On theories of gravitation with nonsymmetric connection

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For a large class of theories of gravitation with nonsymmetric connection, based on general nonlinear Lagrangians, it is proved that the theory is equivalent to the standard Einstein theory of gravitation interacting with additional matter fields.

I. INTRODUCTION

In the previous papers we proved that theories of gravitation based on nonlinear Lagrangians depending on the Ricci tensor are essentially equivalent to the standard Einstein theory. All the effects due to nonlinear Lagrangians can be implemented by introducing additional matter fields. More precisely: to a collection of fields of a theory with a nonlinear Lagrangian \( L \) we are able to assign a new metric tensor and new matter fields in such a way that they satisfy the Euler-Lagrange equations derived from the original Lagrangian \( L \). The above result was proved for metric, metric-affine, and purely affine theories with a symmetric connection. In the present paper we show that this result can be easily extended to the case of nonsymmetric connections.

As a simple example we prove that Moffat's theory with both nonsymmetric metric and connection is equivalent to the Einstein theory of a symmetric metric tensor interacting with two additional matter fields. This is a mathematical result and does not depend on the physical (or philosophical) interpretation of the variables (e.g., the problem of whether the metric tensor we introduce is the "true" metric or not). The relevance of such a result consists in the fact, that it enables us to analyze the dynamical content of the theory (Cauchy problem, energy positivity, stability, etc.) using standard tools (results concerning the Cauchy problem for non-Einsteinian theories are usually extremely difficult and give very weak results of the Cauchy-Kowalewskaya type).

II. LAGRANGIAN THEORY OF NONSYMMETRIC CONNECTION

We begin with a purely affine theory based on a general (nonsymmetric) connection \( \Gamma \). It is known that the connection splits uniquely into the three independent geometric objects:

\[
\Gamma_{\mu}^{\lambda}_{\nu} = \Gamma_{\mu}^{\lambda}_{\nu} + \Delta_{\mu}^{\lambda}_{\nu} + \frac{1}{2} \delta_{\nu}^{\lambda} (A_{\nu} - \Gamma_{\nu}^{\sigma}_{\sigma}),
\]

where \( \Gamma \) is a symmetric connection (in holonomic coordinates: \( \Gamma_{\mu}^{\lambda}_{\nu} = \Gamma_{\nu}^{\lambda}_{\mu} \)), \( \Delta \) is a skew-symmetric, traceless tensor field \( (\Delta_{\mu}^{\lambda}_{\nu} + \Delta_{\nu}^{\lambda}_{\mu} = 0, \Delta_{\mu}^{\lambda}_{\lambda} = 0) \), and \( A_{\nu} = \Gamma_{\nu}^{\lambda}_{\sigma} \) is a connection in the bundle of scalar densities over space-time. We use the following notation for the curvature tensor \( R \) of the connection \( \Gamma \):

\[
R_{\mu}^{\lambda}_{\nu\sigma}(j^l \Gamma) = \Gamma_{\mu}^{\lambda}_{\nu\sigma} - \Gamma_{\mu}^{\lambda}_{\nu\rho} + \Gamma_{\rho}^{\lambda}_{\nu\sigma} - \Gamma_{\rho}^{\lambda}_{\nu\mu} + \frac{1}{2} \delta_{\nu}^{\lambda} (A_{\nu} - \Gamma_{\nu}^{\sigma}_{\sigma}).
\]

Here for a field of geometric objects \( f \) we denote by \( j^l f \) the first jet of \( f \) (the value of \( f \) and its derivatives). Whenever \( (f^\lambda_\mu) \) is a coordinate representation of the field \( f \) then \( (f^\lambda_\mu, f^\mu_\lambda) \) is the coordinate representation of \( j^l f \), where we denote \( f^\lambda_\mu = \partial_\mu f^\lambda, (e.g., \Gamma_{\mu}^{\lambda}_{\nu\rho} = \partial_\rho \Gamma_{\mu}^{\lambda}_{\nu}) \).

We consider the following three independent traces of the curvature:

\[
K_{\mu\nu} = R_{\mu}^{\lambda}_{\nu\lambda} = \frac{1}{2} (R_{\mu}^{\lambda}_{\lambda\nu} + R_{\nu}^{\lambda}_{\lambda\mu}),
\]

\[
F_{\mu\nu} = R_{\lambda}^{\lambda}_{\mu\nu},
\]

\[
P_{\mu\nu} = 2R_{\mu}^{\lambda}_{\nu\lambda} - \frac{1}{2} F_{\mu\nu} = R_{\mu}^{\lambda}_{\lambda\nu} - R_{\nu}^{\lambda}_{\lambda\mu} - \frac{1}{2} P_{\mu\nu}.
\]

We consider an affine theory based on an affine connection \( \Gamma_{\mu}^{\lambda}_{\nu} \) and a matter field \( \phi = (\phi^A) \). Moreover, we assume that the theory is based on a Lagrangian \( L \) depending on derivatives of the connection via the above traces of the curvature only:

\[
L(j^l \Gamma, j^l \phi) = L\phi (K_{\mu\nu}(j^l \Gamma), P_{\mu\nu}(j^l \Gamma), F_{\mu\nu}(j^l \Gamma), \Gamma_{\mu\nu}, j^l \phi).
\]

It is easy to calculate that

\[
K_{\mu\nu} (j^l \Gamma) = K_{\mu\nu} (j^l \Gamma) + \Delta_{\mu}^{\lambda}_{\nu} \Delta_{\nu}^{\sigma}_{\lambda},
\]

\[
F_{\mu\nu}(j^l \Gamma) = A_{\mu\nu} - A_{\nu\mu},
\]

\[
P_{\mu\nu}(j^l \Gamma) = P_{\mu\nu}(j^l \Gamma) + 2D^A \Delta_{\mu}^{\lambda}_{\nu},
\]

where by \( K_{\mu\nu} \) (\( P_{\mu\nu} \)) we denote the symmetric (skew-symmetric) part of the Ricci tensor of the symmetric connection \( \Gamma \):

\[
K_{\mu\nu} = R_{\mu\nu}(j^l \Gamma) = \frac{1}{2} (R_{\mu}^{\lambda}_{\lambda\nu} + R_{\nu}^{\lambda}_{\lambda\mu}),
\]

\[
P_{\mu\nu} = R_{\mu\nu}(j^l \Gamma) = \frac{1}{2} (R_{\mu}^{\lambda}_{\lambda\nu} - R_{\nu}^{\lambda}_{\lambda\mu}) = \frac{1}{2} (\Gamma_{\mu}^{\lambda}_{\nu\lambda} - \Gamma_{\nu}^{\lambda}_{\mu\lambda}),
\]

and by \( D^A \) we denote the covariant derivative with respect to the symmetric connection \( \Gamma \).

From the point of view of field dynamics, nonsymmetric connections do not introduce new phenomena; indeed, the torsion can always be incorporated into \( \phi \) as an additional matter field. This way our theory can be treated as an affine theory based on the symmetric connection \( \Gamma_{\mu}^{\lambda}_{\nu} \) interacting with the following "matter fields": \( \Delta_{\mu}^{\lambda}_{\nu}, A_{\mu\nu}, \) and \( \phi^A \).
following our method, we use the following notation for canonical momenta:

\[ \rho^{\mu\nu} = \rho^{(\mu\nu)} = \frac{\partial L_A}{\partial \dot{K}_{\mu\nu}} = \frac{\partial L_A}{\partial \dot{P}_{\mu\nu}}, \] (12)

\[ \tau^{\mu\nu} = \tau^{(\mu\nu)} = \frac{\partial L_A}{\partial \dot{P}_\mu}, \] (13)

\[ \phi^{\mu\nu} = \phi^{(\mu\nu)} = \frac{\partial L_A}{\partial \dot{F}_{\mu\nu}}, \] (14)

\[ p_\mu^\nu = \frac{\partial L}{\partial \phi^{\mu\nu}}, \] (15)

The above formulas do not define uniquely our currents unless we impose the symmetries corresponding to the symmetries of \( \Gamma \) and \( \Delta \). The currents \( j, d, \) and \( r \) are tensor densities. The character of the momentum \( \rho \) depends on the character of the matter field \( \phi \). For a tensorial matter field \( \phi \), the momentum \( \rho \) is a tensor density, too.

The Euler–Lagrange equation \( \delta L/\delta \Gamma = 0 \) has the form

\[ \frac{D}{D\tau^{\mu\nu}} + \frac{3}{2} \delta_\lambda^\nu \frac{D}{D\rho^\nu} \gamma^\lambda = \frac{1}{4} \partial_{\mu\nu} - \frac{1}{3} \partial_{\mu\nu} - \frac{1}{3} \partial_{\mu\nu}, \] (16)

The Euler–Lagrange equations for "matter" fields \( \Delta, A, \) and \( \phi \) can be written as

\[ \frac{D}{D\tau^{\mu\nu}} + \frac{3}{2} \delta_\lambda^\nu \frac{D}{D\rho^\nu} \gamma^\lambda = \frac{1}{4} \partial_{\mu\nu} - \frac{1}{3} \partial_{\mu\nu} - \frac{1}{3} \partial_{\mu\nu}, \] (17)

which is equivalent to \( \delta L/\delta A = 0 \) and

\[ \partial_\mu \omega^{\mu\nu} = - \frac{1}{2} r_\mu, \quad \partial_\nu p_\mu = p_\mu, \] (19)

which is equivalent to \( \delta L/\delta A = 0 \) and \( \delta L/\delta \phi = 0 \), respectively.

III. METRIC THEORY AS A RESULT OF LEGENDRE TRANSFORMATION

Now we perform the complete Legendre transformation between \( \rho \) and the traceless part \( \Sigma \) of \( \Gamma \) (\( \Sigma^{\mu\nu} = - \frac{1}{2} \delta_{\mu\nu} \Gamma^{\sigma\nu} - \frac{1}{2} \delta_{\mu\nu} \Gamma^{\sigma\mu} \)). Numerically, the transformation consists in subtracting the term \( \partial_\mu (\Sigma^{\mu\nu} \rho^{\nu\sigma}) \) from the Lagrangian. Then both \( \Sigma^{\mu\nu} \) and its derivatives have to be calculated in terms of \( \rho^{\nu\sigma} \), its derivatives and the fields that were not involved into the transformation. As a result of such an operation we obtain a noninvariant Lagrangian that can be improved by adding a complete divergence. Such an improved, second-order (in \( \rho^{\nu\sigma} \)) Lagrangian is already an invariant scalar density. Following the ideas of the purely affine theories of gravitation we interpret \( \rho \) as a dynamically defined metric tensor. More precisely, we define a new metric tensor \( h^{\mu\nu} \) and the inverse (contravariant) tensor \( h_{\mu\nu} \) by the formula

\[ \rho^{\nu\sigma} = - (1/2\kappa) \sqrt{- \det h_{\rho\sigma} h^{\mu\nu}}, \] (20)

where \( \kappa = 8\pi G \) is the gravitational constant. The definition is meaningful if we impose an additional assumption that \( \det(-\rho^{\nu\sigma}) \neq 0 \). We denote by \( \{\mu, \nu\} \) the Christoffel symbols of the metric \( h \) and by \( \nabla \) the corresponding covariant derivative. After the Legendre transformation, our theory may be treated as a metric theory based on the metric tensor \( h \) coupled to the following matter fields: \( \Delta, A, \phi, \) and \( a, \) where the covector field \( a_{\mu} := \Gamma^{\lambda}_{\mu \lambda} - (\_A)_{\mu} \) describes the nonmetricity of the trace of \( \Gamma \). It was proved that the new Lagrangian of the theory is equal numerically to the following expression:

\[ \mathcal{L} = \partial_\mu \left[ \left( a_{\mu, \nu} - \frac{1}{2} a_{\mu} a_{\nu} \right) \rho^{\nu\sigma} \right] + \partial_\mu \left( a_{\mu, \rho} \rho^{\rho\lambda} \right) + \mathcal{L}_\Delta. \] (21)

Due to (11) we have

\[ P_{\mu\nu} = \frac{1}{2} \left( a_{\mu, \nu} - a_{\mu} a_{\nu} \right). \] (22)

In order to eliminate the first jet of \( \mathcal{L} \) from the right-hand side of (21) we have to solve Eqs. (12) and the traceless part of (17) with respect to \( K_{\mu\nu} \) and \( \Sigma^{\mu\nu} \). Let us denote this solution by \( \mathcal{L}_{\mu\nu} \) and \( \Sigma_{\mu\nu} \), respectively. The formula (7) implies

\[ K^{\mu\nu} = \mathcal{L}^{\mu\nu} \left( j^j h^j h^j a_j \Delta_j, j^j A_j, j^j \phi \right) - \mathcal{L}_{\mu\nu} a_{\lambda} \partial_\lambda \Delta_\nu \partial_\sigma \Delta_\rho \partial_\tau \Delta_\omega. \] (23)

Due to the definitions of \( \Sigma^{\mu\nu} \) and \( a_{\mu} \) we have

\[ \mathcal{L}^{\mu\nu} = \mathcal{L}_{\mu\nu} \left( j^j h^j h^j a_j \Delta_j, j^j A_j, j^j \phi \right) + \frac{3}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} \Delta_{\rho\sigma} \partial_\rho \Delta_\sigma \partial_\tau \Delta_\omega. \] (24)

We introduce the following tensor-valued function:

\[ \theta^{\mu\nu} = \mathcal{L}_{\mu\nu} \left( j^j h^j h^j a_j \Delta_j, j^j A_j, j^j \phi \right) + \frac{3}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} \Delta_{\rho\sigma} \partial_\rho \Delta_\sigma \partial_\tau \Delta_\omega. \] (25)

Formula (24) implies that numerically \( \theta^{\mu\nu} = \Gamma^{\mu\nu} - \{\mu, \nu\} \) is the nonmetricity of \( \Gamma \).

Observe that derivatives of \( a_{\mu} \) and \( \Delta_{\rho\sigma} \) enter into the original Lagrangian via \( P_{\mu\nu} \) only. Rewriting formula (9) in terms of the new covariant derivative \( \nabla \) and using (22) and (25) we have

\[ P_{\mu\nu} = \frac{1}{2} \left( a_{\mu, \nu} - a_{\mu} a_{\nu} \right) + 2 \mathcal{L}_{\mu\nu} \left( \frac{1}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} \Delta_{\rho\sigma} \partial_\rho \Delta_\sigma \partial_\tau \Delta_\omega \right) \] (26)

It is convenient to code information carried by the tensor \( \Delta_{\rho\sigma} \) and the covector \( a_{\mu} \) into a single tensor field:

\[ B^{\mu, \nu} := 2 \mathcal{L}_{\mu\nu} \frac{\delta^{\mu\nu} - \delta^{\mu\nu} \Delta_{\rho\sigma} \partial_\rho \Delta_\sigma \partial_\tau \Delta_\omega}{\delta^{\mu\nu} - \delta^{\mu\nu}}, \] (27)

We see that derivatives of \( B \) enter into the theory via the following skew-symmetric tensor field only:

\[ B^{\mu, \nu} = \nabla_\nu B_{\mu}. \] (28)

Now formula (26) can be rewritten as

\[ P_{\mu\nu} = B_{\mu, \nu} - \partial_\nu \Delta_{\mu, \nu} - \partial_\nu \partial_\rho \Delta_\rho, a_{\mu, \nu} \] (29)

We have therefore \( \mathcal{L}_{\mu\nu} = \mathcal{L}_{\mu\nu} \left( j^j h^j B^j F, B^j A, j^j \phi \right) \) and \( \theta^{\mu\nu} = \mathcal{L}_{\mu\nu} \left( j^j h^j B^j F, B^j A, j^j \phi \right) \). It was proved that the new invariant Lagrangian \( \mathcal{I} \) given by (21) is the Einstein–Hilbert Lagrangian of the metric \( h \) with the matter part depending on matter fields \( A, B, \) and \( \phi \).
The function \( \mathcal{L} \) is defined by (25). We stress that the quantities \( P \) and \( F \) are given by (29) and (8), respectively.

However, the definition of \( \mathcal{L} \) was possible only after the first Legendre transformation. To the traceless part \( \Delta \) of the torsion we add a part of its trace represented by the covector \( a_\mu \). However, the definition of \( a_\mu \) depends on the metric tensor \( h_{\mu \nu} \). The transformation from the initial variables \( h, A, B, \phi \) to the new variables \( h, A, B, \phi \) is not a "point transformation" but is a canonical transformation. This way we have shown that the theory with the affine Lagrangian (6) is equivalent to the Einstein theory based on the Einstein–Hilbert Lagrangian (30).

However, it is possible to simplify further the theory reducing the number of independent matter fields. For this purpose we perform the Legendre transformation between the configuration \( B_\mu ^\lambda \), and the momentum \( \tau ^{\mu \nu} \). This way we replace a 24-component matter field \( B \) by a 6-component field \( \tau \). The transformation consists in subtracting the term \( \nabla_\lambda (\tau ^{\mu \nu} B_\mu ^\lambda ) = \partial_\lambda (\tau ^{\mu \nu} B_\mu ^\lambda ) \) from the Lagrangian and calculating \( \langle B_\mu ^\lambda v, \nabla_\lambda B_\mu ^\lambda \rangle \) in terms of \( (\nabla_\lambda \tau ^{\mu \nu}, \tau ^{\mu \nu}) \). We use for this purpose the equation

\[
\frac{\partial \mathcal{L}}{\partial B_\mu ^\lambda} = \tau ^{\mu \nu},
\]

equivalent to (13) and the Euler–Lagrange equation

\[
\frac{\partial \mathcal{L}}{\partial B_\mu ^\lambda} = \nabla_\lambda \tau ^{\mu \nu}.
\]

equivalent to (18) and the trace of (17). Finally we obtain Einstein theory of the metric \( h \) interacting with the following matter fields: \( \tau, A, \phi \). The field equations of the theory can be derived from the Einstein–Hilbert Lagrangian with the following Legendre transformation:

\[
\mathcal{L} = -\frac{1}{2}\sqrt{-\det h_{\alpha \beta}} h^{\mu \nu} \mathcal{K}_{\mu \nu} (\dot{f} h) = \sum_\text{mat} (j^f h, B, F, A, j^f \phi),
\]

where

\[
\mathcal{L}_\text{mat} = \frac{1}{2\kappa\sqrt{-\det h_{\alpha \beta}}} h^{\mu \nu} \left[ \mathcal{S}_{\mu \nu} - \mathcal{A} \partial_\lambda \left( \tau ^{\mu \nu} B_\mu ^\lambda \right) \right] + L_A (\mathcal{S}, \mathcal{P}, \mathcal{F}, \mathcal{G}, B, A, j^f \phi) - \nabla_\lambda \left( \tau ^{\mu \nu} B_\mu ^\lambda \right).
\]

We remember that the above numerical value of \( \mathcal{L}_\text{mat} \) has to be expressed in terms of \( \tau, A, \phi, \) and other derivatives.

For calculational convenience the last Legendre transformation between \( B \) and \( \tau \) can also be performed in two steps. First we perform the Legendre transformation between \( a_\mu \) and \( \tau ^{\mu \nu} \). Then we subtract the complete divergence \( \partial_\mu \left( 2\tau ^{\mu \nu} \Delta_{\nu \sigma} \right) \) from the Lagrangian. Finally we observe that the Lagrangian depends algebraically on \( \Delta \). Equations \( \delta \mathcal{L} / \delta \Delta = 0 \) are therefore algebraic with respect to \( \Delta \). In a generic situation these equations can be treated as a definition of \( \Delta \) in terms of \( j^f h \) and \( j^f \tau \).

IV. SPECIAL CASE: LAGRANGIAN WITHOUT POTENTIALS

As an example of the above procedure we consider a Lagrangian (6) which does not depend on \( \Gamma _{\nu} ^{\lambda \mu} \). We have therefore \( j_{\nu} ^f \tau ^{\nu} = 0 \) and \( a_\mu ^{\lambda \nu} = 0 \). One can check that the nonmetricity Eq. (17) for \( \Gamma \) can be solved algebraically with respect to \( \Delta \). This way we obtain an explicit form of the function \( \mathcal{S} \). Putting this solution into formula (25) we get the following expression:

\[
\mathcal{S}_\text{mat} (j^f h, B, F, A, j^f \phi) = \left( 1/2\kappa \sqrt{-\det h_{\alpha \beta}} \right) (h^{\mu \nu} \mathcal{S}_{\mu \nu} + 3 h^{\mu \nu} a_\mu a_\nu) + \mathcal{W}_1 (h, \Delta, \tau) + L_A (\mathcal{S}, \mathcal{P}, \mathcal{F}, \mathcal{G}, B, A, j^f \phi),
\]

where by \( U(h, \Delta) \) and \( W_1 (h, \Delta, \tau) \) we denote the algebraic functions of the metric \( h \) (the traceless part of \( B \)) and of \( \tau = \tau (\mathcal{S}, \mathcal{P}, \mathcal{F}, \mathcal{G}, B, A, j^f \phi) \).

Now we perform the Legendre transformation between the configuration \( B \) and the momentum \( \tau \). The new matter Lagrangian \( \mathcal{L}_\text{mat} \) equals

\[
\mathcal{L}_\text{mat} = \mathcal{L}_\text{mat} (j^f h, j^f \tau, F, A, j^f \phi) = \mathcal{S}_\text{mat} - \partial_\lambda \left( \tau ^{\mu \nu} B_\mu ^\lambda \right) = \mathcal{S}_\text{mat} - \tau ^{\mu \nu} P_{\mu \nu} + (\partial_\lambda \tau ^{\mu \lambda}) a_\mu - 2 \mathcal{V}_\lambda (\tau ^{\mu \nu} \Delta_{\nu \sigma} ^\lambda).
\]

In order to express the Lagrangian in terms of legal variables we have to solve Eqs. (13) with respect to \( B \). Moreover, we have to solve Eqs. (18) and the trace of (17) with respect to \( B \) (i.e., \( \Delta \) and \( a \)). Let us denote the algebraic solution of (13) with respect to \( B \) by \( \mathcal{S} \). Due to (29) we have

\[
B_{\mu \nu} = \mathcal{S}_{\mu \nu} (j^f h, j^f \tau, j^f A, j^f \phi) + \mathcal{S}_{\mu \sigma} B_\sigma ^\lambda + \mathcal{S}_{\sigma \nu} B_\nu ^\lambda - 3 B_\sigma ^\lambda B_\sigma ^\lambda + \mathcal{S}_{\mu \sigma} B_\sigma ^\lambda - 3 B_\sigma ^\lambda B_\sigma ^\lambda.
\]

The trace of Eq. (17) can be rewritten as

\[
a_\mu = \frac{4\kappa}{\sqrt{-\det h_{\alpha \beta}}} (h_{\alpha \nu} \partial_\nu \tau ^{\mu \nu} - \partial_\nu h_{\alpha \nu} \partial_\nu \tau ^{\mu \nu}).
\]

Equation (18) with \( \Gamma _{\nu} ^{\lambda \mu} \) defined by (24) is linear with respect to \( \Delta_{\nu \sigma} ^\lambda \). Let \( \Delta_{\nu \sigma} ^\lambda = \Delta_{\nu \sigma} ^\lambda (j^f h, j^f \tau) \) be the solution of this equation. Easy calculations lead to the following result:
\[ L_{\text{mat}} = (1/2k) \sqrt{-\det h_{ab}} \, h^{\mu \nu} \mathcal{R}_{\mu \nu} - \tau^{\alpha \beta} \mathcal{R}_{\mu \nu} \]

\[ + L_4 (\mathcal{S}, \mathcal{J}, \mathcal{F}, \partial, \Delta, J, \phi) \]

\[ + (4k/\sqrt{-\det h_{ab}}) (h, \partial, \tau^{\alpha \beta} \mathcal{M}_{\mu \nu} \phi, \partial_\nu \tau^{\alpha \beta}) \]

\[ - \frac{1}{3} h_{aa} \partial_\alpha \partial^{\alpha \beta} \partial_\mu \phi \]

\[ - 2 \Delta^{\alpha \beta} \nabla_\tau \tau^{\mu \nu} + U(h, \Delta) - W_2 (h, \Delta, \tau) \quad (42) \]

where by \( W_2 (h, \Delta, \tau) \) we denote the algebraic function of the fields \( h, \tau \) and of \( \Delta = (J h, J^\tau) \):

\[ W_2 = W_1 (h, \Delta, \tau) + (12k/\sqrt{-\det h_{ab}}) h^{\mu \nu} h_{\alpha \beta} \lambda_{\eta \eta} \]

\[ \times \tau^{\mu \nu} \tau^{\alpha \beta} \Delta^{\epsilon \delta}_{\mu \nu} \Delta^{\eta \eta}_{\epsilon \delta} \]

\[ = (4k/\sqrt{-\det h_{ab}}) \tau^{\mu \nu} \tau^{\alpha \beta} (h_{\alpha \beta} \Delta^{\epsilon \delta}_{\mu \nu} \Delta^{\eta \eta}_{\epsilon \delta} \phi) \]

\[ + 2h_{\alpha \beta} \Delta^{\epsilon \delta}_{\alpha \beta} \Delta^{\eta \eta}_{\epsilon \delta} \phi \]

\[ + h_{\alpha \beta} \Delta^{\epsilon \delta}_{\alpha \beta} \Delta^{\eta \eta}_{\epsilon \delta} \phi - h^{\mu \nu} [h_{\alpha \beta} \Delta^{\epsilon \delta}_{\mu \nu} \Delta^{\eta \eta}_{\epsilon \delta} \phi] \]

\[ + (h_{\alpha \beta} h_{\gamma \eta} - h_{\alpha \gamma} h_{\beta \eta}) \Delta^{\epsilon \delta}_{\mu \nu} \Delta^{\eta \eta}_{\epsilon \delta} \phi] \quad (43) \]

**V. MOFFAT’S THEORY**

The above considerations can be applied to Moffat’s theory (the gravitation theory with a nonsymmetric “metric”). The theory is based on the following Lagrangian \( L_{\text{mat}} \):

\[ L_{\text{mat}} = g^{\mu \nu} \mathcal{B}_{\mu \nu} (j^\Gamma \Gamma) + L_{\text{mat}} (g^{\mu \nu}, W_\mu, j^\phi) \quad (44) \]

where \( g^{\mu \nu} \) is a (nonsymmetric) tensor density,

\[ \mathcal{B}_{\mu \nu} (j^\Gamma \Gamma) = R_{\mu \nu} (j^\Gamma \Gamma) + \frac{1}{2} (\mathcal{E}^\alpha \mu - \mathcal{E}^\alpha \mu) \quad (45) \]

and

\[ W_\mu = \frac{2}{3} (\mathcal{E}^\alpha \mu - \mathcal{E}^\alpha \mu) = \frac{2}{3} (\mathcal{E}^\alpha \mu - A_\mu) \quad (46) \]

The variation is meant with respect to \( \Gamma \) and \( q \) independently. Here \( L_{\text{mat}} \) is a Lagrangian density for the “phenomenological matter sources.”

Due to the decomposition of \( g^{\mu \nu} \) into symmetric and skew-symmetric parts:

\[ \rho^{\mu \nu} = g^{(\mu \nu)} \quad (47) \]

\[ \tau^{\mu \nu} = - \frac{1}{2} g^{(\mu \nu)} \quad (48) \]

we can rewrite formula (44):

\[ L_{\text{mat}} = \rho^{\mu \nu} K_{\mu \nu} (j^\Gamma \Gamma) + \tau^{\mu \nu} [P_{\mu \nu} (j^\Gamma \Gamma) - 2 F_{\mu \nu} (j^\Gamma \Gamma) + 6 \Delta^{\alpha \beta} \Delta^{\mu \nu} + L_{\text{mat}} \quad (49) \]

This Lagrangian differs from (6) by the term containing the derivatives of the torsion. However, the derivatives enter via the same combinations as the ones contained already in \( P_{\mu \nu} \). Therefore, our techniques can be applied also in this case.

The variation with respect to \( q \) can be replaced by the independent variations with respect to \( \rho \) and \( \tau \). The Euler-Lagrange equations \( \delta L / \delta A = 0 \), \( \delta L / \delta \Gamma = 0 \), and \( \delta L / \delta \Delta = 0 \) have the form:

\[ \partial_\rho \tau^{\mu \nu} = \rho^{\mu \nu} \quad (50) \]

\[ D_\rho \tau^{\mu \nu} = - 8 \Delta_{\mu \nu} \rho^{\alpha \beta} \tau^{\alpha \beta} \quad (51) \]

\[ D_\alpha \tau^{\mu \nu} + \frac{1}{8} \partial_\alpha \tau^{\mu \nu} = \frac{1}{2} \delta_{\alpha \mu} \rho^{\nu \nu} \quad (52) \]

Equation (51) implies

\[ \delta_{\mu \nu} = -(8k/\sqrt{-\det h_{ab}}) h_{\alpha \beta} \Delta^{\mu \nu} \rho^{\alpha \beta} \quad (53) \]

Comparing the above result with formula (41) we see that \( \delta_{\mu \nu} \) is already expressed algebraically. Consequently, after the first Legendre transformation between \( \Sigma_\mu ^{\alpha \beta} \) and \( \rho^{\mu \nu} \), reexpressing \( D_\alpha \Delta^{\mu \nu} \) in terms of \( j^\rho, \tau, j^\Delta \) and using (53) we obtain the following numerical value of the matter Lagrangian:

\[ \Omega_{\text{mat}} = \tau^{\mu \nu} [P_{\mu \nu} (j^\Gamma \Gamma) - 2 F_{\mu \nu} (j^\Gamma \Gamma) - 4 \Delta^{\alpha \beta} \Delta^{\mu \nu} + U(h, \Delta) - 4 W_2 (h, \Delta, \tau) + L_{\text{mat}} \quad (54) \]

Moreover, it is convenient to use the field \( W_\mu \) instead of a couple of fields \( A_\mu \) and \( \Gamma^{\alpha \beta} \). Finally, adding the complete divergence \( 4V_\mu (\tau^{\mu \nu} \Delta^{\rho \mu}) \) we obtain the matter Lagrangian for the Moffat’s theory:

\[ \mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}} (j^\rho, j^\tau, j^\Delta, W_\mu, j^\phi) \]

\[ = - \frac{1}{2} \delta_{\mu \nu} W^{\mu \nu} + 4 \Delta^{\alpha \beta} \nabla_\gamma \tau^{\mu \nu} + U(h, \Delta) - 4 W_2 (h, \Delta, \tau) + L_{\text{mat}} \quad (55) \]

We notice that \( \Delta \) enters only algebraically into the above Lagrangian. Equations \( \delta L / \delta \Delta = 0 \) are therefore algebraic equations for \( \Delta \). In a generic situation these equations can be treated as a definition of \( \Delta \) in terms of \( j^\rho, \tau, j^\Delta \). This way we can eliminate \( \Delta \) as an independent variable of the theory. Analytically, the function \( \Delta (j^\rho, j^\tau, j^\Delta) \) is very complicated. Finally, we obtain the theory with two independent matter fields: \( \tau^{\mu \nu} \) and \( W_\mu \) coupled to the (symmetric) metric tensor \( h_\mu \). The field equations can be derived from the following Einstein–Hilbert Lagrangian:

\[ \mathcal{L}(j^\rho, j^\tau, j^\Delta, W_\mu) = -(1/2k) \sqrt{-\det h_{ab}} \, h^{\mu \nu} K_{\mu \nu} (j^\rho) \]

\[ + \frac{1}{2} \delta_{\mu \nu} \rho^{\alpha \beta} (j^\rho) \quad (56) \]

Observe, that both metric tensor \( h \) and matter fields \( \tau \) and \( W_\mu \) are original Moffat’s fields. We therefore proved, that subtracting the complete divergence from the Lagrangian, and reexpressing \( \Gamma \) in terms of \( (h, \tau, W) \) Moffat’s theory becomes Einstein theory.

The situation here is similar as in classical mechanics, where the variational formula based on the Lagrangian \( L = - \dot{q} \dot{p} - H(q, p) \) can be used with \( p \) and \( q \) independent. Instead, it is better to add the complete time derivative \((d/dt) (pq)\) and express \( p \) in terms of \( q \), using the part of equations of motion. This way we prove that \( p \) are not independent degrees of freedom of the theory.


