On theories of gravitation with nonlinear Lagrangians

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It is shown that any theory of gravitation with a nonlinear Lagrangian depending on the Ricci tensor is equivalent to the Einstein theory of gravitation interacting with additional matter fields.

Consider a metric theory of gravitation (possibly interacting with a bosonic matter field $\phi^A$) based on a nonlinear Lagrangian:

$$L(g, \gamma, \phi, \partial \phi) = -(2\kappa)^{-1}(-\det g)^{1/2}F(g_{\mu\nu}, R_{\mu\nu})$$

(1)

where $L_{\text{mat}}$ is a matter Lagrangian, $R_{\mu\nu} = R_{\mu\nu}(g)$ is the Ricci curvature of a metric tensor $g$, $\gamma$ is the Levi-Civita connection of $g$, and $\kappa = 8\pi G$ is the gravitational constant. The choice of the linear function

$$F(g_{\mu\nu}, R_{\mu\nu}) = R_{\mu\nu}$$

(2)

leads to the standard Einstein theory. Recently, e.g., a lot of interest has been devoted to quadratic Lagrangians,

$$F(g_{\mu\nu}, R_{\mu\nu}) = aR + bR^2 + Cg^{\alpha\beta}g_{\mu\nu}R_{\alpha\beta}R_{\mu\nu},$$

(3)

which will be discussed in the sequel as a special case of our problem. Now we consider a general nonlinear case (1). The Euler-Lagrange equations,

$$\frac{\delta L}{\delta g_{\mu\nu}} = 0,$$

(4)

$$\frac{\delta L}{\delta \phi^A} = 0,$$

(5)

form a system of fourth-order differential equations for the metric $g$ unless $F$ is linear. Equation (4) can be rewritten as

$$0 = \delta L/\delta g_{\mu\nu} = \partial L/\partial g_{\mu\nu} - \frac{1}{2}D_{\alpha}[g^{\lambda\mu}(\partial L/\partial g^{\lambda\nu}) + g^{\lambda\nu}(\partial L/\partial g^{\lambda\mu}) - g^{\alpha\beta}(\partial L/\partial g^{\lambda\nu})]$$

$$+ \frac{1}{2}D_{\beta}D_{\alpha}[g^{\rho\nu}(\partial L/\partial g_{\rho\nu}) + g^{\rho\nu}(\partial L/\partial g_{\rho\nu}) - g^{\alpha\beta}(\partial L/\partial g_{\rho\nu})],$$

(6)

where by $D$ we denote the covariant derivative with respect to $\gamma$. We introduce the auxiliary quantity

$$\mathcal{A}^{\mu\nu} = (-\det g)^{1/2}\partial F/\partial R_{\mu\nu},$$

(7)

which is at the moment merely an abbreviation for a tensor density built up for $g_{\mu\nu}$ and $R_{\mu\nu}$:

$$\mathcal{A}^{\mu\nu} = \mathcal{A}^{\mu\nu}(g_{\alpha\beta}, R_{\alpha\beta}).$$

(8)

The specific form of relation (8) depends on the choice of the function $F$, i.e., on the choice of our Lagrangian (1) [e.g., for a quadratic Lagrangian (3) $\mathcal{A}^{\mu\nu}$ depends linearly on $R_{\mu\nu}$; this example will be discussed in detail]. Relation (8) is a second-order differential equation for $g_{\mu\nu}$. This enables us to rewrite Eq. (6) as

$$\delta L_{\text{mat}}/\delta g_{\mu\nu} = -(2\kappa)^{-1}\frac{\partial}{\partial g_{\mu\nu}}[-(\det g)^{1/2}F(g_{\mu\nu}, R_{\mu\nu})] + (4\kappa)^{-1}D_{\alpha}[D_{\beta}g^{\rho\nu}\mathcal{A}^{\rho\nu} + g^{\rho\nu}\mathcal{A}^{\rho\nu} - g^{\rho\nu}\mathcal{A}^{\rho\nu} - g^{\rho\nu}\mathcal{A}^{\rho\nu}] = 0. $$

(9)

The above fourth-order differential equation for $g_{\mu\nu}$ is equivalent to the system (8) and (9) of second-order differential equations for independent quantities $g_{\mu\nu}$ and $\mathcal{A}^{\mu\nu}$. It is interesting to notice that the second-order differential operator acting on $\mathcal{A}^{\mu\nu}$ is universal and does not depend on a special choice of a Lagrangian function (1). This leads to the following result, which we prove in this paper: field equations (4)–(5) are equivalent to the standard Einstein equations for the new metric tensor $h_{\mu\nu}$ uniquely defined by the formula

$$\mathcal{A}^{\mu\nu} = (-\det h_{\alpha\beta})^{1/2}h^{\mu\nu},$$

(10)

($h^{\mu\nu}$ is the inverse to the tensor $h_{\mu\nu}$). More precisely, field theory based on Lagrangian (1) is completely
where

\[ L_1 = (2\kappa)^{-1} \left[ \left( -\det h \right)^{1/2} R(h) \right] + L_{\text{mat}}(h, \partial h, g, \partial g, \phi, \partial \phi) \]

(11)

Here \( R(h) \) denotes the scalar curvature of the new "metric" \( h \). The new matter Lagrangian \( L_{\text{mat}} \) in formula (11) will be defined in the sequel. We do not want to discuss the "philosophical" question which of the fields \( h \) or \( g \) is the "true" metric and which one is merely an additional matter field. An important argument for \( h \) being the true metric is the role of light cones of \( h \) in the causal properties of the Einstein theory (11). The mathematical analysis of our theory based on the Hilbert Lagrangian is very useful because we have at our disposal a lot of techniques for studying the Cauchy problem, the problem of stability of the theory, etc. For example, the energy of the entire system is composed of the gravitational energy of the metric \( h \) and the matter energy. To check the stability of the evolution it is, therefore, sufficient to check the algebraic properties (e.g., positivity) of the energy-momentum tensor obtained from the matter Lagrangian \( L_{\text{mat}} \). Otherwise, the study of the Cauchy problem for the fourth-order equation (6) is extremely difficult and depends very much on the specific form of the Lagrangian.

Generically, a solution of (5), (8), and (9) admits singularities, i.e., points where the signature of the metric becomes nonphysical. The transition between physical and nonphysical regions of space-time corresponds to extremal matter densities. Usually, regions which are nonphysical with respect to the metric \( g \) do not coincide with nonphysical regions for \( h \). A deep analysis of these phenomena could probably help us to decide which one of the two metric tensors is more physical. In most examples, however, regions corresponding to relatively weak densities of matter are equally good for both \( g \) and \( h \).

The construction of the new matter Lagrangian \( L_{\text{mat}} \) is based on the following regularity conditions which we impose on \( F \):

\[ \text{det}(\partial^2 F/\partial R_{\mu \nu} \partial R_{\alpha \beta}) \neq 0 \]  
\( \text{det}(\partial F/\partial R_{\mu \nu} \partial R_{\alpha \beta}) \neq 0 \),  

(12)

(13)

where the second derivative of \( F \) is treated as a \( 10 \times 10 \) matrix. Condition (13) guarantees the existence of a solution of the 10 algebraic equations (8) with respect to \( R_{\mu \nu} \). This way the choice of the function \( F \) in the original Lagrangian uniquely defines a tensor-valued function \( R \) such that

\[ R_{\mu \nu} = R_{\mu \nu}(h, g) \]

is equivalent to Eq. (8). We set

\[ L_{\text{mat}}(h, \partial h, g, \partial g, \phi, \partial \phi) = L_{\text{mat}}(g, \partial g, \phi, \partial \phi) + L_1(h, g) + L_2(h, \partial h, g, \partial g) \]

(15)

where

\[ L_1 = (2\kappa)^{-1} \left[ \left( -\det h \right)^{1/2} h_{\mu \nu} R_{\mu \nu}(h, g) \right] - \left( -\det g \right)^{1/2} F(g_{\mu \nu}, R_{\mu \nu}(h, g)) \]

(16)

and

\[ L_2 = (8\kappa)^{-1} \left[ \left( -\det h \right)^{1/2} h_{\mu \nu} g^{\alpha \beta} g^{\lambda \eta} \right. \]

\[ \times \left[ 2 \nabla_{\delta} g_{\mu \nu} \left( \nabla_{\gamma} g_{\rho \sigma} - \nabla_{\gamma} g_{\rho \sigma} \right) \right. \]

\[ + \left. \nabla_{\delta} g_{\alpha \beta} \left( \nabla_{\gamma} g_{\mu \nu} - \nabla_{\gamma} g_{\mu \nu} \right) + \nabla_{\delta} g_{\alpha \beta} \nabla_{\gamma} g_{\mu \nu} \right] \]

(17)

Here by \( \nabla \) we denote the covariant derivative with respect to the Levi-Civita connection of the metric \( h \). We stress that \( L_2 \) is universal and does not depend on the choice of \( F \). The function \( L_1 + L_2 \) in formula (15) plays the role of a matter Lagrangian for the new matter field \( g \) (\( L_1 \) depends algebraically on \( g \) and may be treated as a "Higgs" part of the Lagrangian and \( L_2 \) is a kinetic energy part) whereas \( L_{\text{mat}} \) describes the interaction between \( g \) and \( \phi \).

The formula (16) is analogous to the classical Legendre transformation (\( H = \partial L/\partial \dot{\theta} - L \)). Indeed, the transformation of the theory (1) to (11) is a Legendre transformation between \( R \) and \( h \). This aspect is discussed elsewhere.

The Euler-Lagrange equations for the Lagrangian \( L \) have the form

\[ \delta L / \delta \phi^A = \delta L_{\text{mat}} / \delta \phi^A = \delta L_{\text{mat}} / \delta \phi^A = 0 \],

(18)

\[ \delta L / \delta g_{\mu \nu} = \delta L_{\text{mat}} / \delta g_{\mu \nu} \]

\[ = \delta L_{\text{mat}} / \delta g_{\mu \nu} + \delta L_1 / \delta g_{\mu \nu} + \delta L_2 / \delta g_{\mu \nu} = 0 \],

(19)

\[ \delta L / \delta h^{\mu \nu} = 0 \].

(20)

Equations (18) and (5) are identical. Now we will prove the equivalence of Eq. (8) with the Einstein Eq. (20). Because of Eq. (11) the latter reads

\[ (2\kappa)^{-1} \left( -\det h \right)^{1/2} \left[ R_{\mu \nu}(h) - \frac{1}{2} h_{\mu \nu} h^{\alpha \beta} R_{\alpha \beta}(h) \right] \]

\[ = \delta L_{\text{mat}} / \delta h^{\mu \nu} = \delta L_1 / \delta h^{\mu \nu} + \delta L_2 / \delta h^{\mu \nu} \].

(21)

Moreover,

\[ \partial L_1 / \partial h^{\mu \nu} = (2\kappa)^{-1} \left( -\det h \right)^{1/2} \left[ R_{\mu \nu}(h) - \frac{1}{2} h_{\mu \nu} h^{\alpha \beta} R_{\alpha \beta}(h) \right] \]

\[ + (2\kappa)^{-1} \left( -\det h^{1/2} h^{\alpha \beta} - (-\det g)^{1/2} \partial F / \partial R_{\alpha \beta} \right) \partial R_{\alpha \beta} / \partial h^{\mu \nu} \].

(22)
Because of Eqs. (7) and (10) the second term in the above formula vanishes (this is the standard Legendre transformation effect).

The Lagrangian $L_2$ can be rewritten as

$$L_2 = (2\kappa)^{-1}(-\det h)^{1/2}h^{\mu\nu} (A^\mu_\alpha A^\nu_\beta - A^\alpha_\mu A^\beta_\nu),$$

where $A$ is a difference between Christoffel's symbols for metric tensors $g$ and $h$:

$$A^\alpha_\mu (h, \partial h, g, \partial g) = \gamma^\alpha_\mu (g) - \gamma^\alpha_\mu (h) = \frac{1}{2} g^{\lambda\eta} \left( \nabla_\mu g_{\eta\nu} + \nabla_\nu g_{\eta\mu} - \nabla_\eta g_{\mu\nu} \right).$$

Using a formula analogous to (6) we may calculate $\delta L_2 / \delta h$:

$$\delta L_2 / \delta h = \partial L_2 / \partial h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} \left( \nabla_\alpha h^{\beta\gamma} \frac{\partial L}{\partial h^{\beta\gamma}} + h^{\beta\gamma} \frac{\partial^2 L}{\partial h^{\beta\gamma}} \right) - h^{\alpha\beta} \left( \nabla_\mu h^{\alpha\beta} + \frac{1}{2} \nabla_\nu h^{\alpha\beta} \right) + \frac{1}{2} \nabla_\nu A^\alpha_\mu A^{\beta}_\nu - A^{\alpha}_\mu A^\beta_\nu - \nabla_\alpha A^\beta_\nu + \frac{1}{2} \nabla_\mu A^\alpha_\nu.$$  

(25)

Finally, subtracting the trace from the left-hand side of (21) and using (22) and (25) we obtain

$$R^{\mu\nu}(h) + \nabla^{\alpha} A^{\mu}_\alpha - \frac{1}{2} \nabla^{\alpha} A^{\nu}_\alpha - \frac{1}{2} \nabla^{\nu} A^{\alpha}_\mu + A^{\alpha}_\nu A^{\beta}_\mu - A^{\alpha}_\mu A^{\beta}_\nu = R^{\mu\nu}(h, g).$$  

(26)

However, the left-hand side of (26) is equal to $R^{\mu\nu}(g)$. Hence, Eq. (26) is equivalent to (14) which, in turn, was equivalent to (8). To complete our proof we have to show the equivalence between (19) and (9). Again, the Legendre-transformation argument shows that the second term (9) is equal to $\delta L_2 / \delta g$. Finally, one has to check that the last term in (9) is equal to $\delta L_2 / \delta h$. This is a lengthy but straightforward calculation which we omit here.

Using more advanced methods based on the theory of symplectic relations one can prove the equivalence with the Einstein theory of a theory based on a Lagrangian $L(g, \partial g, R^{\mu\nu}, \partial \phi)$, more general than (1). These results will be published soon. 3

Finally, we demonstrate our result for the case of the quadratic Lagrangian given by formula (3). The regularity condition (13) implies $c (4b + c) \neq 0$. In this case Eq. (8) reads

$$\mathcal{A}^{\mu\nu} = (-\det g)^{1/2} (a g^{\mu\nu} + 2b g^{\mu\nu} R + 2cg a^\mu g^{\beta\nu} R_{\alpha\beta}),$$

(27)

or equivalently

$$\mathcal{A}^{\mu\nu} = (-\det g)^{1/2} (a g^{\mu\nu} + 2b g^{\mu\nu} R + 2cg a^\mu g^{\beta\nu} R_{\alpha\beta}),$$

(28)

Consequently, the "Higgs part" (16) of the matter Lagrangian for $g$ equals

$$L_1 (h, g) = (2\kappa)^{-1} \left[ a (4b + c) - 1 \right] (-\det g)^{-1/2} \left[ (a g^{\mu\nu} A^{\mu\nu} - b (4b + c) - 1) g^{\mu\nu} \right].$$

(29)

Historically, the first result of this type was given by Ferraris. 4 He considered nonlinear Lagrangians depending on scalar curvature only: $F = F(R)$. We assume that $F' > 0$ and $F'' > 0$. In this case

$$\mathcal{A}^{\mu\nu} = (-\det g)^{1/2} g^{\mu\nu} F'(R).$$

(30)

Consequently, $h^{\mu\nu} = e^ {\Psi} g^{\mu\nu}$, where

$$\Psi = \ln F'(R).$$

(31)

The entire information about the matter field g is, therefore, carried by the scalar field $\Psi$. In this case condition (13) is not satisfied and we are not able to calculate $R^{\mu\nu}$ in terms of $h$ and $g$. Nevertheless, the scalar curvature can be calculated in terms of $\Psi$. This way the theory reduces to the Einstein theory for $h$ interacting with a scalar field $\Psi$ (and possibly with the preexisting matter field $\Phi$). Let $R = f(\Psi)$ be a solution of (31). Then the Higgs term (16) reads

$$L_1 = (2\kappa)^{-1} (-\det h)^{1/2} e^{-\Psi} [f(\Psi) - e^{-\Psi} F(f(\Psi))].$$

(32)

Similarly, one can calculate $L_2$ in terms of $\Psi$:

$$L_2 = (3/4\kappa)(-\det h)^{1/2} h^{\mu\nu} (g^\mu_\nu \partial_\mu \Psi) \partial_\nu \Psi.$$

(33)

In the case of original Lagrangian (3) with $c = 0$, we have $F' = a + 2b R$ and $f(\Psi) = (2b)^{-1} (e^\Psi - a)$. The Higgs term (32) reduces to

$$L_1 = (8b\kappa)^{-1} (-\det h)^{1/2} (1 - a e^{-\Psi})^2.$$

(34)

Our considerations show that all the dynamical effects due to nonlinear Lagrangians can be implemented by introducing new matter fields.

