ON POSITIVITY OF GRAVITATIONAL ENERGY

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A simple argument for energy positivity is presented.

1. INTRODUCTION

Positivity of gravitational energy has already been proved by several different methods but still there is some mystery about it. I would like to present here a simple argument which, in my opinion, highly clarifies the problem. It turns out that the surface integral defining the ADM energy can be replaced by a volume integral which contains manifestly positive and manifestly negative terms only. The gauge freedom enables us to annihilate the latter.

We consider the simplest case of a Cauchy surface \( \Sigma \) which is topologically \( \mathbb{R}^3 \). Cauchy data on \( \Sigma \) are described by an asymptotically flat positive 3-metric tensor \( g_{kl} \) and an ADM momentum \( p_{kl} \). By asymptotic flatness we mean the condition which is strong enough to eliminate the supertranslations:

\[
g_{kl} = f(\delta_{kl} + \rho_{kl})
\]

where \( \delta_{kl} \) is the flat metric, \( \rho = O(1/r^5) \), \( \lambda \rho = O(1/r^4) \). The conformal factor \( f \) is canonically conjugate to the gauge fixing parameter \( p = \xi^{kl} g_{kl} \). From the symplectic point of view it is therefore illegal to impose any condition on the asymptotic behaviour of \( f \) except its asymptotic value at infinity which we may fix to be 1. This information is sufficient to calculate \( f \) from Lichnerowicz equation.

Our method is based on 2+1 decomposition of the Cauchy surface \( \Sigma \). We suppose that there exists a foliation \( \{\Sigma_\tau\} \) of \( \Sigma \) by a family of 2-surfaces \( \Sigma_\tau \) parametrized by the real parameter \( \tau \). Consider a new metric

\[
h_{kl} = e^{-\phi} g_{kl}
\]

which is conformally equivalent to \( g \). We choose the function \( \phi \) in such a way that the following condition is satisfied:

\[
|\nabla \tau| = 1
\]

For the sake of simplicity choose a coordinate system \((x^1, x^2, x^3)\) compatible with the foliation, i.e. such that \( x^2 = \tau \) and \((x^1, x^2)\) is a coordinate chart on each leaf \( x^3 = \text{const.} \)
In terms of these coordinates condition (3) reads
\[ |\nabla \tau|^2 = h^{kh}(\partial_k \tau)(\partial_h \tau) = h^{33} = 1 \]  \hspace{1cm} (4)
and therefore
\[ \psi = -\log g^{33} \]  \hspace{1cm} (5)
The conformal rescaling of the metric induces the following change of the scalar curvature:
\[ e^\psi R(g) = R(h) - \frac{1}{2} h^{kl} (\partial_k \phi)(\partial_l \phi) - 2\Delta \phi \]  \hspace{1cm} (6)
where the Laplace-Beltrami operator \( \Delta \) is taken with respect to the metric \( h \):
\[ \Delta \phi = \frac{1}{\sqrt{\det h}} a_k (\sqrt{\det h} h^{kl} \partial_l \phi) = \nabla_k \nabla^k \phi \]  \hspace{1cm} (7)
Also the covariant derivative \( \nabla \) is taken with respect to \( h \).

The metric \( h^{kl} \) induces a metric \( \gamma_{AB} \); \( A,B = 1,2 \); on each leaf \( \Sigma_\tau \). In our system of coordinates we have
\[ \gamma_{\alpha B} = h_{\alpha B} \]  \hspace{1cm} (8)
By \( \kappa_{AB} \) we denote the second fundamental form (extrinsic curvature tensor) of each leaf \( \Sigma_\tau \) with respect to the metric \( h \):
\[ \kappa_{AB} = (h^{33})^{-1/2} \gamma_{AB} = -\nabla_A \nabla_B \tau \]  \hspace{1cm} (9)
The curvature \( R(h) \) can be calculated in terms of geometry of leaves:
\[ R(h) = R(\gamma) + \kappa^2 - \kappa_{AB} \kappa^{AB} + \nabla_k (\kappa \nabla^k \tau + a^k) \]  \hspace{1cm} (10)
where \( R(\gamma) \) is the curvature of a 2-geometry, \( \kappa \) is the trace of \( \kappa_{AB} \) with respect to \( \gamma \) and
\[ a^k = (\nabla_l \nabla^l \tau)(\nabla^k \tau) \]  \hspace{1cm} (11)
Due to the equality
\[ 1 = h^{33} = h^{kl}(\partial_k \tau)(\partial_l \tau) = (\nabla_k \tau)(\nabla^k \tau) \]  \hspace{1cm} (12)
we have
\[ a^k \nabla_k \tau = a^3 = 0 \]  \hspace{1cm} (13)
Now we combine formulae (6) and (10). Moreover, we use the scalar constraint which is satisfied by our Cauchy data:
\[ R(g) = \frac{1}{\det g} (p^k \nabla^L_k - \frac{1}{2} p^2) = \frac{1}{\det h} \frac{e^{2\psi}}{p^2} (p^k \nabla^L_k - \frac{1}{2} p^2) \]  \hspace{1cm} (14)
We obtain finally:
\[ 2 \kappa^2 \nabla^L_k (\partial_k \phi)(\partial^L_k \phi) + \nabla^L_{\kappa} R(\gamma) = \frac{1}{\sqrt{\det h}} e^{-2\psi} p^k \nabla^L_k \]
\[ + \frac{1}{2} \sqrt{\det h} (\partial_k \phi)(\partial^L_k \phi) + \sqrt{\det h} \kappa_{AB} \kappa^{AB} - \sqrt{\det h} \kappa^2 = \frac{1}{2e^{2\psi} \sqrt{\det h}} p^2 \]  \hspace{1cm} (15)
We are going to prove in the sequel that the integral of the left-hand side of the above equation is equal to the energy. To annihilate the negative terms on the right-hand side we use the gauge freedom in choosing both the Cauchy surface $\Sigma$ and its foliation $\{\Sigma_t\}$. Assuming maximality of $\Sigma$ we obtain $E \neq 0$ as the last term vanishes. To analyze the term containing $\kappa$ observe that

$$\sqrt{\det h} \kappa = \sqrt{\det h} \gamma^{AB} \nabla_A \nabla_B \kappa = \sqrt{\det h} \nabla_k \nabla^k \kappa =$$

$$= \partial_k (\sqrt{\det h} \nabla^k \kappa) \kappa = \partial_k (\sqrt{\det h} |\nabla\kappa|) \nabla^k \kappa = \partial_k D^k \kappa$$

The vector density $D^k \kappa$ will be called the conformal gradient density of the function $\kappa$. It is invariant with respect to conformal rescalings of the metric:

$$D^k \kappa = \sqrt{\det h} \sqrt{\det g} (\partial^m (\partial^m \kappa)) |\nabla\kappa| \kappa^{kl} \partial_k \nabla_l \kappa =$$

$$= \sqrt{\det g} \sqrt{\det h} (\partial^m (\partial^m \kappa)) g^{kl} \partial_k \nabla_l \kappa$$

The condition $\kappa = 0$ is equivalent to the 2-order nonlinear elliptic equation for the function $\kappa$:

$$\partial_k D^k \kappa = 0$$

The function which fulfills (18) will be called conformally harmonic. The operator $\partial_k D^k \kappa$ is the only conformally invariant second order operator which maps scalar functions to scalar densities and may be called the conformal generalization of the Laplacian. The equation (18) can be derived from the variational principle $\delta L = 0$ with the Lagrangian

$$L(\nabla \kappa) = \sqrt{\det h} |\nabla\kappa|^3$$

We now consider two important topological cases corresponding to different boundary conditions for the equation (18).

2. TOPOLOGICALLY FLAT FOLIATIONS

Suppose that $x^k$ is one of asymptotically flat coordinates $x^k$. In this case each leaf $\Sigma_t$ is topologically $\mathbb{R}^2$. Due to the Gauss-Bonnet theorem the integral of $R(\gamma)$ over $\Sigma_t$ vanishes. Integrating both sides of (15) over the whole $\Sigma$ we obtain the positivity of the quantity $E$ defined by the formula:

$$4\pi E = 2\int_{S} \sqrt{\det h} (\partial^k - \partial^k \partial \varphi) \partial_k \varphi =$$

$$= \int_{\Sigma} \left\{ \frac{1}{\sqrt{\det h}} e^{-\varphi} \partial^l \partial^k \partial^l \partial_k + \frac{1}{2} \sqrt{\det h} (\partial^k \varphi)(\partial^k \varphi) + \sqrt{\det h} \kappa_{AB} \kappa^{AB} \right\} \partial^2 \varphi$$

where $S$ is the sphere at infinity. Due to asymptotic flatness we have $\partial^k \varphi = 0(1/r^3)$
The first term gives thus no contribution to the surface integral at infinity. Consequently
\[ 4\pi E = -2\int_{\Sigma} \sqrt{\text{det} g} \ (\partial K) \ d\sigma K \geq 0 \] (21)

Moreover, \( E = 0 \) implies \( F^{kl} = 0 \), \( \kappa_{AB} = 0 \) and \( \phi = \text{const} \) due to Lichnerowicz equation (15). This happens in the case of Minkowski space-time only.

The Lichnerowicz equation implies the asymptotic behaviour of \( \phi \):
\[ \phi = \frac{2E}{r} + O(1/r^2) \] (22)
or equivalently
\[ g_{kl} = \left( 1 + \frac{2E}{r} + O(1/r^2) \right) \left( \delta_{kl} + O(1/r^2) \right) \] (23)
The reader can easily check that \( E \) is equal to the ADM energy.

3. NESTED SPHERES

The function \( \tau = \log r, \ r = (x^2 + y^2 + z^2)^{1/2} \), fulfills the equation (18) in flat space \( \mathbb{R}^3 \). In general case we are looking for a function \( \tau \) which behaves asymptotically like \( \log r \) at infinity and at a chosen point \( x_0 \). Surfaces \( \Sigma_\tau \) are topologically 2-spheres. Denote \( r = e^\tau \). We have
\[ \Theta^{33} = (\nabla \tau | \nabla \tau) = \frac{1}{r^2} (\nabla r | \nabla r) = \frac{1}{r^2} e^{-2\tau} \] (24)

Denote \( \psi = -\log g^{rr} \). We have \( \phi = \psi + 2\tau = \psi + 2x \) and
\[ e^{x^2} \partial_\tau (\sqrt{\text{det} h} \nabla^\tau \psi) = e^{x^2} \partial_\tau (\sqrt{\text{det} h} \nabla^\tau \psi) + 2 e^{x^2} \partial_\tau \partial^\tau \psi \] (25)

Due to equation (18) the function \( \phi \) can be replaced by \( \psi \) in the right-hand side of the equation (15). We multiply both sides of (15) by \( r \). We have
\[ e^{x^2} \partial_\tau (\sqrt{\text{det} h} (a^k + \kappa e^\tau - \kappa \psi)) = e^{x^2} \partial_\tau (\sqrt{\text{det} h} (a^k + \kappa e^\tau - \kappa \psi)) \]
\[ - \sqrt{\text{det} h} (a^k + \kappa e^\tau - \kappa \psi) \delta_\tau \delta^\tau \] (26)

Equation (13) is again valid. Moreover, we use equation (25).

Due to Gauss-Bonnet theorem for 2-spheres the integral of the scalar curvature term over each \( \Sigma_\tau \) is equal to:
\[ \int_{\Sigma_\tau} \sqrt{\text{det} \gamma} R(\gamma) = \int_{\Sigma_\tau} 2\sqrt{\text{det} \gamma} \] (27)

On the other hand
\[ \sqrt{\text{det} h} (\partial_\tau \psi) (a^k) = \sqrt{\text{det} h} \left\{ (\partial_\tau \psi) (a^k) + \delta (a^k) + 4 \right\} \] (28)

The last term of the above expression multiplied by the factor \( \frac{1}{2} r \) cancels the quantity (27) when integrated over \( \Sigma_\tau \). The second term cancels the second term of (26). Putting together all these observations and integrating over \( \Sigma \) we obtain a manifestly positive quantity.
\[ d\pi E = - \lim_{\tau \to \infty} \int_{\Sigma_\tau} (r \sqrt{\det h} \ 3^2 \psi_x^2) dx^1 dx^2 = \]
\[ = \int \left\{ \frac{r^{-3}}{\sqrt{\det h}} e^{-2\psi} \frac{\partial^2}{\partial r^2} + \frac{r^2}{2} \sqrt{\det h} (\partial_r \psi^2) + r \sqrt{\det h} \nabla^A \nabla^B \right\} dx^3 \]

which is equal to ADM energy. Again
\[ \psi = \frac{2E}{r} + O(1/r^2) \]

and
\[ g_{kl} = r^2 e^{2\psi} (\eta_{kl} + O(1/r^2)) \]

where \( \eta_{kl} \) represents the flat metric in \( R^3 \)
\[ ds^2 = r^2 (dt^2 + d\Omega^2) \]

and \( d\Omega^2 \) is the standard metric on the unit sphere \( S^2 \).

4. CONCLUDING REMARKS

The highly nonlinear equation (18) has been thoroughly investigated by P. Chruściel\(^3,4\). These results are very encouraging. It seems plausible to expect the existence of the function \( \psi \) (both "flat" and "radial") satisfying equation (18) in all reasonable cases, at least when the metric \( g_{kl} \) does not differ too much from the flat metric. The method described here enables us to describe in a simple way the dynamical (unconstrained) degrees of freedom of the gravitational field and to analyze the dynamical role of the energy as a hamiltonian\(^5\).

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