theta, \phi\} \) as the new ‘interior’.

Once again the resulting Lagrangian does not depend upon \( M_1, t_1 \) so that from the Euler-Lagrange equation for \( M_1 \) is possible to obtain \( M_1 \) explicitly:

\[
M_1 = \frac{1}{2} \psi_1 \left\{ 1 - \left( \sqrt{1 - \frac{2M_1}{\psi_1}} \cosh \mu_1 - \frac{m_1^2}{\psi_1^2} + \left( 1 - \frac{2M_0}{\psi_1} \right) \sinh \mu_1 \right)^2 \right\}.
\] (10)

Once more \( m_1 \) plays the role of an equation of state describing the interaction between the particles on the interior shell and \( \mu_1 \) is the hyperbolic angle this time between the surfaces \( t_1 = \text{const.} \) on the intermediate Schwarzschild side and the surfaces \( t_0 = \text{const.} \) on the new interior side. Again, the hyperbolic angle \( \mu_1 \) is given as an implicit function of \( \psi_1, \psi_1, t_1 \):

\[
\frac{\sinh \mu_1}{\cosh \mu_1} = \frac{\psi_1, t_1}{1 - \frac{2M_1}{\psi_1}} = 0.
\] (11)

The total action for the system composed of two shells evaluated in the ‘external’ time-frame \( t_2 \) can finally be rewritten as:

\[
A = \int_{t_2}^{t_2} L(\psi_1, \dot{\psi}_1, \psi_2, \dot{\psi}_2)dt_2 = \int_{t_2}^{t_2} (\psi_1 \dot{\psi}_1 + \psi_2 \dot{\psi}_2 - M_2)dt_2.
\] (12)

where \( M_2 = M_2(\psi_2, \mu_2(\psi_2, \dot{\psi}_2); M_1) \) and \( M_1 \) is now given from (eq. 10) as a function depending on the new configuration variables \( \psi_1, \dot{\psi}_1 \) and on the parameter \( M_0 \).

To evaluate the Hamiltonian of the system we must now calculate the momenta canonically conjugated to \( \psi_1 \) and \( \psi_2 \):

\[
p_1 = \frac{\partial L}{\partial \dot{\psi}_1} = \psi_1 \mu_1,
\] (13)

\[
p_2 = \frac{\partial L}{\partial \dot{\psi}_2} = \psi_2 \mu_2
\] (14)

and perform the Legendre transformation:

$$H = p_1 \dot{\psi}_1 + p_2 \dot{\psi}_2 - L = M_2.$$  \hspace{1cm} (15)

This proves that the value of our Hamiltonian function is equal to the total energy of the system (the external Schwarzschild mass) and it is constant due to the dynamics.

4. Conclusions

Starting from an appropriate Hilbert action that includes boundary terms, we have constructed the dynamics for a system composed of two gravitating massive thin shells and found that the corresponding Hamiltonian correctly reproduces the total energy of the system for an observer at infinity. The equations of motion are given by the usual Hamilton equations:

$$\dot{\psi}_1 = \frac{\partial H}{\partial p_1},$$  \hspace{1cm} (16)

$$\dot{\psi}_2 = \frac{\partial H}{\partial p_2},$$  \hspace{1cm} (17)

$$\dot{p}_1 = -\frac{\partial H}{\partial \psi_1},$$  \hspace{1cm} (18)

$$\dot{p}_2 = -\frac{\partial H}{\partial \psi_2},$$  \hspace{1cm} (19)

of which the first two are equivalent to the constraint equations (eqs. 11 and 7), while the last two give the entire dynamics of the system. Based on the choice of the state equations for the two shells, a variety of scenarios can occur, from complete gravitational collapse to shell crossing (eventually with some mechanism that describes energy transfer between the shells, cf. [11], [12]) to the formation of stable configurations. From an astrophysical perspective the future study of the dynamics of these multiple shells system shall give further insights into the still relatively open field of gravitational collapse and black hole or naked singularity formation. Furthermore, since our procedure can be iterated, we see how to extend the formalism to a system composed of \( n \) gravitating shells.

REFERENCES


