

Fractional Fourier Transform and Geometric Quantization

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Abstract

Generalized Fourier transformation between the position and the momentum representation of a quantum state is constructed in a coordinate independent way. The only ingredient of this construction is the symplectic (canonical) geometry of the phase-space: no linear structure is necessary. It is shown that the “fractional Fourier transform” provides a simple example of this construction. As an application of this techniques we show that for any linear Hamiltonian system, its quantum dynamics can be obtained *exactly* as the lift of the corresponding classical dynamics by means of the above transformation. Moreover, it can be deduced from the *free quantum evolution*. This way new, unknown symmetries of the Schrödinger equation can be constructed. It is also argued that the above construction defines in a natural way a connection in the bundle of quantum states, with the base space describing all their possible representations. The non-flatness of this connection would be responsible for the non-existence of a quantum representation of the complete algebra of classical observables.

Keywords: Fractional Fourier transform; geometric quantization; linear quantum system; Schrödinger equation

MSC: 81S10, 53D50, 35Q41, 43A32

1 Introduction

Correct mathematical description of a specific quantum system was in many cases obtained *via* an appropriate “quantization procedure” from the corresponding “classical theory”. The first example of this type is the Heisenberg approach to quantum mechanics. Most of the field-theoretical models, like quantum electrodynamics, have also been constructed this way.

Geometric quantization (see e. g. [1]) was an attempt to formalize the above analogy between classical and quantum systems and to provide a tool to construct uniquely quantum theory once its classical counterpart is known. In particular, it has been noticed that some mathematical structures used in

mechanics and classical field theory on one side and in quantum mechanics and quantum field theory on the other are very similar. It was obvious from the very beginning that this analogy cannot go too far because quantum physics *cannot be reduced* to classical physics. Nevertheless, such a unifying point of view has led to important mathematical results in theory of group representations, theory of analytic functions, differential geometry and other branches of mathematics (see e. g. [2]).

Seen from the physical context, quantum mechanics (both the Schrödinger and the Heisenberg version) has, *a priori* very little to do with the symplectic structure of the underlying classical phase space. Indeed, it *is not* invariant with respect to (non-linear) canonical transformations!

A priori it is even non-invariant with respect to non-linear point transformations! However, this invariance may be easily restored if we use metric structure $g = g_{kl}dx^k dx^l$ of the configuration space Q , define the Hilbert space of pure states as $\mathcal{H} := L^2(Q, \sqrt{\det g_{kl}} \cdot d^n x)$ and take the Laplace-Beltrami operator (with minus sign) as the kinetic energy.

This is a relatively nice framework (called “covariant quantum mechanics”), which was recently thoroughly analyzed e.g. by M. Modugno (cf. [3]). In particular, no assumptions concerning the topology of the configuration space Q are necessary here. However, there are severe restrictions for the applicability of this approach: the time must be absolute, only non-relativistic Hamiltonians (i.e. “kinetic plus potential energy”) are allowed, no momentum representation is available etc.

A deep analysis of quantum mechanics and relativistic quantum field theory, first performed by J. M. Souriau (cf. [1]) and, independently, by W. Tulczyjew (cf. [4]), lead to the formulation of “geometric quantization theory”, based on the phase space and its symplectic structure.

A popular approach to “geometric quantization” consists in defining the quantum dynamics in terms of the “reproducing kernels” (see [5, 6] and the monographs [7] or [8]), which are carried by the geometric structure of the phase space of the system with finite number of degrees of freedom (The corresponding structures arising in field theory was analyzed in [9] and [11]).

Unfortunately, the complete symplectic structure cannot be represented on the quantum level, even if more and more sophisticated mathematical tools are introduced. In particular, the classical observable algebra (equipped with the Poisson bracket) has no appropriate *irreducible* representation in the algebra of operators acting in the Hilbert space. Here, “appropriate” means that it reduces to the standard quantum mechanics when restricted to the (finite or infinite-dimensional) Heisenberg algebra. Different functional-analytic frameworks can be chosen in order to convert the above “meta-mathematical” statement into a precise theorem, but non of them (e.g.: bounded or unbounded, continuous, smooth or only measurable observables) leads to a satisfactory representation. The notion of a *prequantization*, introduced by J. M. Souriau, even if mathematically beautiful, does not help

much, because it leads to the representation which is highly *reducible* and, therefore, cannot be used when calculating e.g. atomic optical spectra.

Physicists, chemists and quantum opticians, who try to model physical properties of complicated multi-molecular systems *via* Schrödinger equation, often use specific “quantization rules”, formulated in terms of specific “orderings” imposed on products of operators (e.g.: normal, anti-normal, Weyl etc., cf. [10] and references therein). They observe that the calculated spectra depend upon the ordering chosen. This fact may be considered as the “practical proof” that the entire classical observable algebra has no quantum representation.

There are deep mathematical results due to geometric quantization theory (e.g.: representation theory, “reproducing kernels”, metaplectic structure, the Maslov index etc.). These results are based on highly sophisticated mathematical tools. In spite of that, they are not very useful for applications. As a consequence, they remain virtually unknown to physicists. On the other hand, quantum physicists often discover some elements of this symplectic *Atlantis* but, in most cases, they are not fully aware of the consequences of these discoveries. This was recently the case of the “fractional Fourier transform”, an old mathematical structure rediscovered in quantum optics.

The goal of our paper is to present the basic structures which are necessary to formulate quantum mechanics in a simplest language and to analyze the symplectic invariance of the theory. To make our presentation as simple as possible we limit ourselves to the topologically trivial case (i.e. when the physical phase space is topologically equal to \mathbb{R}^{2n}). This is the case of most physical applications. Moreover, we use only those representations of the quantum Hilbert space which correspond to the so called “real polarizations”. This excludes some interesting issues like e.g. Bargmann representation, but allows us to simplify considerably the mathematical framework which does not go beyond the geometric interpretation of what the physicists know from the very beginning of quantum mechanics and may be found in standard textbooks.

Even if mathematically not sophisticated, our approach describes all the essential features of quantum mechanics. In particular, we prove that in case of linear dynamics, geometric quantization cannot fail: the *entire information* about quantum dynamics can be retrieved from its classical counterpart. We show that the correct evolution kernels can be obtained from the classical dynamics *via* a universal formula which is nothing but a properly geometrized Fourier transformation, superposed with (again: properly geometrized) Galilei transformation. In particular case of a harmonic oscillator, the “fractional Fourier transform” can be obtained as a specific example. This way we prove that dynamics of various quantum systems, which look apparently very different (like e. g.: free motion, harmonic oscillator, motion in a constant electric or magnetic field) provide specific examples of a single, universal formula. Moreover, the classical isomorphism relating any two

cases of classical linear dynamics on the quantum level is represented by a *local* (with respect to space and time) isomorphism between the corresponding Hilbert spaces. In particular, unexpected symmetries of the Schrödinger equation are obtained (some of them were known already long time ago, see e. g. [12] and references herein). Because both the Fourier transformation and the Galilei transformation describe statics (change of the representation and change of the reference frame) we conclude that the dynamics of linear systems is entirely implied by their static properties.

In the last part of the paper we show how a generic, non-linear classical evolution can be lifted to the quantum evolution *via* a natural connection in the bundle of quantum states. The connection is, however, non-flat and this is why the entire canonical structure of the phase space cannot be represented on the quantum level.

2 Fractional Fourier transform

The fractional Fourier transform (FrFT) is known e.g. from Namias paper [13]. It gives an important tool in classical optics [14, 15, 16, 17], quantum optics [18, 19, 20, 21, 22, 23] and signal processing [24]. But the idea of such an integral transformation appeared much earlier in mathematical literature, see e.g. [25]. The 1-dimensional FrFT is given by the following formula (the coefficients have been chosen in a way which is suitable for purposes of quantum mechanics):

$$(\mathcal{F}_\gamma f)(x') = \int K(\gamma, x, x') f(x) dx, \quad (1)$$

where the kernel $K(\gamma, x, x')$ is given by

$$K(\gamma, x, x') = \frac{e^{i\frac{\gamma}{2}}}{\sqrt{i \sin \gamma}} e^{i\pi \left((x^2 + x'^2) \cot \gamma - \frac{2xx'}{\sin \gamma} \right)}. \quad (2)$$

Here, f is a complex-valued function (in applications f can describe a quantum-mechanical wave-function, or a fully coherent, quasimonochromatic, classical electromagnetic wave). The constant γ is a real number [13, 26]. The formula (2) is, *a priori* meaningless for $\gamma = 0$ but its limit for $\gamma \rightarrow 0$ does exist and is equal to the Dirac distribution $\delta(x - x')$. Hence, the corresponding limit of the transformation (1) is equal to identity: $\mathcal{F}_0 f = f$.

The transformation is called “fractional”, because it provides an interpolation between the identity operator \mathcal{F}_0 and the ordinary Fourier transform which we obtain for $\gamma = \frac{\pi}{2}$. Indeed, operators \mathcal{F}_γ depend continuously upon the parameter γ and satisfy the group property (see [13, 26]):

$$\mathcal{F}_{\mu+\nu} = \mathcal{F}_\mu \mathcal{F}_\nu. \quad (3)$$

Observe that the formula for the quantum-mechanical propagator of the harmonic oscillator with frequency ω and mass m :

$$G(t, x, x') = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega t}} e^{\frac{im\omega}{2\hbar} \left((x^2 + x'^2) \cot \omega t - \frac{2x'x}{\sin \omega t} \right)}, \quad (4)$$

reduces (up to a constant phase factor) to (2) if we choose $\gamma = \omega t$ and re-scale appropriately coordinates x, x' . This observation is already known from the Namias' work [13]. In the present paper we show that this transformation is a specific example of a Generalized Fourier Transformation which will be defined in a purely geometric, coordinate-invariant way.

Before we present this construction in subsequent Sections, we are going to show that formula (4) can be simply understood as a superposition of the following two standard operations: 1) the conventional Fourier transformation between the position and the momentum representations and 2) the Galilei transformation changing the phase of the wave function as a consequence of the change of a reference frame.

To prove the above statement let us consider the classical dynamics of the harmonic oscillator¹:

$$\begin{aligned} x(t) &= x(0) \cos \omega t + p(0) \frac{1}{m\omega} \sin \omega t, \\ p(t) &= -x(0) m\omega \sin \omega t + p(0) \cos \omega t. \end{aligned} \quad (5)$$

Denote $x := x(0)$, $p := p(0)$, $x' := x(t)$ and $p' := p(t)$. Consider first the particular case $\omega t = \frac{\pi}{2}$. We have:

$$x' = \frac{p}{m\omega}. \quad (6)$$

Hence, $p = m\omega x'$ is the momentum canonically conjugate to x . Therefore, transition between the $x(0)$ -representation and $x(t)$ -representation of the quantum state must be given in terms of the transition between the position and the momentum representation. Indeed, formula (4) reduces to

$$G\left(t, x, \frac{p}{m\omega}\right) = \sqrt{\frac{m\omega}{2\pi i \hbar}} e^{-\frac{ipx}{\hbar}} = \sqrt{-im\omega} \sqrt{\frac{1}{2\pi \hbar}} e^{-\frac{ipx}{\hbar}}, \quad (7)$$

which, essentially, is the Fourier kernel defining the transition to the momentum representation. However, we have an extra coefficient “ $\sqrt{-im\omega}$ ” on the right hand side. Its constant phase factor “ $\sqrt{-i}$ ” is due to the convention used and has no physical meaning. But its modulus is necessary because the wave function is not a scalar object but a *half-density*. Without going

¹We use here the “Heisenberg picture”: points of the phase space do not move during the evolution and represent entire *histories* of the system. Evolution applies to *observables*. Hence, $(x(0), p(0))$ and $(x(t), p(t))$ have to be understood as two different coordinate systems in the same phase space. Souriau calls this phase space “espace des mouvements”.

too far into mathematical subtleties, which will become obvious in the next Section, the above statement means that the square of the modulus of a wave function is a density. Hence, the coefficient “ $\sqrt{m\omega}$ ” is necessary because its square “ $m\omega$ ” represents the change of the volume due to the reparametrization $p \mapsto x' = \frac{p}{m\omega}$ of the momentum space.

Now, consider an arbitrary value of the time variable t . The same formula (namely: $x' = x \cos \omega t + p \frac{1}{m\omega} \sin \omega t$) can be rewritten as:

$$p + x \cdot m\omega \cot \omega t = \frac{m\omega}{\sin \omega t} x' =: \tilde{p} . \quad (8)$$

We conclude that the quantity \tilde{p} may be taken as a momentum canonically conjugate to x . The argument used above explains the multiplicative factor $\sqrt{\frac{m\omega}{\sin \omega t}}$ in formula (4) and the last term in the exponent. But, there is an additional phase factor, namely $\exp\left(\frac{im\omega}{2\hbar}(x^2 + x'^2) \cot \omega t\right)$. We are going to show that it is a consequence of the Galilei transformation corresponding to formula (8).

Indeed, formula (8) is a particular example of a canonical transformation between the old canonical variables $(x, p) \mapsto (x, \tilde{p})$, where the new momentum is given by:

$$\tilde{p} = p + F(x) . \quad (9)$$

Such a “momentum translation” arises e.g. when performing a Galilei transformation:

$$\tilde{x} := x - t \cdot V ,$$

where V denotes the velocity of the new reference frame. Consequently, we have $\dot{\tilde{x}} = \dot{x} - V$ and, therefore,

$$\tilde{p} = m\dot{\tilde{x}} = m(\dot{x} - V) = p - mV ,$$

whereas $\tilde{x} = x$ at $t = 0$.

Transformation (9) is called a *generalized Galilei transformation*, the name *proper Galilei transformation* being reserved for the case when the function $F(x)$ is constant.

In a generic, multidimensional case, transformation $(x^i, p_i) \mapsto (x^i, \tilde{p}_i)$, with

$$\tilde{p}_i = p_i + F_i(x) . \quad (10)$$

is canonical if and only if the differential 1-form $\alpha := F_i dx^i$ is closed. Due to topological triviality of the configuration space this is equivalent to the fact that α must be exact, i.e. we have:

$$F_i(x) = \frac{\partial}{\partial x^i} S(x) . \quad (11)$$

The quantum version of the generalized Galilei transformation (10) is obvious. It consists in multiplying the wave function by the phase factor

$\exp\left(\frac{i}{\hbar}S(x)\right)$:

$$\tilde{\psi}(x) := \psi(x) \cdot \exp\left(\frac{i}{\hbar}S(x)\right), \quad (12)$$

which, together with the Schrödinger representation of the momenta

$$p_i = \frac{\hbar}{i} \frac{\partial}{\partial x^i}, \quad (13)$$

reproduces, indeed, formula (10). We stress that the phase S is implied by (11) up to an additive constant only. This agrees with the fact that the global phase of the wave function has no physical significance.

In particular case of the transformation (8), we have $F(x) = x \cdot m\omega \cot \omega t$ and, therefore:

$$S(x) = \frac{m\omega}{2} x^2 \cot \omega t, \quad (14)$$

which explains the phase factor $\exp\left(\frac{im\omega}{2\hbar}x^2 \cot \omega t\right)$ in formula (4).

To explain the remaining phase factor, namely: $\exp\left(\frac{im\omega}{2\hbar}x'^2 \cot \omega t\right)$, let us first summarize the sequence of operations which have to be applied to the wave function $\psi(x)$ in order to reproduce the transformation defined by the integral kernel (4).

1. Wave function $\psi(x)$ represents the quantum state with respect to the Heisenberg algebra generated by observables (x, p) . Its representation $\tilde{\psi}(x)$ with respect to (x, \tilde{p}) , where $\tilde{p} = p + \frac{d}{dx}S(x)$, is obtained *via* the generalized Galilei transformation (12):

$$\tilde{\psi}(x) := \psi(x) \cdot \exp\left(\frac{im\omega}{2\hbar}x^2 \cot \omega t\right). \quad (15)$$

2. We pass to the momentum representation using the ordinary Fourier transformation $\hat{\tilde{\psi}}(\tilde{p})$ of the function $\tilde{\psi}(x)$. This way we exchange the role of x and \tilde{p} , which corresponds to the canonical transformation $(x, \tilde{p}) \mapsto (\tilde{p}, -x)$. The momentum canonically conjugate to \tilde{p} , namely $-x$, is represented now by the operator $\frac{\hbar}{i} \frac{\partial}{\partial \tilde{p}}$ acting on the wave function $\hat{\tilde{\psi}}(\tilde{p})$.
3. Next step consists in using Ansatz (8), i.e.: $\tilde{p} := \frac{m\omega}{\sin \omega t} x'$. This means that we implement the canonical transformation:

$$\left(\frac{m\omega}{\sin \omega t} x', -x\right) \mapsto \left(x', -\frac{m\omega}{\sin \omega t} x\right).$$

For this purpose only the density factor $\sqrt{\frac{m\omega}{\sin \omega t}}$ is necessary and we obtain the new wave function:

$$\phi(x') := \sqrt{\frac{m\omega}{\sin \omega t}} \cdot \hat{\tilde{\psi}}\left(\frac{m\omega}{\sin \omega t} x'\right) \quad (16)$$

4. Finally, we want to replace the “fictitious” momentum $q := -\frac{m\omega}{\sin \omega t} x$ by the “true” momentum p' , canonically conjugate to x' . For this purpose we use again formulae (5) and (8):

$$\begin{aligned} p' &= -x \cdot m\omega \sin \omega t + p \cdot \cos \omega t = -x \cdot \frac{m\omega}{\sin \omega t} + x' \cdot m\omega \cot \omega t \\ &= q + x' \cdot m\omega \cot \omega t . \end{aligned}$$

We see that, again, a Galilei transformation is necessary, with the phase factor $\exp\left(\frac{i}{\hbar}S(x')\right)$ defined by equation

$$p' - q = x' \cdot m\omega \cot \omega t = \frac{d}{dx'}S(x') ,$$

and, whence, given by formula (14). This way we obtain the final wave function

$$\psi'(x') := \phi(x') \cdot \exp\left(\frac{im\omega}{2\hbar}x'^2 \cot \omega t\right) . \quad (17)$$

We conclude that the entire missing phase factor in formula (4) comes from the above Galilei transformation.

The above procedure shows that the “fractional Fourier transform” kernel (4) is nothing but the *ordinary* Fourier kernel (step 2.), appropriately superposed with two Galilei transformations (steps 1. and 4.) and one obvious transformation coming from rescaling of the corresponding configuration space (step 3.). These are standard, *local* transformations of the wave function, implied by the necessary rearrangements of the phase-space coordinates. In the present paper we are going to show that the above construction does not depend upon specific choice of coordinates used in the above example but has a deep geometric meaning. This way not only harmonic oscillator, but any linear quantum system evolves according to a similar law. In fact, formula (2) is a special case of kernels which arise in a natural way in geometric quantization [5, 6], whenever we want to describe transformation between two representations of a quantum state. All these kernels may be defined in a geometric, coordinate-independent way. They arise as superpositions of two standard building blocks: 1) the (appropriately geometrized) Fourier transformation and 2) the generalized Galilei transformation. To prove this fact, we analyze in the next Section the geometric structure of a quantum-mechanical wave function in terms of the phase-space geometry.

3 Geometric quantization

Consider the classical phase space (\mathcal{P}, Ω) of a system with n -degrees of freedom. This means that $\dim \mathcal{P} = 2n$. By Ω we denote the canonical symplectic

form. Locally, a coordinate system (x^i, p_i) , $i = 1, \dots, n$, may be found, such that Ω reduces to the following expression:

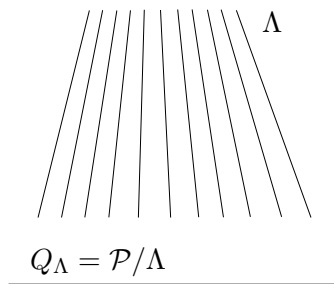
$$\Omega = dp_i \wedge dx^i, \quad (18)$$

where the summation convention is always used. Such coordinates are called “canonical coordinates”. In this paper we consider the simplest, topologically trivial case $\mathcal{P} \simeq \mathbb{R}^{2n}$, when canonical coordinates exist globally.

As a consequence of the Heisenberg uncertainty relation, quantum-mechanical wave function can not depend upon all these phase-space coordinates but only upon a half of them. Physically, this means that a representation of quantum states in terms of wave functions is possible only *with respect* to a “complete system of commuting observables”. Examples, such as the “position representation” (wave functions depend upon position variables (x^i)) or the “momentum representation” (wave functions depend upon momenta (p_i)) are well known. Geometrically, a “system of commuting observables” may be considered as a foliation Λ of \mathcal{P} by the congruence of all n -dimensional surfaces $\{(x^i, p_i) : x^i = \text{const.}\}$ for the position representation and surfaces $\{(x^i, p_i) : p_i = \text{const.}\}$ for the momentum representation, respectively. The leaves of the above foliations are *Lagrangian* submanifolds of \mathcal{P} . This means that: 1) they are isotropic with respect to the canonical 2-form (18) and 2) they have maximal dimension which is possible for isotropic surfaces, namely a half of the dimension of \mathcal{P} . Geometric quantization is a theory which describes the intrinsic properties of the quantum state in a geometric, coordinate-independent way.

3.1 Quantum states and generalized Galilei transformation

To give geometric definition of a quantum state, the following three observations have to be taken into account:



1) Whenever a global, Lagrangian foliation Λ of the phase space has been chosen, the space of fibers

$$Q_\Lambda = \mathcal{P}/\Lambda \quad (19)$$

plays the role of a generalized configuration space. Physically, it describes independent variables (control parameters) of the system. The wave function

is an object living on Q_Λ . Description of a quantum state *via* such a wave function will be called the Q_Λ -representation. As an example we can take the position or the momentum representation at different instants t of time.

2) To be able to calculate probabilities or transition amplitudes, we have to integrate over the configuration space Q_Λ . For this purpose people usually assume that a measure ρ on Q_Λ has been chosen, such that the transition probability between ψ_1 and ψ_2 is given by their scalar product in $L^2(Q_\Lambda, \rho)$, namely:

$$(\psi_1|\psi_2) := \int_{Q_\Lambda} \psi_1^* \psi_2 \, d\rho . \quad (20)$$

Usually one chooses the Lebesgue measure carried by any system of coordinates on the configuration space. Unfortunately, such a description depends upon an arbitrary choice of coordinates. Even a change of units (i.e. centimeters *versus* inches) must be compensated by an appropriate re-scaling of the wave function, cf. formula (16). But only the quantity $\psi_1^* \psi_2 \, d\rho$ has a physical meaning. This quantity is independent upon all these (arbitrary) re-scalings. An obvious simplification of the formalism consists in “incorporating” the “square root of the measure” into the wave function. Namely, we consider the intrinsic half-density

$$\Psi := \psi \cdot \sqrt{d\rho} \quad (21)$$

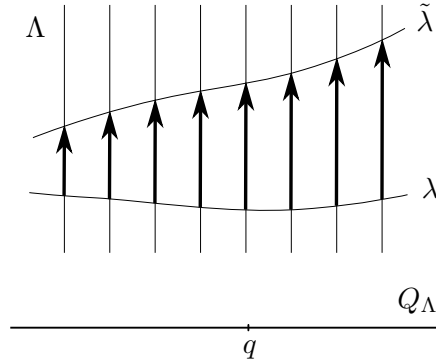
instead of the scalar function ψ . This object does not depend upon any choice of coordinates nor the choice of any measure on Q_Λ . Assuming that the above half-density is locally absolutely continuous with respect to the square root of the Lebesgue measure carried by any system of coordinates (x^i) on Q_Λ , we may recover the traditional, scalar wave function ψ as the ratio between Ψ and the reference half-density $\sqrt{|dx^1 \wedge dx^2 \wedge \dots \wedge dx^n|}$. Complex half-densities, square-integrable, absolutely continuous with respect to Lebesgue, form a Hilbert space which will be called $L^2(Q_\Lambda)$.

3) The above Hilbert space can not, however, be identified with the physical space of states, because it does not reflect properly the Galilei transformations of the wave function, due to the change of the reference frame. Indeed, quantum representation of the momentum p_i canonically conjugate to the position x^i is given by formula (13). We have seen in the previous Section that a generalized Galilei transformation: $\tilde{p}_i = p_i + \frac{\partial}{\partial x^i} S(x)$, must be implemented on the level of quantum mechanics by the multiplication of the wave function by the phase factor, as in formula (12).

Observe that the above Galilei transformation consists in shifting the value of p by a constant value $\frac{\partial}{\partial x^i} S(x)$ in each fiber $q \in Q_\Lambda$ independently. In order to choose a specific one among all the possible canonical momenta p_i , we have to choose at each fiber $q \in Q_\Lambda$ the point where this observable vanishes. The collection of all these points forms a Lagrangian surface $\lambda := \{p_i = 0\} \subset \mathcal{P}$ which is transversal with respect to the foliation Λ . We

conclude that a choice of such a surface corresponds to a choice of a reference frame.

We are going to show in the next Section that each fiber $q \in Q_\Lambda$ carries a natural *affine* structure. Choosing a specific “reference point” $\lambda \cap q \in q$ transforms it into a *vector* space. Moreover, we prove that this vector space (i.e. space tangent to the fiber q) is canonically equivalent to the cotangent space $T_q^*Q_\Lambda$. Choice of a reference frame implies, therefore, that the fibration Λ acquires the vector-bundle structure isomorphic to the cotangent bundle T^*Q_Λ .



Suppose now that another Lagrangian surface $\tilde{\lambda} := \{\tilde{p}_i = 0\} \subset \mathcal{P}$ (i.e. another reference frame) has been chosen. The difference between the two points: $\tilde{\lambda} \cap q$ and $\lambda \cap q$, defines in each affine space q a tangent vector or, equivalently, a covector on Q_Λ attached at q . The collection of these covectors forms a differential 1-form on the configuration space Q_Λ , which we denote by $\tilde{\lambda} - \lambda$. As a consequence of the fact that both surfaces $\tilde{\lambda}$ and λ were Lagrangian we obtain an obvious

Corollary: The form $\tilde{\lambda} - \lambda$ is closed.

Due to our topological assumption we have, therefore:

$$\tilde{\lambda} - \lambda = dS_{\tilde{\lambda},\lambda} , \quad (22)$$

which, otherwise, would be true only locally. The function $S_{\tilde{\lambda},\lambda}$ is defined up to an additive constant.

As we have already discussed in the previous Sections, the elementary quantum mechanics implies that the wave functions describing the same quantum state with respect to different reference frames: $\lambda := \{p_i = 0\}$ and $\tilde{\lambda} := \{\tilde{p}_i = 0\}$, differ by a phase factor, namely:

$$\Psi_{\Lambda,\lambda} = \Psi_{\Lambda,\tilde{\lambda}} \cdot e^{\frac{i}{\hbar} S_{\tilde{\lambda},\lambda}} , \quad (23)$$

where the function $S_{\tilde{\lambda},\lambda}$ is uniquely (up to an additive constant) defined by the two submanifolds: λ and $\tilde{\lambda}$.

We see that to assign a wave function to a quantum (pure) state, it is not sufficient to fix a “complete system of commuting observables” (i.e. a foliation Λ) but it is also necessary to choose a reference frame (i.e. a Lagrangian surface transversal to Λ). The same quantum state, within the same representation Λ (i.e. position or momentum representation) is represented by different wave functions with respect to different reference frames. This suggests the following

Definition: Quantum state in a representation Λ is a class of equivalent wave functions:

$$\mathfrak{Q}_\Lambda := [\Psi_{\Lambda,\lambda}]$$

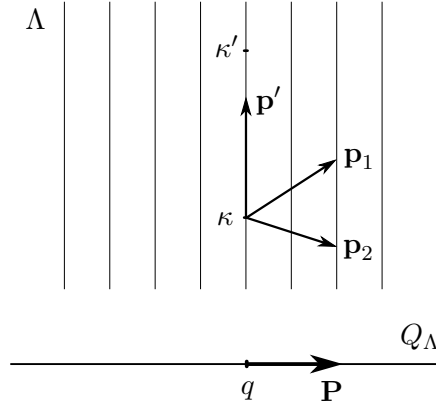
where the equivalence relation is given by the generalized Galilei transformation:

$$\Psi_{\Lambda,\lambda} \sim \Psi_{\Lambda,\tilde{\lambda}} \iff \left\{ \Psi_{\Lambda,\lambda} = \Psi_{\Lambda,\tilde{\lambda}} \cdot e^{\frac{i}{\hbar} S_{\tilde{\lambda},\lambda}} \ ; \ dS_{\tilde{\lambda},\lambda} = \tilde{\lambda} - \lambda \right\} . \quad (24)$$

Observe that the space \mathcal{H}_Λ composed of all quantum states is a projective Hilbert space because a constant phase factor e^{ic} , $c \in \mathbb{R}$, of the wave function (i. e. an additive constant of $S_{\tilde{\lambda},\lambda}$) is always out of control.

3.2 Proof of the vector-bundle structure of a Lagrangian foliation

The coordinate-free construction of the vector-bundle structure of Λ and of the phase function $S_{\tilde{\lambda},\lambda}$ in terms of the phase-space geometry was given in [6]. It may be briefly sketched as follows:



Given a fiber $q \in \Lambda$ and a point on it, $\kappa \in q$, vectors tangent to q at κ can be canonically identified with covectors on Q_Λ , attached at q . The identification is given by the formula:

$$\langle \mathbf{P} | \mathbf{p}' \rangle := \Omega(\mathbf{p}, \mathbf{p}') \quad (25)$$

Here, $\mathbf{P} \in T_q Q_\Lambda$ is a vector tangent to Q_Λ at q and $\mathbf{p}' \in T_\kappa q \subset T_\kappa \mathcal{P}$ is a vector tangent to the fiber q . By $\mathbf{p} \in T_\kappa \mathcal{P}$ we denote any vector which

projects onto \mathbf{P} with respect to the canonical projection in the fiber bundle $\pi : \mathcal{P} \rightarrow Q_\Lambda$. The value of \mathbf{p}' on \mathbf{P} is, therefore, equal to its “symplectic scalar product” with any representative \mathbf{p} of \mathbf{P} , i.e. with a vector \mathbf{p} fulfilling: $\pi_*\mathbf{p} = \mathbf{P}$. Of course, such a vector is not unique. But for any pair of such vectors, say \mathbf{p}_1 and \mathbf{p}_2 , their difference projects on zero, i. e. must be tangent to the fiber q . Consequently, we have:

$$\Omega(\mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}') = 0$$

because both $\mathbf{p}_1 - \mathbf{p}_2$ and \mathbf{p}' are tangent to q which is Lagrangian. This proves that the left hand side of (25) is defined uniquely. This way we have constructed a mapping

$$F_\kappa : T_\kappa q \rightarrow T_q^* Q_\Lambda .$$

The non-degeneracy of Ω implies that F_κ is an isomorphism.

The above construction defines an auto-parallelism (a flat connection) on every fiber $q \in \Lambda$. Indeed, given two points $\kappa, \kappa' \in q$, their tangent spaces $T_\kappa q$ and $T_{\kappa'} q$ are canonically isomorphic to the same cotangent space $T_q^* Q_\Lambda$ and, therefore, may be canonically identified. Moreover, it is easy to check that vector fields which are constant along fibers, are Hamiltonian vector fields generated by functions on Q_Λ (i.e. functions constant on fibers of Λ). But the Poisson bracket of two such functions vanishes identically. This proves that constant vector fields do commute, i.e. the connection is flat. We conclude that every fiber $q \in \Lambda$ may be treated as a subset of an affine space. For pedagogical reasons we assume in the sequel that the topology of the fibration is trivial, i.e. the fiber covers the entire affine space.

Now, we are going to assign to every pair $(\tilde{\lambda}, \lambda)$ of sections of the bundle $\mathcal{P} \rightarrow Q_\Lambda$ a covector field on the configuration space Q_Λ . We denote it by $\tilde{\lambda} - \lambda$. It is defined by the formula:

$$(\tilde{\lambda} - \lambda)(q) := \tilde{\lambda} \cap q - \lambda \cap q , \quad (26)$$

where the right hand side is a vector tangent to the fiber q , connecting the two points, i.e. a covector on Q_Λ . Because both sections are Lagrangian submanifolds, the resulting form is closed:

$$d(\tilde{\lambda} - \lambda) = 0 .$$

Hence, locally, it satisfies:

$$\tilde{\lambda} - \lambda = dS_{\tilde{\lambda}, \lambda} .$$

Due to the trivial topology of Q_Λ , the potential $S_{\tilde{\lambda}, \lambda}$ exists globally and is defined uniquely up to an additive constant.

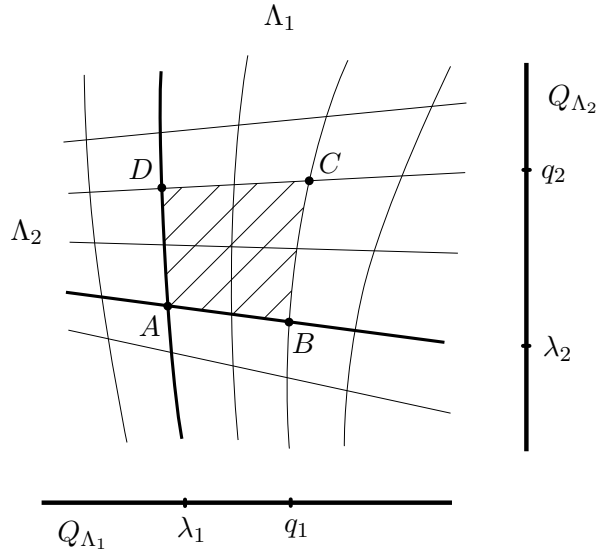
3.3 Generalized Fourier transformation

The only arbitrary element which remains in the description of a quantum state is the Lagrangian foliation Λ , representing a complete set of commuting observables. Now, we are going to describe the transformation which undergoes the wave function of a given quantum state when we pass from one foliation to the other. This will cover i. g. transformation from the position to the momentum representation. But, we may also consider two foliations corresponding to the position representation $\Lambda(t) := \{x(t) = \text{const.}\}$ at two different instants of time: t_1 and t_2 . The transformation between these two foliations represents quantum dynamics.

Assume, therefore, that we have two different foliations Λ_1 and Λ_2 of the symplectic space \mathcal{P} . We are going to define the transformation from \mathcal{H}_{Λ_1} to \mathcal{H}_{Λ_2}

$$\mathcal{F}_{\Lambda_2\Lambda_1} : \mathcal{H}_{\Lambda_1} \rightarrow \mathcal{H}_{\Lambda_2}. \quad (27)$$

as an integral operator acting on corresponding wave functions (cf. [5], [6]). Here, we limit ourselves to a simplified version, which works for *transversal* foliations. In this case any fiber $\lambda_1 \in \Lambda_1$ defines a reference frame for the description of a quantum state with respect to Λ_2 and *vice versa*. Choose, therefore, a pair (λ_1, λ_2) , $\lambda_i \in \Lambda_i$; of such reference frames. Now, for any other pair (q_1, q_2) , $q_i \in \Lambda_i$; consider the four intersection points: 1) $A = \lambda_1 \cap \lambda_2$, 2) $B = \lambda_2 \cap q_1$, 3) $C = q_1 \cap q_2$ and, finally, 4) $D = q_2 \cap \lambda_1$. Because every fiber carries an affine structure, every pair of subsequent inter-



section points defines uniquely an interval of a “straight line” connecting them (e.g. we connect A with B along a straight line in λ_2 and so on). This way we obtain uniquely the oriented “rectangle” $ABCD$ which will be denoted $(\lambda_2, q_1, q_2, \lambda_1)$, where the orientation is defined by the sequence $(ABCD)$.

Define its ‘‘symplectic surface’’ $k(\lambda_2, q_1, q_2, \lambda_1)$ by:

$$k(\lambda_2, q_1, q_2, \lambda_1) := \int_S \Omega \quad (28)$$

where S is any (oriented) 2-surface spanned by the rectangle, i.e. satisfying the condition: $\partial S = (\lambda_2, q_1, q_2, \lambda_1)$. The definition does not depend upon a choice of such a surface because the symplectic form Ω is closed. Indeed, if S_1 and S_2 are two such surfaces, then there is a 3-volume V such that $\partial V = S_2 - S_1$ and, consequently, we have:

$$\int_{S_2} \Omega - \int_{S_1} \Omega = \int_{\partial V} \Omega = \int_V d\Omega = 0 .$$

The mapping (27) is defined as the integral transformation of the corresponding wave functions:

$$\Psi_{\Lambda_2, \lambda_1}(q_2) = \int_{Q_{\Lambda_1}} \Psi_{\Lambda_1, \lambda_2}(q_1) K_{\lambda_1, \lambda_2}(q_1, q_2) , \quad (29)$$

where the kernel K is defined as follows:

$$K_{\lambda_1, \lambda_2}(q_1, q_2) = \sqrt{\left| \left(\frac{1}{i\hbar} \Omega \right)^n \right|} \cdot e^{-\frac{i}{\hbar} k(\lambda_2, q_1, q_2, \lambda_1)} . \quad (30)$$

Here, $2n = \dim \mathcal{P}$, $\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega$ is a $2n$ -form (scalar density) on \mathcal{P} , $\sqrt{|\Omega^n|}$ is the corresponding half-density.

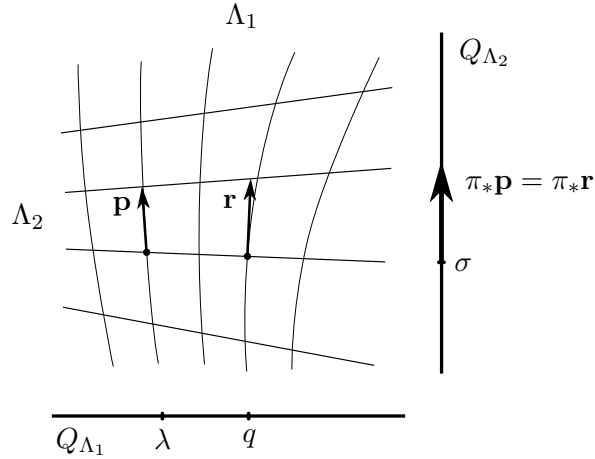
If (x^i) , $i = 1, \dots, n$, are coordinates on Q_{Λ_1} and (y^i) , $i = 1, \dots, n$, are coordinates on Q_{Λ_2} , then (x^i, y^j) define a coordinate chart on the phase space \mathcal{P} . Hence, the $2n$ -form Ω^n is proportional to $dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n$. Consequently, we have:

$$\sqrt{|\Omega^n|} = f(x, y) \sqrt{|dx^1 \wedge \cdots \wedge dx^n|} \sqrt{|dy^1 \wedge \cdots \wedge dy^n|} \quad (31)$$

Because wave function $\Psi_{\Lambda_1, \lambda_2}$ is a half-density on Q_{Λ_1} , it contains already the factor $\sqrt{|dx^1 \wedge \cdots \wedge dx^n|}$. Together with the same factor from (31) it produces the scalar density on Q_{Λ_1} which we integrate according to formula (29). The result of this integration contains the remaining factor $\sqrt{|dy^1 \wedge \cdots \wedge dy^n|}$ from (31), i.e. produces a half-density on Q_{Λ_2} .

The operator $\mathcal{F}_{\Lambda_2 \Lambda_1}$ is called the generalized Fourier transformation. It is well defined for any pair of transversal foliations. In the present paper we shall use it thoroughly in a specific case, when the two foliations are *compatible*. It turns out that this covers all the cases of *linear dynamics* (e.g. free motion, harmonic oscillator and a constant electric or magnetic fields). As will be seen in the next Section, the entire information about the quantum dynamics can be obtained from its classical counterpart by means of the operator $\mathcal{F}_{\Lambda_2 \Lambda_1}$.

For the sake of completeness we shall now formulate the compatibility condition, which implies specific properties of the generalized Fourier kernel (30). For this purpose observe that, given two transversal foliations, there is a unique and natural way to transport vectors tangent to fibers of Λ_1 along the fibers of Λ_2 . Indeed, given a fiber $\sigma \in \Lambda_2$, two vectors \mathbf{p} and \mathbf{r} , tangent to $\lambda \in \Lambda_1$ and $q \in \Lambda_1$ at the points $\lambda \cap \sigma$ and $q \cap \sigma$ respectively, may be identified if they project onto the same vector tangent to Q_{Λ_2} , i.e. if $\pi_*\mathbf{p} = \pi_*\mathbf{r}$.



On the other hand, the affine structure of the fibers allows us to transport them parallelly along fibers of Λ_1 .

Definition: Two mutually transversal, Lagrangian foliations Λ_1 and Λ_2 are called *compatible* if parallel translations along Λ_1 commute with those along Λ_2 .

An obvious example of compatible foliations is given by the position foliation $\{x^i = \text{const.}\}$ and the momentum foliation $\{p_i = \text{const.}\}$, if (x^i, p_i) are canonical variables. In case of compatible foliations the operator $\mathcal{F}_{\Lambda_2\Lambda_1}$ is unitary and fulfills the chain rule: $\mathcal{F}_{\Lambda_3\Lambda_2}\mathcal{F}_{\Lambda_2\Lambda_1} = \mathcal{F}_{\Lambda_3\Lambda_1}$.

The proof of this property may be sketched as follows. For *compatible* foliations the function f in formula (31) factorizes and we have: $f(x, y) = h(x) \cdot k(y)$. On the other hand, both Q_{Λ_1} and Q_{Λ_2} carry an affine structure and the corresponding vector spaces are in canonical duality. It is easy to see that the phase factor $k(\lambda_2, q_1, q_2, \lambda_1)$ is given by the above duality form

$$k(\lambda_2, q_1, q_2, \lambda_1) := \langle (q_1 - \lambda_1)(\lambda_2) | (q_2 - \lambda_2)(\lambda_1) \rangle. \quad (32)$$

Hence, the entire kernel (30) factorizes and reduces to the standard Fourier kernel written in linear coordinates compatible with the affine structure carried by the two foliations. This implies the group properties of the transformation.

4 Symmetries between linear quantum systems

We stress that there was no linear structure of the configuration or the phase spaces assumed *a priori*. The symplectic form implies the affine structure of the fibers of the Lagrange'an foliation Λ . However, if we take two *compatible* foliations Λ_1 and Λ_2 , then the entire phase space \mathcal{P} acquires an affine structure.

In this context the *linear* dynamics has to be understood as a specific situation, for which the “position-foliations” $\Lambda_t := \{x(t) = \text{const.}\}$ remain mutually compatible for different times t_1 and t_2 . It is easy to check that this happens if and only if there are canonical variables in \mathcal{P} , such that the Hamiltonian is at most quadratic.

In this section we analyze examples of linear dynamics in the geometric quantization context. We prove that our generalized Fourier transformation gives the correct quantum evolution. We begin with the classical analysis which shows that the configuration foliations $\Lambda_t := \{x(t) = \text{const.}\}$ are, in fact, the same for all possible cases of linear dynamics. This implies that any solution of the Schrödinger equation with at most quadratic potential (e.g. harmonic oscillator, constant electric or magnetic fields) is uniquely given by a corresponding solution describing the free motion. We conclude that different linear quantum systems are, essentially, all the same.

4.1 Harmonic oscillator vs. free motion

Consider classical dynamics of a free particle (for simplicity we limit ourselves to 1 degree of freedom)²:

$$\begin{aligned} x(t) &= x(0) + \frac{t}{m}p(0), \\ p(t) &= p(0), \end{aligned} \tag{33}$$

and of a harmonic oscillator³:

$$\begin{aligned} \tilde{x}(\tau) &= \cos \omega\tau \tilde{x}(0) + \frac{1}{m\omega} \sin \omega\tau \tilde{p}(0), \\ \tilde{p}(\tau) &= -m\omega \sin \omega\tau \tilde{x}(0) + \cos \omega\tau \tilde{p}(0), \end{aligned} \tag{34}$$

²Hamiltonian equations for free particle:

$$\dot{\mathbf{r}}(t) = \frac{1}{m}\mathbf{p}(t), \quad \dot{\mathbf{p}}(t) = 0.$$

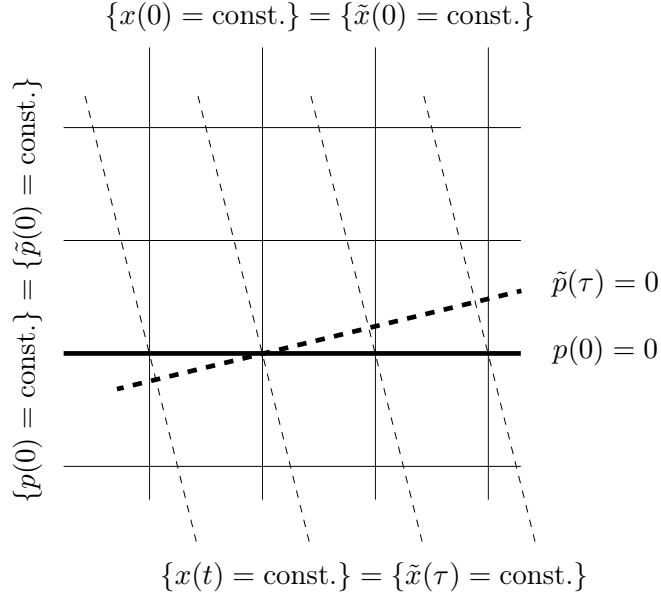
³Hamiltonian equations for harmonic oscillator:

$$\dot{\mathbf{r}}(\tau) = \frac{1}{m}\tilde{\mathbf{p}}(\tau), \quad \dot{\tilde{\mathbf{p}}}(\tau) = -k\tilde{\mathbf{r}}(\tau).$$

where $\omega = \sqrt{\frac{k}{m}}$. Suppose that positions and momenta coincide at initial time $t = 0$ (i.e. we have: $\{x(0) = \text{const.}\} = \{\tilde{x}(0) = \text{const.}\}$ and $\{p(0) = \text{const.}\} = \{\tilde{p}(0) = \text{const.}\}$). Observe that for

$$t = \frac{1}{\omega} \tan \omega\tau,$$

the configuration foliations of the two systems coincide. Indeed, we have:



$$\sin \omega\tau = \frac{\omega t}{\sqrt{1 + \omega^2 t^2}}, \quad \cos \omega\tau = \frac{1}{\sqrt{1 + \omega^2 t^2}},$$

and, therefore, equations (34) imply the following relations:

$$\tilde{x}(\tau) = \frac{1}{\sqrt{1 + \omega^2 t^2}} x(t), \quad (35)$$

$$\tilde{p}(\tau) = -\frac{m\omega^2 t}{\sqrt{1 + \omega^2 t^2}} x(t) + \sqrt{1 + \omega^2 t^2} p(t). \quad (36)$$

This proves that foliations $\{x(t) = \text{const.}\}$ and $\{\tilde{x}(\tau) = \text{const.}\}$ do coincide. If, therefore, $\phi(t, x)\sqrt{dx}$ is the wave function of a free moving particle, and $\psi(\tau, \tilde{x})\sqrt{d\tilde{x}}$ is a wave function of a harmonic oscillator, both starting with the same initial condition at $t = 0$, then both wave functions must coincide up to a Galilei transformation. Such a transformation is necessary because, due to (36), the reference section $\sigma := \{\tilde{p}(\tau) = 0\}$ for the harmonic oscillator corresponds to

$$p(t) = \frac{m\omega^2 t}{1 + \omega^2 t^2} x(t) .$$

But the wave function of the free motion describes the quantum state with respect to the reference $\lambda := \{p(t) = 0\}$. We have, therefore:

$$\sigma - \lambda = \frac{m\omega^2 t}{1 + \omega^2 t^2} x = d \left(\frac{1}{2} \frac{m\omega^2 t}{1 + \omega^2 t^2} x^2 \right) .$$

The quantity in brackets describes the phase of the Galilei factor. We conclude that the following equality holds:

$$\phi(t, x) \sqrt{dx} = (1 + \omega^2 t^2)^{-\frac{1}{4}} \psi \left(\frac{1}{\omega} \arctan \omega t, \frac{x}{\sqrt{1 + \omega^2 t^2}} \right) e^{\frac{i}{2\hbar} \frac{m\omega^2 t}{1 + \omega^2 t^2} x^2} \sqrt{dx}, \quad (37)$$

because $\sqrt{d\tilde{x}} = (1 + \omega^2 t^2)^{-\frac{1}{4}} \sqrt{dx}$. It is easy to check that the function ϕ satisfies the free Schrödinger equation if and only if ψ satisfies the Schrödinger equation for the harmonic oscillator.

The above *local* transformation between solutions of the free Schrödinger equation and the harmonic oscillator was first found by U. Niederer (see [27, 28]) and then reinterpreted by A. O. Barut in terms of the conformal group representation (see [29]). In paper [12] the same symmetries were derived as the only local symmetries in the multisymplectic formulation of the Schrödinger equation. The formula (37) is also known as the *lens transform* in the field of nonlinear Schrödinger equations (see [30, 31]).

4.2 Motion in constant electric field vs. free motion

A similar relation between free motion and the motion of a charged particle in a constant electric (or gravitational) field⁴ can be proved (again, for the maximal simplicity we limit ourselves to the 1D case):

$$\begin{aligned} \tilde{x}(\tau) &= \tilde{x}(0) + \frac{\tau}{m} \tilde{p}(0) + \frac{eE\tau^2}{2m}, \\ \tilde{p}(\tau) &= \tilde{p}(0) + eE\tau. \end{aligned} \quad (38)$$

We see that the position foliations $\{x(t) = \text{const.}\}$ and $\{\tilde{x}(\tau) = \text{const.}\}$ coincide for $t = \tau$. Indeed, comparing (33) with (38) we obtain the following relations

$$\begin{aligned} \tilde{x}(\tau) &= x(t) + \frac{eEt^2}{2m}, \\ \tilde{p}(\tau) &= p(t) + eEt. \end{aligned} \quad (39)$$

⁴Hamiltonian equations for a charged particle moving in a constant electric field \mathbf{E} :

$$\dot{\mathbf{r}}(\tau) = \frac{1}{m} \tilde{\mathbf{p}}(\tau), \quad \dot{\tilde{\mathbf{p}}}(\tau) = e\mathbf{E}.$$

The last equation implies that the Galilei transformation from the reference surface $\lambda := \{p(t) = 0\}$ to the new reference surface $\sigma := \{\tilde{p}(\tau) = 0\}$ consists in a simple shift by the constant (in variable x) value “ eEt ”. Consequently, the corresponding phase is linear in x and equals eEt .

Let $\phi(t, x)\sqrt{dx}$ be a wave function of a free particle, and $\psi(\tau, \tilde{x})\sqrt{d\tilde{x}}$ – a wave function of a charged particle, both starting from the same initial value at $t = 0$. Then, geometric quantization implies the following identity:

$$\phi(t, x)\sqrt{dx} = \psi\left(t, x + \frac{eEt^2}{2m}\right) e^{-\frac{i}{\hbar}eEt} \sqrt{dx}, \quad (40)$$

because $\sqrt{d\tilde{x}} = \sqrt{dx}$.

It is easy to check that the wave function ϕ satisfies the free Schrödinger equation if and only if ψ satisfies the following Schrödinger equation:

$$i\hbar\partial_\tau\psi(\tau, \tilde{x}) = -\frac{\hbar^2}{2m}\partial_{\tilde{x}}^2\psi(\tau, \tilde{x}) + \left(-eE\tilde{x} + \frac{e^2E^2\tau^2}{m}\right)\psi(\tau, \tilde{x}). \quad (41)$$

It describes the motion of our charged particle in the linear potential:

$$U(\tilde{x}) := -eE\tilde{x} + \frac{e^2E^2\tau^2}{m}, \quad (42)$$

i.e. in the constant electric field. The non-vanishing additive constant $\frac{e^2E^2\tau^2}{m}$ may be eliminated by the gauge transformation

$$U \rightarrow U' = U + \partial_\tau\chi, \quad (43)$$

whereas the wave function transforms as follows:

$$\psi' = \psi e^{-\frac{i}{\hbar}\chi}. \quad (44)$$

In our case we have

$$\chi = -\frac{e^2E^2\tau^3}{3m}. \quad (45)$$

We conclude that in the following equality:

$$\phi(t, x)\sqrt{dx} = \psi'\left(t, x + \frac{eEt^2}{2m}\right) e^{-\frac{i}{\hbar}eEt} e^{-\frac{i}{\hbar}\frac{e^2E^2t^3}{3m}} \sqrt{dx} \quad (46)$$

the wave function ϕ satisfies the free Schrödinger equation if and only if ψ' satisfies the Schrödinger equation with the standard potential

$$U'(\tilde{x}) := -eE\tilde{x}.$$

We stress, however, that these manipulations have no physical meaning: both the additive constant in the potential and the constant phase in the wave function have no physical interpretation and cannot be controlled within the framework we have defined in the present paper.

The transformation (46) is also known in the literature as the Avron-Herbst formula (see [32]).

4.3 Motion in constant magnetic field vs. free motion

Finally, we are going to show a relation between solutions of the Shrödinger equation for a free particle and for a charged particle moving in a constant magnetic field.⁵ Putting $\mathbf{B} = (0, 0, B)$, we obtain the non-trivial motion in the (x, y) plane:

$$\begin{aligned}\tilde{x}(\tau) &= \frac{1}{2}(\cos \omega\tau + 1)\tilde{x}(0) + \frac{1}{2}\sin \omega\tau\tilde{y}(0) \\ &\quad + \frac{1}{\omega m}\sin \omega\tau\tilde{p}_x(0) - \frac{1}{\omega m}(\cos \omega\tau - 1)\tilde{p}_y(0), \\ \tilde{y}(\tau) &= -\frac{1}{2}\sin \omega\tau\tilde{x}(0) + \frac{1}{2}(\cos \omega\tau + 1)\tilde{y}(0) \\ &\quad + \frac{1}{\omega m}(\cos \omega\tau - 1)\tilde{p}_x(0) + \frac{1}{\omega m}\sin \omega\tau\tilde{p}_y(0),\end{aligned}\tag{47}$$

$$\begin{aligned}\tilde{p}_x(\tau) &= -\frac{1}{4}\omega m\sin \omega\tau\tilde{x}(0) + \frac{1}{4}\omega m(\cos \omega\tau - 1)\tilde{y}(0) \\ &\quad + \frac{1}{2}(\cos \omega\tau + 1)\tilde{p}_x(0) + \frac{1}{2}\sin \omega\tau\tilde{p}_y(0), \\ \tilde{p}_y(\tau) &= -\frac{1}{4}\omega m(\cos \omega\tau - 1)\tilde{x}(0) - \frac{1}{4}\omega m\sin \omega\tau\tilde{y}(0) \\ &\quad - \frac{1}{2}\sin \omega\tau\tilde{p}_x(0) + \frac{1}{2}(\cos \omega\tau + 1)\tilde{p}_y(0),\end{aligned}\tag{48}$$

where $\omega = \frac{eB}{m}$. Suppose, as usual, that the appropriate foliations for the free motion and for the motion in magnetic field overlap at the beginning: $\{(x(0), y(0)) = \text{const.}\} = \{(\tilde{x}(0), \tilde{y}(0)) = \text{const.}\}$ and $\{(p_x(0), p_y(0)) = \text{const.}\} = \{(\tilde{p}_x(0), \tilde{p}_y(0)) = \text{const.}\}$. Then, the configuration foliations after time t also overlap for

$$t = \frac{2\sin \omega\tau}{\omega(\cos \omega\tau + 1)},$$

or, equivalently,

$$\sin \omega\tau = \frac{4\omega t}{4 + \omega^2 t^2}, \quad \cos \omega\tau = \frac{4 - \omega^2 t^2}{4 + \omega^2 t^2}.$$

We obtain, therefore, relations:

$$\begin{aligned}\tilde{x}(\tau) &= \frac{4}{4 + \omega^2 t^2}x(t) + \frac{2\omega t}{4 + \omega^2 t^2}y(t), \\ \tilde{y}(\tau) &= -\frac{2\omega t}{4 + \omega^2 t^2}x(t) + \frac{4}{4 + \omega^2 t^2}y(t),\end{aligned}$$

⁵In the case of a constant magnetic field \mathbf{B} we can choose the vector potential in the following form: $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \tilde{\mathbf{r}}$. Then, the Hamiltonian $H = \frac{(\tilde{\mathbf{p}} - e\mathbf{A})^2}{2m}$ implies the following equations

$$\dot{\tilde{\mathbf{r}}}(\tau) = \frac{e}{2m}\tilde{\mathbf{r}}(\tau) \times \mathbf{B} + \frac{1}{m}\tilde{\mathbf{p}}(\tau), \quad \dot{\tilde{\mathbf{p}}}(\tau) = -\frac{e^2}{8m}\nabla(\mathbf{B} \times \tilde{\mathbf{r}}(\tau))^2 + \frac{e}{2m}\tilde{\mathbf{p}}(\tau) \times \mathbf{B}.$$

$$\begin{aligned}\tilde{p}_x(\tau) &= -\frac{m\omega^2 t}{2(4 + \omega^2 t^2)} x(t) - \frac{m\omega^3 t^2}{2(4 + \omega^2 t^2)} y(t) + p_x(t) + \frac{1}{2}\omega t p_y(t), \\ \tilde{p}_y(\tau) &= \frac{m\omega^3 t^2}{2(4 + \omega^2 t^2)} x(t) - \frac{m\omega^2 t}{2(4 + \omega^2 t^2)} y(t) - \frac{1}{2}\omega t p_x(t) + p_y(t).\end{aligned}$$

Suppose now that $\phi(t, x, y)\sqrt{dx dy}$ describes the free particle evolution, and $\psi(\tau, \tilde{x}, \tilde{y})\sqrt{d\tilde{x} d\tilde{y}}$ the evolution of a charged particle in the constant magnetic field. If both functions satisfy the same initial conditions at $t = 0 = \tau$, then we have the identity:

$$\begin{aligned}\phi(t, x, y)\sqrt{dx dy} &= \sqrt{\frac{4}{4 + \omega^2 t^2}} \\ \times \psi\left(\frac{1}{\omega} \arcsin \frac{4\omega t}{4 + \omega^2 t^2}, \frac{4}{4 + \omega^2 t^2} x + \frac{2\omega t}{4 + \omega^2 t^2} y, -\frac{2\omega t}{4 + \omega^2 t^2} x + \frac{4}{4 + \omega^2 t^2} y\right) \\ &\times e^{\frac{i}{2\hbar} \frac{m\omega^2 t}{4 + \omega^2 t^2} (x^2 + y^2)} \sqrt{d\tilde{x} d\tilde{y}}. \quad (49)\end{aligned}$$

The density factor comes from the identity: $\sqrt{d\tilde{x} d\tilde{y}} = \sqrt{\frac{4}{4 + \omega^2 t^2}} \sqrt{dx dy}$. The phase factor comes from the generalized Galilei transformation (24), because “zero” surface $\{\mathbf{p}(t) = 0\}$ has to be replaced by the “new zero”: $\{\tilde{\mathbf{p}}(\tau) = 0\}$. It is easy to check that ϕ fulfills the free Schrödinger equation if and only if ψ fulfills the Schrödinger equation for a charged particle moving in a constant magnetic field.

5 Reproducing kernels for linear dynamics

Using our techniques we were able to show that any linear evolution is isomorphic to the free evolution. For this purpose we were not obliged to solve the Schrödinger equations. In the present section we prove that also the solution of the initial value problem can be easily obtained in terms of the generalized Fourier and the Galilei transformations.

5.1 Initial value problem for the free particle

Consider an initial quantum state at time $t = 0$, which is described by the wave function $\Psi_{\Lambda, \lambda}(x)$, where Λ is the corresponding configuration foliation, i.e. the collection of all the fibers $\{(x, p) : x = \text{const.}\}$, and $\lambda = \{p = 0\}$ is the “zero” (reference) surface. After the lapse of the t , the classical evolution of the system leads to the new variables $(x', p') := (x(t), p(t))$. We are going to prove that the corresponding quantum evolution leads *exactly* to the wave function $\Psi_{\Lambda', \lambda'}(x')$, where Λ' is the corresponding configuration foliation, i.e. the collection of the fibers $\{(x', p') : x' = \text{const.}\}$, whereas $\lambda' = \{p' = 0\}$.

Indeed, the (purely “static”) recalculation of the same quantum state from the old representation $\Psi_{\Lambda, \lambda}(x)$ to the new representation $\Psi_{\Lambda', \lambda'}(x')$ can be performed in three steps:

1. The generalized Galilei transformation between the reference $\lambda = \{p = 0\}$ to the new reference $\mu' = \{x' = 0\}$. Due to (33) we have:

$$(\lambda - \mu')(x) = \frac{m}{t}x dx = d\left(\frac{m}{2t}x^2\right),$$

which determines (up to an additive constant) the phase function $S_{\mu',\lambda} = \frac{m}{2t}x^2$. Hence, according to (24), we have:

$$\Psi_{\Lambda,\mu'}(x) = \Psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}\frac{m}{2t}x^2}. \quad (50)$$

2. In the second step, we perform the generalized Fourier transformation (29). Starting from the wave function $\Psi_{\Lambda,\mu'}$ we obtain $\Psi_{\Lambda',\mu}$, where $\mu = \{x = 0\}$:

$$\Psi_{\Lambda',\mu}(x') = \int \Psi_{\Lambda,\mu'}(x) e^{-\frac{i}{\hbar}\frac{m}{t}x'x} \sqrt{\frac{m}{i\hbar t}} \sqrt{dx} \sqrt{dx'}, \quad (51)$$

because $\frac{1}{i\hbar}\Omega = \frac{1}{i\hbar}dp \wedge dx = \frac{m}{i\hbar t}dx' \wedge dx$.

3. Finally, in order to calculate the wave function $\Psi_{\Lambda',\lambda'}$, we have to apply again the Galilei transformation from the reference $\mu = \{x = 0\}$ to the new reference $\lambda' = \{p' = 0\}$. Due to (33) we have

$$(\mu - \lambda')(x') = \frac{m}{t}x' dx' = d\left(\frac{m}{2t}x'^2\right).$$

This determines the generating function $S_{\lambda',\mu} = \frac{m}{2t}x'^2$. Hence, according to (24), we have:

$$\Psi_{\Lambda',\lambda'}(x') = \Psi_{\Lambda',\mu}(x') e^{\frac{i}{\hbar}\frac{m}{2t}x'^2}. \quad (52)$$

As a superposition of the three subsequent transformations: (50), (51) and (52), we finally obtain:

$$\Psi_{\Lambda',\lambda'}(x') = \sqrt{\frac{m}{i\hbar t}} \int \Psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}\frac{m}{2t}(x-x')^2} \sqrt{dx} \sqrt{dx'}. \quad (53)$$

To translate this formula to the standard textbook language, where the quantum state is usually represented by a *scalar* wave function, we make the following *Ansatz*:

$$\Psi_{\Lambda,\lambda}(x) := \psi_{\Lambda,\lambda}(x) \sqrt{dx}; \quad \Psi_{\Lambda',\lambda'}(x') := \psi_{\Lambda',\lambda'}(x') \sqrt{dx'}. \quad (54)$$

As a result, we obtain *exactly* the well-known resolution kernel for the free Schrödinger equation:

$$\psi_{\Lambda',\lambda'}(x') = \sqrt{\frac{m}{i\hbar t}} \int \psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}\frac{m}{2t}(x-x')^2} dx. \quad (55)$$

This is where the (arbitrary!) choice of a measure on the configuration space arises. We stress, however, that formula (53), where the quantum state is correctly represented by a *half-density*, is perfectly invariant with respect to any change of such a measure.

5.2 Initial value problem for the harmonic oscillator

The same three steps lead to the resolution kernel for the harmonic oscillator. We stress, that the information about the quantum evolution is entirely encoded in the classical evolution (34). Indeed, the first step consists in the generalized Galilei transformation:

$$(\lambda - \mu')(x) = m\omega \cot \omega t x dx = d\left(\frac{1}{2}m\omega \cot \omega t x^2\right),$$

and, whence, according to (24) we have:

$$\Psi_{\Lambda, \mu'}(x) = \Psi_{\Lambda, \lambda}(x) e^{\frac{i}{\hbar} \frac{1}{2} m\omega \cot \omega t x^2}. \quad (56)$$

Then, we apply the generalized Fourier transformation (29):

$$\Psi_{\Lambda', \mu}(x') = \int \Psi_{\Lambda, \mu'}(x) e^{-\frac{i}{\hbar} \frac{m\omega}{\sin \omega t} x' x} \sqrt{\frac{m\omega}{i\hbar \sin \omega t}} \sqrt{dx} \sqrt{dx'}, \quad (57)$$

because $\frac{1}{i\hbar} \Omega = \frac{1}{i\hbar} dp \wedge dx = \frac{m\omega}{i\hbar \sin \omega t} dx' \wedge dx$. Finally, we recalculate the wave function from the reference μ' to the reference λ . Formula (34) implies:

$$(\mu - \lambda')(x') = m\omega \cot \omega t x' dx' = d\left(\frac{1}{2}m\omega \cot \omega t x'^2\right),$$

and, consequently:

$$\Psi_{\Lambda', \lambda'}(x') = \Psi_{\Lambda', \mu}(x') e^{\frac{i}{\hbar} \frac{1}{2} m\omega \cot \omega t x'^2}. \quad (58)$$

Superposition of the three transformations: (56), (57) and (58) gives us:

$$\Psi_{\Lambda', \lambda'}(x') = \sqrt{\frac{m\omega}{i\hbar \sin \omega t}} \int \Psi_{\Lambda, \lambda}(x) e^{\frac{i}{\hbar} \frac{1}{2} m\omega \left(\cot \omega t (x^2 + x'^2) - \frac{2x'x}{\sin \omega t} \right)} \sqrt{dx} \sqrt{dx'}. \quad (59)$$

Again, when translated to the language of *scalar* wave functions by the *Ansatz* (54), the formula gives us the resolution kernel for the harmonic oscillator. As already discussed in Section 2, it is equal to the fractional Fourier transform (2).

5.3 Initial problem for the charged particle in constant electric field

Analogously, we consider the case of the charged particle in constant electric field. Again, using only the classical evolution (38) we recover its quantum version in the following three steps: 1) the Galilei transformation:

$$(\lambda - \mu')(x) = \left(\frac{m}{t}x + \frac{eEt}{2}\right) dx = d\left(\frac{m}{2t}x^2 + \frac{eEt}{2}x\right),$$

which implies:

$$\Psi_{\Lambda,\mu'}(x) = \Psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}(\frac{m}{2t}x^2 + \frac{eEt}{2}x)} ; \quad (60)$$

2) the generalized Fourier transformation (which looks similarly as in case of the free particle, cf. (51))

$$\Psi_{\Lambda',\mu}(x') = \int \Psi_{\Lambda,\mu'}(x) e^{-\frac{i}{\hbar}\frac{m}{t}x'x} \sqrt{\frac{m}{i\hbar t}} \sqrt{dx} \sqrt{dx'} , \quad (61)$$

because $\frac{1}{i\hbar}\Omega = \frac{1}{i\hbar}dp \wedge dx = \frac{m}{i\hbar t}dx' \wedge dx$ and 3) once more the Galilei transformation

$$(\mu - \lambda')(x') = \left(\frac{m}{t}x' + \frac{eEt}{2}\right)dx' = d\left(\frac{m}{2t}x'^2 + \frac{eEt}{2}x'\right),$$

and, consequently:

$$\Psi_{\Lambda',\lambda'}(x') = \Psi_{\Lambda',\mu}(x') e^{\frac{i}{\hbar}(\frac{m}{2t}x'^2 + \frac{eEt}{2}x')} . \quad (62)$$

Superposing the three transformations: (60), (61) and (62), we get

$$\Psi_{\Lambda',\lambda'}(x') = \sqrt{\frac{m}{i\hbar t}} \int \Psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}\frac{m}{2t}(x-x')^2} e^{\frac{i}{\hbar}\frac{eEt}{2}(x+x')} \sqrt{dx} \sqrt{dx'} . \quad (63)$$

The above wave function satisfies the Schrödinger equation in the linear, time-dependent potential

$$U(x) := -eEx - \frac{e^2E^2t^2}{8m} . \quad (64)$$

As already discussed in Section 4.2, the time dependence of the potential *via* an irrelevant constant $-\frac{e^2E^2t^2}{8m}$ can be removed by an appropriate gauge transformation. For this purpose the (physically irrelevant) phase factor

$$\exp\left(-\frac{i}{\hbar}\frac{e^2E^2t^3}{24m}\right),$$

can be applied. Finally, we obtain the formula

$$\Psi_{\Lambda',\lambda'}(x') = \sqrt{\frac{m}{i\hbar t}} \int \Psi_{\Lambda,\lambda}(x) e^{\frac{i}{\hbar}\frac{m}{2t}(x-x')^2} e^{\frac{i}{\hbar}\frac{eEt}{2}(x+x')} e^{-\frac{i}{\hbar}\frac{e^2E^2t^3}{24m}} \sqrt{dx} \sqrt{dx'} , \quad (65)$$

which, with the help of the *Ansatz* (54), may be easily translated to the language of scalar wave functions.

5.4 Initial problem for the charged particle in constant magnetic field

For the sake of completeness we discuss also the charged particle in constant magnetic field. Using classical dynamics (47) and (48) we obtain: 1) the Galilei transformation from $\lambda = \{p_x = 0, p_y = 0\}$ to $\mu' = \{x' = 0, y' = 0\}$:

$$\begin{aligned} (\lambda - \mu')(x, y) &= \frac{1}{2}m\omega \cot \frac{\omega t}{2} x dx + \frac{1}{2}m\omega \cot \frac{\omega t}{2} y dy \\ &= d\left(\frac{1}{4}m\omega \cot \frac{\omega t}{2} (x^2 + y^2)\right), \end{aligned}$$

hence

$$\Psi_{\Lambda, \mu'}(x, y) = \Psi_{\Lambda, \lambda}(x, y) e^{\frac{i}{\hbar} \frac{1}{4} m \omega \cot \frac{\omega t}{2} (x^2 + y^2)}; \quad (66)$$

2) the generalized Fourier transformation:

$$\begin{aligned} \Psi_{\Lambda', \mu}(x', y') &= \int \Psi_{\Lambda, \mu'}(x, y) e^{-\frac{i}{\hbar} \frac{1}{2} m \omega \left(\cot \frac{\omega t}{2} x' x - y' x + x' y + \cot \frac{\omega t}{2} y' y \right)} \\ &\quad \times \sqrt{\frac{m^2 \omega^2}{2(i\hbar \sin \frac{\omega t}{2})^2}} \sqrt{dx dy} \sqrt{dx' dy'}, \quad (67) \end{aligned}$$

because $\frac{1}{i\hbar} \Omega = \frac{1}{i\hbar} dp_x \wedge dx + \frac{1}{i\hbar} dp_y \wedge dy = \frac{m\omega}{2i\hbar} (\cot \frac{\omega t}{2} dx' \wedge dx - dy' \wedge dx + dx' \wedge dy + \cot \frac{\omega t}{2} dy' \wedge dy)$ and

3) the Galilei transformation from $\mu = \{x = 0, y = 0\}$ to $\lambda' = \{p'_x = 0, p'_y = 0\}$:

$$\begin{aligned} (\mu - \lambda')(x', y') &= \frac{1}{2}m\omega \cot \frac{\omega t}{2} x' dx' + \frac{1}{2}m\omega \cot \frac{\omega t}{2} y' dy' \\ &= d\left(\frac{1}{4}m\omega \cot \frac{\omega t}{2} (x'^2 + y'^2)\right), \end{aligned}$$

hence

$$\Psi_{\Lambda', \lambda'}(x', y') = \Psi_{\Lambda', \mu}(x', y') e^{\frac{i}{\hbar} \frac{1}{4} m \omega \cot \frac{\omega t}{2} (x'^2 + y'^2)}. \quad (68)$$

Finally, formulae (66), (67) and (68) imply:

$$\begin{aligned} \Psi_{\Lambda', \lambda'}(x', y') &= \sqrt{\frac{m^2 \omega^2}{2(i\hbar \sin \frac{\omega t}{2})^2}} \\ &\times \int \Psi_{\Lambda, \lambda}(x, y) e^{\frac{i}{\hbar} \frac{1}{4} m \omega \cot \frac{\omega t}{2} ((x-x')^2 + (y-y')^2)} e^{-\frac{i}{\hbar} \frac{1}{2} m \omega (x' y - y' x)} \sqrt{dx dy} \sqrt{dx' dy'}, \quad (69) \end{aligned}$$

which is the correct resolution kernel of the quantum initial value problem.

6 Quantum connection

Given two mutually transversal, compatible Lagrangian foliations Λ_1 and Λ_2 of the phase space \mathcal{P} , it is always possible to choose linear canonical variables (x^i, p_i) in such a way that the symplectic form reduces to (18). This way our construction does not go beyond the Heisenberg group. However, if we take Λ_3 compatible with Λ_2 , this does not imply compatibility of Λ_1 with Λ_3 . Transforming the quantum state form Λ_1 first to Λ_2 and then from Λ_2 to Λ_3 in a way defined in this paper, we finally obtain transformation between foliation which may be far from being compatible. As an example take again $\dim \mathcal{P} = 2$ and consider a generalized Galilei transformation of the form

$$\tilde{p} = p + \varphi(x) .$$

Take

$$\Lambda_1 = \{p = \text{const.}\} , \quad \Lambda_2 = \{x = \text{const.}\} , \quad \Lambda_3 = \{\tilde{p} = \text{const.}\} .$$

Because both (x, p) and (x, \tilde{p}) are canonical variables, the pairs (Λ_1, Λ_2) and (Λ_2, Λ_3) are mutually compatible. But (Λ_1, Λ_3) are, in general, non-compatible⁶.

Another example is provided by any non-linear transformation of positions: $x := f(X)$. We have

$$\Omega = dp \wedge dx = dp \wedge f'(X)dX = dP \wedge dX , \quad (70)$$

where $f'(X) := \frac{df}{dX}(X)$, $P(x, p) := p \cdot (f' \circ f^{-1}(x))$. Take:

$$\Lambda_1 = \{p = \text{const.}\} , \quad \Lambda_2 = \{x = \text{const.}\} = \{X = \text{const.}\} , \quad \Lambda_3 = \{P = \text{const.}\} .$$

Of course, (Λ_1, Λ_2) and (Λ_2, Λ_3) are pairwise compatible. But, in general, (Λ_1, Λ_3) is not and the canonical transformation $(x, p) \mapsto (X, P)$ may be highly non-linear. Using our techniques we are, however, able to construct uniquely the quantum counterpart of such non-linear canonical transformations. This construction can be extended to an arbitrary sequence of foliations: $(\Lambda_1, \Lambda_2, \dots, \Lambda_i, \Lambda_{i+1}, \dots, \Lambda_N)$, such that two subsequent foliations are mutually compatible.

The following questions arise:

1. Starting from a foliation $\Lambda_{initial}$, can we reach this way *any* foliation Λ_{final} ?
2. Does the result depend upon the path, if Λ_{final} can be reached from $\Lambda_{initial}$ in two different ways?

⁶They are compatible if and only if φ is linear.

Let us consider the infinitesimal version of this problem. In the first example we put

$$\tilde{p} = p + \epsilon\varphi(x) ,$$

and consider the resulting Hamiltonian vector field (i.e. an “infinitesimal canonical transformation”)

$$Y_H = \varphi(x) \frac{\partial}{\partial p} ,$$

generated by the “Hamiltonian function” of the form: $H(x, p) = h(x)$ such that $\varphi = -h'$.

In the second example we put

$$x = f(X) = X - \epsilon g(X) .$$

Infinitesimal version of the canonical transformation

$$(X, P) = (f^{-1}(x), p \cdot (f' \circ f^{-1}(x))) ,$$

can be easily obtained if we observe that (in first order in ϵ) we have:

$$X = f^{-1}(x) \simeq x + \epsilon g(x)$$

and, consequently,

$$P = p \cdot (f' \circ f^{-1}(x)) \simeq p (1 - \epsilon g'(x)) .$$

This corresponds to the vector field

$$Y_H = -p \cdot g'(x) \frac{\partial}{\partial p} + g(x) \frac{\partial}{\partial x} ,$$

generated by the Hamiltonian function of the form: $H(x, p) = p \cdot g(x)$.

This way we cover infinitesimal transformations generated by Hamiltonian functions of the type $h(x)$ and $p \cdot g(x)$, where h and g are arbitrary (non-linear) functions. But, we have proved in this paper, that *linear* symplectic group has an *exact* projective representation in the space of quantum states. Superposing this representation with the above two types of generators, we conclude that also generators of the type $h(\xi)$ and $\eta \cdot g(\xi)$, where ξ and η are arbitrary linear combinations of x and p , can be reached this way. It is obvious, that any Hamiltonian can be approximated by sums of such functions. We see that our construction enables us to lift any *infinitesimal* canonical transformation to the space of quantum states.

The bundle of all possible quantum states over all possible Lagrangian foliations acquires, therefore, a unitary connection. Consequently, any one-parameter family of foliations generated from any $\Lambda_{initial}$ by a classical dynamics in \mathcal{P}) can be “quantized”, i.e. lifted to the space of quantum states.

It can be checked that this connection is non-flat, i.e. such a quantization is path-dependent. This means that finite canonical transformations cannot be quantized, i.e. the representation of the *linear* symplectic group cannot be extended to a representation of the complete (non-linear) group.

7 Conclusions

In this paper we have proved that for linear systems, the quantum evolution is uniquely and unambiguously generated by its classical counterpart in terms of the (appropriately geometrized) Fourier and Galilei transformations. This construction leads to a unique *projective* representation of the linear symplectic group. This construction is virtually unknown, although both its ingredients (the standard Fourier transformation and the generalized Galilei transformation) belong to the classical repertoire of quantum mechanics. It implies that solutions of any Schrödinger equation corresponding to a *linear* classical evolution can be obtained from solutions of the *free* Schrödinger equation *via* a local (in space and time) transformation. Such an observation may provide a valuable mathematical tool in quantum optics.

In case of a generic, non-linear evolution, the above construction cannot work because the evolution does not preserve the *compatibility* of the corresponding phase-space foliations. Nevertheless, generalized Galilei transformations allow us to go beyond linear symplectic structure, at least infinitesimally. This way a (non-flat) connection in the bundle of quantum states is uniquely constructed. It allows us to “quantize” any classical evolution, i.e. one-parameter family of symplectomorphisms. The non-flatness of the connection implies the non-existence of an extension of the above representation of the linear symplectic group to representation of the complete symplectic group.

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