Gravitational Energy: a quasi-local, Hamiltonian approach

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Abstract

Hamiltonian evolution of gravitational field within a spatially compact world tube with non-vanishing boundary is discussed. A universal Hamiltonian identity is proved which relates gravitational Cauchy data (internal and external 3-geometry of a Cauchy surface) with boundary data (internal and external 3-geometry of the tube). It is shown how different ways to control the boundary data lead to different quasi-local definitions of gravitational energy (mass).

1 Introduction

Energy integral – if there exists any – provides an important tool in the theory of partial differential equations. The key point in most "existence and uniqueness" proofs is based on *a priori* estimations. These estimations are usually possible due to positivity (or – even better – convexity) of the energy functional. As an example we can take the scalar field theory whose dynamics is governed by the wave equation: $\ddot{\varphi} = \Delta\varphi$. Here, the amount of field energy contained in a 3–volume V is given by the following integral[1]:

$$E_V = \frac{1}{2} \int_V \left[\pi^2 + (\nabla\varphi)^2\right] \mathrm{d}^3x \ , \tag{1}$$

were by $\pi := \dot{\varphi}$ we denote the momentum canonically conjugate to the field configuration φ.

Another example is given by the linear Maxwell electrodynamics, where the field energy is given by the Maxwell formula:

$$E_V = \frac{1}{2} \int_V \left(D^2 + B^2\right) \mathrm{d}^3x \ , \tag{2}$$

with (D, B) being the electric and magnetic induction fields. It is interesting to note that formula (2) reduces to (1). Indeed, evolution of the Maxwell field reduces to two decoupled wave equations. The sum of energies of these two (independent) degrees of freedom, calculated according to (1), is equal precisely to the right hand side of (2). This issue is discussed in Section 5.

Search for an appropriate "energy integral" in mathematically exotic field theories may sometimes look chaotic. However, relativistic theories which can be derived form a variational principle (*Lagrangean* theories) admit a universal procedure, based on Noether theorem, which enables us to define a conserved energy-momentum tensor. When appropriately integrated over a Cauchy surface, energy-momentum tensor assigns a conserved quantity to every symmetry of the theory. But even there, ambiguities relative to the behaviour of the field at the boundary ∂V do persist. As an example consider Maxwell electrodynamics, where at least two different energy-momentum tensors: the "canonical" one and the "symmetric" one, can be defined. The standard, textbook comment concerning this ambiguity says that Noether theorem gives the appropriate energy value (2) only "up to boundary terms". We discuss this issue in the sequel and show that the existence of different energy functionals in Maxwell theory is nothing exceptional: it is a straightforward consequence of the fact that the field boundary data can be controlled in many non-equivalent ways. In particular, formula (2) corresponds to a specific ("Dirichlet") boundary-value problem for the field.

[1]In this paper "dot" always denotes the time derivative. To describe the massive Klein-Gordon field we have to add the potential field energy $U(\varphi) = m^2\varphi^2$, whereas a generic function $U(\varphi)$ corresponds to a generic non-linear scalar field theory.

In General Relativity Theory the role of field energy is even more ambiguous. A generic spacetime does not admit any symmetry which, according to special-relativistic procedures, would be necessary to define a conserved quantity. On the other hand, every spacetime diffeomorphism is a symmetry of the theory. This observation suggests that, maybe, there are too many conserved quantities. Which one among them (if any) describes the true field energy?

When analyzed in terms of the standard, textbook language of a Hamiltonian Field Theory, gravitational field admits no energy-momentum tensor (cf. [1] and [2]). Various "pseudotensors" which have been proposed in this context do not describe correctly the energy content of the field. In our opinion the reason is obvious: the very nature of the gravity implies that the energy cannot be additive:

$$E_{V_1 \cup V_2} \lneqq E_{V_1} + E_{V_2} \ . \tag{3}$$

Indeed, gravitational interaction between the masses (energies) contained in both volumes V_1 and V_2 implies that the union $V_1 \cup V_2$ contains less energy than the sum on the right-hand side. Hence, gravitational energy contained in a volume V cannot be obtained *via* a simple integration of any "energy density" over V.

The failure of the standard "Hamiltonian approach" to General Relativity Theory is due to the fact that, technically, such an approach is based on integration by parts and neglecting boundary integrals arising this way: they are always supposed to vanish. When defining appropriate functional spaces for mathematically rigorous analysis of the field evolution, the above rule is treated implicitly as a sort of a *paradigm*. Indeed, functional-analytic framework for description of Cauchy data is always chosen in such a way that the boundary integrals vanish automatically!

Such a procedure works in Special Relativity, where strong fall-off conditions at spatial infinity may be imposed on matter fields. But in General Relativity Theory strong fall-of conditions are incompatible with the field evolution. In fact, assuming that spacetime metric $g_{\mu\nu}$ approaches the flat metric $\eta_{\mu\nu}$ as fast as $\frac{1}{r^2}$ implies that the metric must be *globally flat*!

Trying to mimic in General Relativity the special relativistic Hamiltonian procedures, we painfully discover that the rule: "surface integrals vanish and the entire information is carried by the volume integrals" must be replaced by the opposite rule: "volume integrals vanish and the entire information is carried by the surface integrals". Consequently, gravitational field energy has to be assigned to the boundary ∂V rather, than to the volume V itself. This point of view, known as the *quasilocal* approach, was first proposed in 1982 by Roger Penrose (see [3]). Meanwhile, over 20 different definitions (to our knowledge) of the "quasilocal mass" have been proposed. Also the present authors took part in this discussion (cf. [17], [18], [4]).

It is not our goal to give here arguments supporting any of these 20 definitions. Instead, we present a general Hamiltonian framework, where boundary integrals are not neglected but are kept as meaningful objects. When applied to gravitational field, this approach leads directly to quasi-local quantities: each of them related to a particular boundary value problem. Any definition of the "quasi-local mass" must be based on a particular choice of a "boundary control mode". All of them are contained in the "homogeneous Hamiltonian formula", which we present in Section 6. This formula is the main subject of our paper and, in our opinion, provides a natural starting point for any serious analysis of gravitational energy.

2 Symplectic relations and their generating functions

Energy is a generating function of the field dynamics with respect to the natural (canonical) symplectic structure carried by the theory. To illustrate this point of view consider first law of thermodynamics of a simple gaz:

$$\mathrm{d}U(V,S) = -p\,\mathrm{d}V + T\mathrm{d}S \ . \tag{4}$$

We interpret this formula in the following way: the four-dimensional space $\mathcal{P}$ labeled by four physical parameters: the volume V, the entropy S, the pressure p and the temperature T, is equipped with an exterior one-form

$$\theta := -p\,\mathrm{d}V + T\mathrm{d}S \ , \tag{5}$$

and, therefore, can be treated as the cotangent bundle over the two-dimensional space $Q := \{(V,S)\}$ describing "control parameters" (V,S). We have, namely:

$$\mathcal{P} \simeq T^*Q \ ,$$

where "$-p$" plays role of the momentum canonically conjugate to V and "T" is conjugate to S. For a given amount of a specific gaz, the collection of all its admissible physical states forms a two-dimensional submanifold $\mathcal{D} \subset \mathcal{P}$ defined as the graph of the differential of a certain function U defined on Q. Indeed, formula (4) can be read in the following way:

$$\mathcal{D} = \mathrm{graph}(\mathrm{d}U) := \{(p,T,V,S) : -p = \frac{\partial U}{\partial V} \ ; \quad T = \frac{\partial U}{\partial S}\} \subset \mathcal{P} \ . \tag{6}$$

Physically, U describes internal energy of the gas.

Instead of internal energy U we may use the Helmholtz free energy $F = F(V,T)$ to describe the same "dynamics" $\mathcal{D}$. For this purpose we rewrite the first law (4) in the following way:

$$\mathrm{d}U(V,S) = -p\,\mathrm{d}V + \mathrm{d}(TS) - S\mathrm{d}T \ . \tag{7}$$

Putting now

$$F := U - TS$$

and treating it as a function of V and T, we finally obtain:

$$\mathrm{d}F(V,T) = -p\,\mathrm{d}V - S\mathrm{d}T \ , \tag{8}$$

or, equivalently:

$$\mathcal{D} = \mathrm{graph}(\mathrm{d}F) := \{(p,T,V,S) : -p = \frac{\partial F}{\partial V} \ ; \quad -S = \frac{\partial F}{\partial T}\} \subset \mathcal{P} \ . \tag{9}$$

This means that now we control the volume and the temperature, i.e. the point of the new configuration space $R := \{(V,T)\}$ and we have

$$\mathcal{P} \simeq T^*R \ ,$$

carrying the canonical one-form

$$\tilde{\theta} := -p\,\mathrm{d}V - S\mathrm{d}T \ . \tag{10}$$

Observe that the following two-form:

$$\omega := \mathrm{d}V \wedge \mathrm{d}p + \mathrm{d}T \wedge \mathrm{d}S \ , \tag{11}$$

satisfies

$$\omega = \mathrm{d}\theta = \mathrm{d}\tilde{\theta} \ , \tag{12}$$

and, therefore, is invariant with respect to the above manipulations. We conclude that ω has to be considered as the fundamental structure of $\mathcal{P}$. This means that $(\mathcal{P}, \omega)$ is a symplectic manifold, whereas the two identifications:

$$T^*Q \simeq \mathcal{P} \simeq T^*R$$

are merely the two (among infinitely many others, which are also possible) "control modes". Physically, a control mode consists in choosing two parameters which we are going to control, whereas the remaining two "momenta" can be measured in the laboratory as the "response parameters". Such a choice must respect the symplectic structure of $\mathcal{P}$. The two examples discussed above are easily obtained by choosing in formula (11) either (V, S) (which leads directly to formula (5)) or (V, T) (which leads directly to formula (10)) as control parameters (see Fig. 1). But infinitely many other control modes, based on combinations of these parameters (also non-linear) are possible.

Physical laws governing the system are described by the submanifold $\mathcal{D} \subset \mathcal{P}$. It encodes the control-response relations. The fundamental paradigm of this approach is that the "dynamics" $\mathcal{D}$ is a *Lagrangean* (i.e. maximal, isotropic) submanifold of $\mathcal{P}$. This implies (cf. [21]) that $\mathcal{D}$ can be treated (at least locally) as the graph of the differential of a certain function of control parameters. Such a function is given uniquely up to an additive constant and is called a *generating function* of $\mathcal{D}$ with respect to the given control mode.

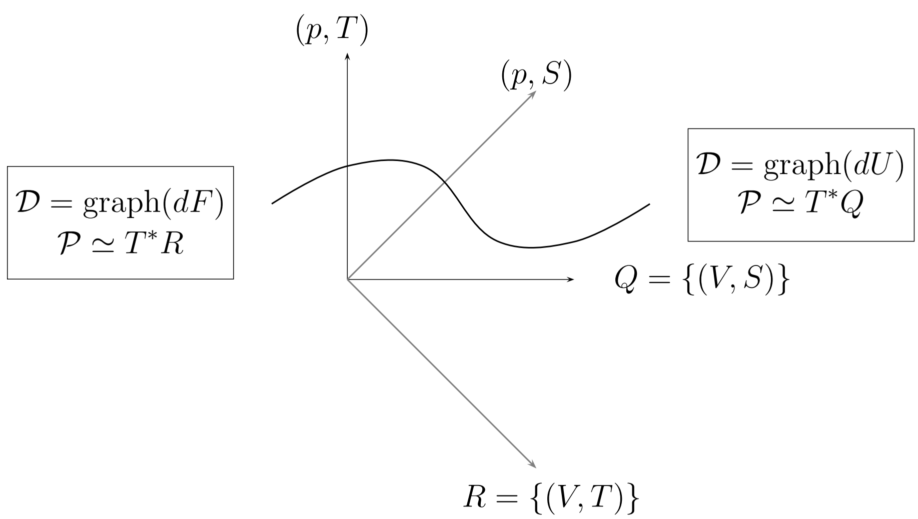

Fig. 1

The mathematical framework presented above can be easily improved in such a way that much more general situations can also be described this way (like e.g.: 1. constrained control modes, where the dynamics $\mathcal{D}$ does not cover the whole control space Q but only its submanifold $\mathcal{C} \subset Q$, or 2. singularities of $\mathcal{D}$ with respect to the foliation T^*Q). Nevertheless, the key points of this approach remain always the same (cf. [21]):

1. The "meta–theory" (i.e. theory of all possible simple thermodynamical bodies) is described by a symplectic manifold $(\mathcal{P}, \omega)$. At a first glance, this space looks too big: it contains twice as many parameters as are necessary to describe all the physical states of a specific physical system belonging to this theory (i.e. four parameters in thermodynamics, whereas only two parameters of a simple body can be freely controlled).

2. Every specific example of this theory (i.e. a specific simple thermodynamical body, with all its physical properties) is described by a Lagrangan submanifold $\mathcal{D} \subset \mathcal{P}$ called the *dynamics* of the theory.

3. A control mode is a symplectomorphism $T^*Q \simeq \mathcal{P}$, where Q is an arbitrary space of control parameters. Then, there is a "generating function" of the dynamics U, such that $\mathcal{D} = \mathrm{graph}(\mathrm{d}U)$.

4. Among all possible control modes there is one which corresponds to the "adiabatic insulation" of the system. The generating function with respect to this mode describes the *true energy* of the system.

To recognize the "true energy" among all possible generating functions of the dynamics, physical arguments are necessary, which go beyond the above mathematical structure. Nevertheless, positivity or convexity of the generating function is usually a decisive criterion.

3 Lagrangian and Hamiltonian formulations of mechanics

Euler-Lagrange equation in classical mechanics, together with the definition of canonical momenta, may be written as:

$$dL(q^i, \dot{q}^i) = \dot{p}_i \, \mathrm{d}q^i + p_i \, \mathrm{d}\dot{q}^i \ , \tag{13}$$

or, equivalently,

$$\mathcal{D} := \left\{ (p_i, \dot{p}_i, q^i, \dot{q}^i) \;\middle|\; \dot{p}_i = \frac{\partial L}{\partial q^i} \quad ; \quad p_i = \frac{\partial L}{\partial \dot{q}^i} \right\} \subset \mathcal{P} \ . \tag{14}$$

Here, by $\mathcal{P}$ we denote the "meta-space" parameterized by the $4N$ parameters: N positions (q^i), N velocities $(\dot{q}^i)$, N momenta (p_i), and, finally, N forces $(\dot{p}_i)$. By L we denote the Lagrangian function depending upon positions and velocities. Again, the space is twice too big to describe a specific mechanical system with N degrees of freedom: its dynamics (14) is a ($2N$-dimensional) collection of all the points which are accessible for such a system.

The Legendre transformation to the Hamiltonian description of the same dynamics is analogous to the Legendre transformation (7) in thermodynamics:

$$dL(q^i, \dot{q}^i) = \dot{p}_i \, \mathrm{d}q^i + \mathrm{d}\left(p_i \, \dot{q}^i\right) - \dot{q}^i \, \mathrm{d}p_i \ . \tag{15}$$

Putting now

$$-H := L - p_i \, \dot{q}^i$$

we obtain:

$$-dH(q^i, p_i) = \dot{p}_i \, \mathrm{d}q^i - \dot{q}^i \, \mathrm{d}p_i \ , \tag{16}$$

or, equivalently:

$$\mathcal{D} := \left\{ (p_i, \dot{p}_i, q^i, \dot{q}^i) \;\middle|\; \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad ; \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \right\} \subset \mathcal{P} \ . \tag{17}$$

We conclude that the two one-forms:

$$\theta := \dot{p}_i \, \mathrm{d}q^i + p_i \, \mathrm{d}\dot{q}^i \ , \qquad \text{and} \quad \tilde{\theta} := \dot{p}_i \, \mathrm{d}q^i - \dot{q}^i \, \mathrm{d}p_i$$

are merely the two (among infinitely many) primitive forms of the symplectic two-form

$$\omega := \mathrm{d}\dot{p}_i \wedge \mathrm{d}q^i + \mathrm{d}p_i \wedge \mathrm{d}\dot{q}^i \ \left(= \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{d}p_i \wedge \mathrm{d}q^i \right) \right) \ , \tag{18}$$

which is invariant with respect to the Legendre transformation and, therefore, can be considered as the fundamental structure of the "meta-phase-space" $\mathcal{P}$. Lagranian and Hamiltonian dynamics are the two possible control modes, describing the same dynamics $\mathcal{D} \subset \mathcal{P}$. In the first one, control parameters $(q^i, \dot{q}^i)$ run over the tangent space TQ of the configuration space Q, whereas momenta (p_i) and forces $(\dot{p}_i)$ parameterize the response parameters, according to (14). In the second one, control parameters (q^i, p_i) run over the co-tangent space T^*Q of the configuration space Q, whereas velocities $(\dot{q}^i)$ and forces $(\dot{p}_i)$ describe the response parameters, according to (17). Geometrically, the meta-phase-space $\mathcal{P}$ can be identified with the tangent bundle to the co-tangent bundle: $\mathcal{P} = T(T^*Q)$. The symplectomorphism

$$T^*(TQ) \simeq \mathcal{P} \simeq T^*(T^*Q) \ ,$$

was was first described by W. M. Tulczyjew (cf. [5] or [6]) and is called the *Tulczyjew triple.*

4 Field dynamics as a symplectic relation

A Lagrangian field theory can easily be formulated within this framework. As an example consider a scalar field theory whose dynamics is generated by a Lagrangian function

$$L = L(\varphi, \varphi_\mu) \ ,$$

where $\varphi_\mu := \partial_\mu \varphi$. Field equations (Euler-Lagrange eqs.), together with definition of the canonical field momenta: $p^\mu = \frac{\partial L}{\partial \varphi_\mu}$, can be written in the following way:

$$\mathcal{D} := \left\{ (p^\mu, \partial_\mu p^\mu, \varphi, \varphi_\mu) \,\middle|\, \partial_\mu p^\mu = \frac{\partial L}{\partial \varphi} \quad ; \quad p^\mu = \frac{\partial L}{\partial \varphi_\mu} \right\} \ , \tag{19}$$

or, equivalently:

$$\delta L(\varphi, \varphi_\mu) = \partial_\mu (p^\mu \delta \varphi) = (\partial_\mu p^\mu) \delta \varphi + p^\mu \delta \varphi_\mu \ . \tag{20}$$

These formulae have the following, symplectic interpretation: at every spacetime point (x^μ) we have a 10-dimensional symplectic "meta-phase-space" $\mathcal{P} = \{(p^\mu, \partial_\mu p^\mu, \varphi, \varphi_\mu)\}$ of quantities living at this point. These quantities are: the field variable φ, its four derivatives φ_μ ("velocities"), the components p^μ of the corresponding canonical momentum and, finally, the "current" $j := \partial_\mu p^\mu$ (divergence of the momentum). By "δ" we denote external derivative within this space (i.e. "vertical" derivative, in contrast to spacetime derivatives ∂_μ). These objects can be organized into invariant, geometric structures: jets of sections of natural bundles over spacetime (see [21]). The theory which we present here in terms of local coordinates is, therefore, perfectly coordinate-invariant.

At each spacetime point $\mathbf{x} = (x^\mu)$ we denote by $\mathcal{D}_\mathbf{x} \subset \mathcal{P}_\mathbf{x}$ the collection of all physically admissible points of this meta-phase-space. Solution of the field equations are those, whose jets belong at every point $\mathbf{x}$ to this subspace. Similarly as in the previous sections, we see that $\mathcal{P} - \mathbf{x}$ carries the canonical symplectic structure

$$\Omega_\mathbf{x} := \delta\theta = \delta j \wedge \delta\varphi + \delta p^\mu \wedge \delta\varphi_\mu = \partial_\mu \left(p^\mu\, \delta\varphi\right) \ , \tag{21}$$

where

$$\theta_\mathbf{x} := \partial_\mu(p^\mu \delta\varphi) = (\partial_\mu p^\mu)\delta\varphi + p^\mu \delta\varphi_\mu \ . \tag{22}$$

Formula (21), analogous to (18), has an intrinsic, geometric meaning (see e.g. [21]) and is coordinate-invariant. Variational formulation of the field dynamics may be considered as a particular control mode in $\mathcal{P}$, where (φ, φ_μ) (i.e. first jet of the field configuration) has been chosen as control parameters.

Another control mode leads to the Hamiltonian formulation of the dynamics. For this purpose we have to choose a "3+1"-decomposition of spacetime. Once chosen such a decomposition, we may rewrite the Lagrangian generating formula (20) in the following way:

$$\delta L(\varphi, \varphi_\mu) = \partial_\mu(p^\mu \delta\varphi) = \partial_0(p^0 \delta\varphi) + \partial_k(p^k \delta\varphi) = \dot{\pi}\delta\varphi + \pi\delta\dot{\varphi} + \partial_k(p^k\delta\varphi) \ , \tag{23}$$

where $\pi := p^0$. Here, dot denotes time derivative (i.e. derivative with respect to $t = x^0$), whereas $k = 1, 2, 3$; labels space-like dimensions. Now, standard "Legendre" manipulations (see e.g. [1]) lead to the following result:

$$\delta L(\varphi, \varphi_\mu) = \dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi + \delta(\pi\dot{\varphi}) + \partial_k(p^k\delta\varphi) \ . \tag{24}$$

Consequently, we have

$$-\delta H = \dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi + \partial_k(p^k\delta\varphi) \ , \tag{25}$$

where we have defined the energy density H in the following way:

$$H = \pi\dot{\varphi} - L \ .$$

It has to be expressed as a function of the new control parameters: $H = H(\varphi, \varphi_k, \pi)$, whereas the remaining quantities (j, p^k, φ) are now treated as response parameters: their values are obtained from the generating formula

(25). Integrating this formula over a finite portion V of the Cauchy surface $\Sigma := \{x^0 = \text{const.}\}$, we obtain

$$-\delta \int_V H = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) + \int_{\partial V} p^\perp \delta\varphi \ , \tag{26}$$

where $p^\perp$ denotes the normal (with respect to the boundary ∂V) component of p. Above formula is usually presented in its simplified version:

$$-\delta \, \mathcal{H}_V = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) \ , \tag{27}$$

where the boundary term has been neglected and $\mathcal{H}_V$ denotes the total energy contained in V:

$$\mathcal{H}_V := \int_V H \ . \tag{28}$$

Formula (27) is treated usually as an infinite-dimensional version of the Hamiltonian formula (16). But the complete formula (26) is a relation between the *three objects*: 1) the initial data (φ, π), 2) their time derivatives $(\dot{\varphi}, \dot{\pi})$ and 3) the boundary data $(\varphi, p^\perp)|_{\partial V}$ (in contrast to the situation in mechanics, where only the two first objects enter into the game). The boundary data (non existent in mechanics) really influence the field evolution and cannot be neglected. To obtain from (26) an infinite-dimensional Hamiltonian system (27), Dirichlet boundary conditions: $\varphi|_{\partial V} = f$, have to be imposed. This means that we limit ourselves to field configurations satisfying this condition. Within this restricted phase space we have $(\delta\varphi)|_{\partial V} \equiv 0$ and, indeed, (26) reduces to (27).

This is not a unique way to fix boundary conditions: we might as well rewrite the boundary term of (26) in the following way:

$$\int_{\partial V} p^\perp \delta\varphi = \delta \int_{\partial V} (p^\perp \varphi) - \int_{\partial V} \varphi \, \delta p^\perp \ . \tag{29}$$

Putting the complete differential on the right-hand-side, we obtain this way:

$$-\delta \, \tilde{\mathcal{H}}_V = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) - \int_{\partial V} \varphi \, \delta p^\perp \ , \tag{30}$$

where we have defined the new Hamiltonian function:

$$\tilde{\mathcal{H}}_V := \mathcal{H}_V + \int_{\partial V} (p^\perp \varphi) \ . \tag{31}$$

Now, to obtain the infinite-dimensional Hamiltonian dynamics, Neumann boundary conditions: $(p^\perp)|_{\partial V} = f$, have to be imposed. This means that we limit ourselves to field configurations satisfying this condition. Within this restricted phase space we have $(\delta p^\perp)|_{\partial V} \equiv 0$ and, indeed, (30) reduces to

$$-\delta\,\tilde{\mathcal{H}}_V = \int_V (\dot{\pi}\delta\varphi - \dot{\varphi}\delta\pi) \ . \tag{32}$$

Both (27) and (32) define infinite dimensional Hamiltonian systems – see Fig. 2. To decide which one among the two Hamiltonian functions describes the field energy (and not the *free energy*) we must have more insight into the physics they describe. Usually, the energy positivity provides a decisive argument. Otherwise, we may ask which one among possible boundary conditions corresponds to the adiabatic insulation of the field contained in V from the external field.

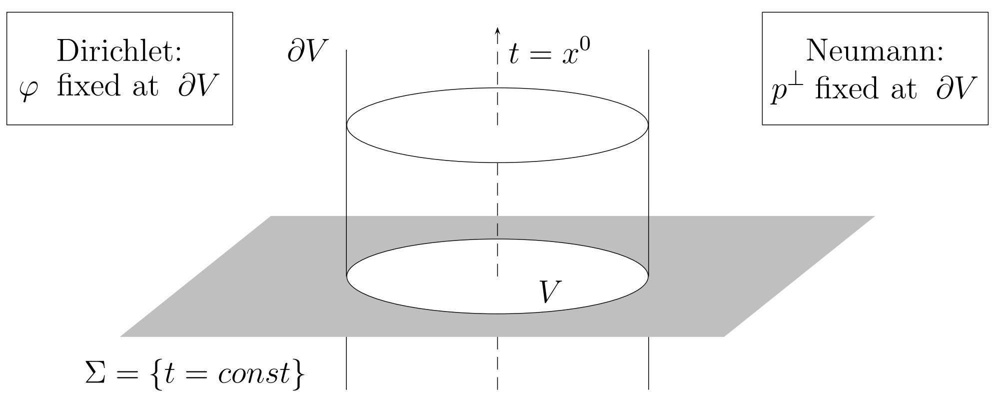

Fig. 2

5 Example: symmetric *versus* canonical energy in Maxwell electrodynamics

Already in the undergraduate course of Maxwell electrodynamics we learn that the naive application of Noether theorem leads to the "canonical" field energy, which is not the "true energy" and must be "improved" by a certain boundary integral in order to obtain the well established Maxwell formula (2). These manipulations provide an excellent example for the role of boundary conditions in the Hamiltonian formulation of field theory. Indeed, we claim that the free Maxwell theory is equivalent to two (mutually decoupled) wave equations. To prove this statement, Maxwell equations:

$$\begin{aligned} \dot{B} &= -\mathrm{curl}\, D \ , \quad \mathrm{div}\, D = 0 \ , \\ \dot{D} &= \mathrm{curl}\, B \ , \quad \mathrm{div}\, B = 0 \ , \end{aligned}$$

will be rewritten in terms of the radial component of both the electric and magnetic fields. We define the following two scalar fields:

$$\begin{aligned} \phi &:= rD^r = x^k D_k \ , \\ \psi &:= rB^r = x^k B_k \ . \end{aligned}$$

Due to the vanishing divergence of D, we have:

$$\begin{aligned} &\partial^j \partial_j \left(x^k D_k \right) = \partial^j \left(\delta^k_j D_k + x^k \partial_j D_k \right) = \\ &\partial_j \left(D^j + x^k \partial^j D_k \right) = 2\partial_j D^j + x^k \partial_j \partial^j D_k \end{aligned}$$

or, consequently,

$$\Delta \left(x^k D_k \right) = x^k \Delta D_k \ . \tag{33}$$

But Maxwell equations imply that every component of D and B satisfies wave equation, e.g. we have: $\Delta D_k = \ddot{D}_k$. Hence, (33) implies

$$\Delta \phi = x^k \Delta D_k = x^k \ddot{D}_k = \ddot{\phi} \ . \tag{34}$$

Analogous formula is valid for the magnetic field. Hence, we have:

$$\Delta \phi = \ddot{\phi} \ , \qquad \Delta \psi = \ddot{\psi} \ . \tag{35}$$

We will show that these are the only dynamical equations of the Maxwell theory and that their initial data $(\phi, \dot{\phi}, \psi, \dot{\psi})$ carry entire information about both fields (D, B). For this purpose we decompose the fields in spherical coordinates, e.g.:

$$D = D^r \frac{\partial}{\partial r} + D^A \frac{\partial}{\partial x^A} \ ,$$

where $A = 1, 2$; label angular coordinates. Radial component D^r is directly encoded by the function $\phi = rD^r$. On each sphere $S_r = \{r = \text{const.}\}$ the tangent part D^A can be further decomposed into the sum of a gradient and a co-gradient (by ϵ we denote the anti-symmetric Levi-Civitta tensor):

$$D_A = \partial_A \alpha + \epsilon_A{}^B \partial_B \beta \ .$$

The gradient part α can be directly reconstructed from the constraint equation $\mathrm{div} D = 0$ which, in spherical coordinates, reads:

$$\partial \left(r^2 D^r \right) + \Delta_{(2)} \alpha = 0 \ ,$$

where by $\Delta_{(2)}$ we denote the two-dimensional Laplasjan on the unit sphere.

The co-gradient part β can be reconstructed from $\dot{B}^r$

$$r^2 \, \Delta_{(2)} \beta = (\mathrm{curl}\, D)^r = -\dot{B}^r = -\frac{1}{r} \dot{\psi} \ .$$

Hence, on each sphere separately we have the "quasi-local" reconstruction of the functions (α, β):

$$\begin{aligned} \Delta_{(2)} \alpha &= -\partial_r (r\phi) \ , \\ \Delta_{(2)} \beta &= -\frac{1}{r^3} \dot{\psi} \ , \end{aligned}$$

and, consequently, of the complete electric field D. An analogous reconstruction of the magnetic field is valid. We conclude that the two wave equations (35) encode the entire Maxwell dynamics. As we have already learned in the previous Section, appropriate boundary conditions have to be imposed in order to obtain an infinite-dimensional Hamiltonian dynamics. In particular, Dirichlet condition for both ϕ and ψ lead to energy formula (1) for both two modes. It is easy to check that the sum of these "Dirichlet energies" reproduces exactly the Maxwell field energy (2), coming form the *symmetric* energy-momentum tensor. But controlling Dirichlet condition for ϕ and ψ means controlling the transversal components $(D^\perp, B^\perp)$ of the electric and magnetic fields[2] at the boundary ∂V.

On the other hand, Neumann condition for the electric field ϕ is obtained when, instead of controlling $D^\perp$, we control the scalar potential A_0 at the boundary (cf. [18]). This also leads to an infinite dimensional Hamiltonian system. The corresponding Hamiltonian for the field ϕ can be obtained from (1) by the Legendre transformation (31). Summing up the Dirichlet energy for ψ and the Neumann energy for ϕ we obtain exactly the "canonical energy" (i.e. the energy obtained from the so called *canonical* energy-momentum tensor). Both energies generate perfectly well defined Hamiltonian dynamics.

[2]Actually, the proof presented above is valid only for V being a ball: $V = K(0, R)$. In [18] the statement was proved for an arbitrary 3-volume V.

However, only the first one is positive definite and – due to physical arguments – describes the "true energy" of the field, the other being merely a "free energy". We conclude that controlling $(D^\perp, B^\perp)$ at the boundary corresponds to the adiabatic insulation of the system, whereas controlling the scalar potential (i.e. covering ∂V with a metal shell and grounding it) is a kind of a "thermal bath", admitting energy transfer between V and the "thermostat".

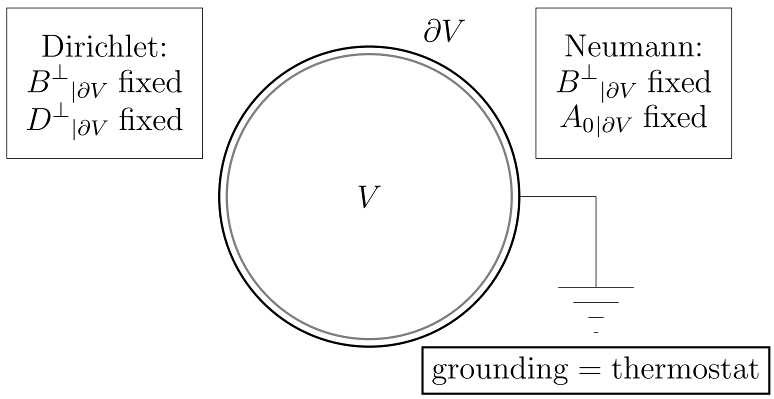

Fig. 3

6 Homogeneous Hamiltonian identity in canonical relativity

During the long history of General Relativity theory, many authors tried to mimic the above "Hamiltonian manipulations" in context of the Einstein's gravity theory. For this purpose different variational formulations of Einstein equations have been taken as the starting point.

In particular, the "purely metric" formulation was often used. It is based on variation with respect to the metric tensor $g_{\mu\nu}$. The corresponding Hilbert Lagrangian function $L = \frac{1}{16\pi}\sqrt{|g|}R$ is of the second differential order. Instead, we may also use the non-invariant, first order, Einstein Lagrangian, obtained by subtracting a complete divergence from the Hilbert Lagrangian. Palatini proposed yet another (the so called "metric-affine") formulation, where variation is taken with respect to both the metric and the connection $\Gamma^\lambda_{\mu\nu}$, treated *a priori* as independent quantities. Finally, one of us (see [20]) proposed a "purely-affine" formulation, where the metric does not appear in the Lagrangian function and variation is taken with respect to the connection only. In this approach, metric tensor arises as a momentum canonically conjugate to the connection. Typically, the affine Lagrangian is of the form $L(\Gamma, \partial\Gamma) = c \cdot \sqrt{|detR_{(\mu\nu)}|}$.

Even more sophisticated formalism was used by those authors, who propose to improve one of the above Lagrangian functions by appropriate boundary terms (see [8] or [10]).

The goal of this paper is to show that none of these manipulations is relevant for the Hamiltonian formulation of the theory. In fact, we are going to present here a result which follows directly from field equations and is perfectly non-sensitive to the choice of any one among the above formalisms.

Indeed, all the above variational formulations of General Relativity theory (and also everything else, which may be obtained from them *via* an arbitrary boundary manipulation) lead to the same *volume part* of Canonical Gravity, first obtained by Arnowitt, Deser and Misner (A.D.M.) in their classical paper [11]. It can be described as follows. Given a space-like hypersurface $V \subset M$ (possibly *with* boundary), embedded in a general relativistic space-time M, denote by $\mathbf{n}$ the unit, time oriented field, normal to V.

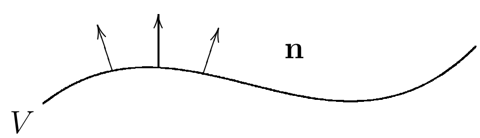

Fig. 4

To simplify the formalism we choose coordinate system (x^μ), $\mu = 0, 1, 2, 3$; in such a way that x^0 is constant on V. Consider the extrinsic curvature tensor

$$K_{mn} = (\nabla_m \partial_n \,|\, \mathbf{n}) = -(\partial_n \,|\, \nabla_m \mathbf{n}) \ , \tag{36}$$

and its trace $K = K_{mn}\, \tilde{g}^{mn}$, where by $\tilde{g}^{mn}$ we denote the three dimensional inverse to the restriction g_{kl} to Σ of the metric $g_{\mu\nu}$ $(k, l = 1, 2, 3;\ \mu, \nu = 0, 1, 2, 3)$. The "ADM-momentum tensor" defined as follows:

$$P^{kl} = \sqrt{\det \tilde{g}}\, (K \tilde{g}^{kl} - K^{kl}) \ , \tag{37}$$

plays role (when divided by the factor 16π) of the momentum canonically conjugate to the 3-metric g_{kl}. This means that the "volume part" of canonical relativity, analogous to the right-hand-side of formulae (27) or (32), equals:

$$\frac{1}{16\pi}\int_V \left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) \ . \tag{38}$$

This result is universal for all possible variational formalisms used as a starting point.

At this point no further information about the theory, except Einstein equations, is necessary. Indeed, time derivatives $(\dot{P}^{kl}, \dot{g}_{kl})$ are uniquely implied by Einstein equations and can be expressed in terms of the canonical variables (P^{kl}, g_{kl}) (see e.g. [12], page 525). When inserted into formula (38) and integrated by parts, these equations lead to the following, universal identity:

$$\begin{aligned} 0 &= \frac{1}{16\pi}\int_V \left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) + \frac{1}{8\pi}\int_{\partial V} (\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda) \\ &+ \frac{1}{16\pi}\int_{\partial V} (2n\delta\mathbf{Q} - 2n^A\delta\mathbf{Q}_A + \mathbf{Q}^{AB}\delta g_{AB}) \ , \end{aligned} \tag{39}$$

where the following notation has been used. By λ we denote the surface measure on the boundary ∂V. To simplify further the formalism, coordinate x^3 can be chosen in such a way, that it remains constant on ∂V during the evolution. Then (x^A), $A = 1, 2$; are coordinates on ∂V and we have:

$$\lambda = \sqrt{\det g_{AB}}\ \mathrm{d}^2x \ . \tag{40}$$

Then, we define the (hyperbolic) angle α between the Cauchy surface V and the world tube T, spanned by ∂V during the evolution. For this purpose, at every point $m \in T$ consider the two dimensional tangent plane orthogonal to ∂V (such a plane carries the structure of the two dimensional Minkowski space). We define the following *normalized* vectors: 1) $\mathbf{N}$ is tangent to the tube T and future-oriented; 2) $\mathbf{M}$ is orthogonal to $\mathbf{N}$, 3) $\mathbf{m}$ is tangent to V, directed outwards and, finally 4) the already defined vector $\mathbf{n}$, orthogonal to V, see Fig. 5.

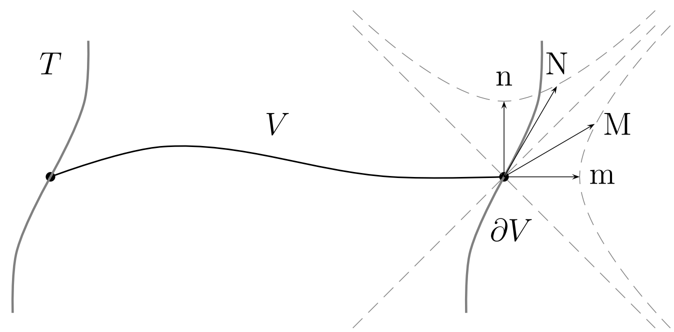

Fig. 5

The hyperbolic angle α between $\mathbf{N}$ and $\mathbf{n}$ is defined as follows:

$$\alpha := \operatorname{arsinh}(\mathbf{N}\,|\,\mathbf{m}) \ . \tag{41}$$

Quantities (n, n^A, g_{AB}) represent the 3-metric of the tube T in the (2+1)-decomposition of T (generated by the Hamiltonian (3+1) spacetime-decomposition). To define them we use (x^a), $a, b = 0, 1, 2$; as a coordinate system on $T = \{x^3 = \text{const.}\}$. Then g_{AB}, $AB = 1, 2$; is the 2-metric of $\partial V \subset T$. The remaining quantities (n, n^A) are defined as the three-dimensional lapse function and the shift vector on the tube T *via* a decomposition of the time-evolution vector on T into the orthogonal and the tangent to ∂V parts:

$$\frac{\partial}{\partial x^0} = n\,\mathbf{N} + n^A \frac{\partial}{\partial x^A} \ . \tag{42}$$

Finally, quantities $\mathbf{Q}, \mathbf{Q}_A, \mathbf{Q}^{AB}$ represent the external curvature of the tube T. Using (x^a), $a, b = 0, 1, 2$; on T the curvature is represented by a tensor L_{ab}, living on T and defined in a way analogous to (36):

$$L_{ab} = (\nabla_a \partial_b \,|\, \mathbf{M}) = -(\partial_b \,|\, \nabla_a \mathbf{M}) \ . \tag{43}$$

Next, we use its ADM-representation, analogous to (37)

$$Q^{ab} := \sqrt{\det \hat{g}}\ (L \hat{g}^{ab} - L^{ab}) \ , \tag{44}$$

where by $\hat{g}^{ab}$ we denote the contravariant metric on T (the inverse of g_{ab}). Finally, we use again the (2+1)-decomposition, i.e. we split the entire information contained in Q^{ab} into the three two-dimensional objects living on ∂V: 1) the scalar density $\mathbf{Q}$, 2) the covector density $\mathbf{Q}_A$ and 3) the tensor density $\mathbf{Q}^{AB}$, according to the formula:

$$\mathbf{Q} := n Q^{00} \ , \tag{45}$$

$$\mathbf{Q}_A := Q^0{}_A \ , \tag{46}$$

$$\mathbf{Q}^{AB} := Q^{AB} + n^A Q^{0B} + Q^{A0} n^B + n^A n^B Q^{00} \ . \tag{47}$$

We stress that formula (39) is an identity and does not depend upon any choice of the "canonical formalism" whatsoever. When compared with standard Hamiltonian formulae in field theory, like e.g. (26) in the scalar field theory, we see that the complete Hamiltonian part here:

$$\frac{1}{16\pi}\int_V \left(\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}\right) + \frac{1}{8\pi}\int_{\partial V} (\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda) \ , \tag{48}$$

does not reduce to the original volume part (38), but is supplemented by an additional surface integral. This phenomenon means, that the symplectic structure in the space of initial data (P^{kl}, g_{kl}) does not reduce to the volume part:

$$\omega_V = \frac{1}{16\pi}\int_V \mathrm{d}P^{kl}(x) \wedge \mathrm{d}g_{kl}(x) \ , \tag{49}$$

but must be appended by the surface part

$$\omega_{\partial V} = \frac{1}{8\pi}\int_{\partial V} \mathrm{d}\lambda(x) \wedge \mathrm{d}\alpha(x) \ . \tag{50}$$

As a result of this modification we obtain the *complete* symplectic form

$$\omega := \omega_V + \omega_{\partial V} \tag{51}$$

which is now invariant with respect to diffeomorphisms which do not move points of T (see Fig 6). It is easy to observe, that ω_V alone *is not invariant* with respect to such gauge transformations.

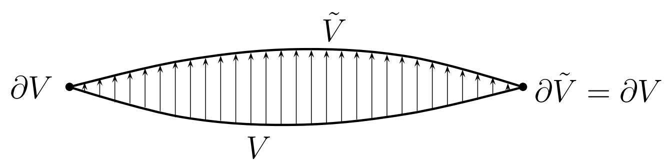

Fig. 6

Comparing further formula (39) with standard Hamiltonian formulae in field theory, like e.g. (26), we notice that the candidate for the Hamiltonian function H on the left hand side vanishes identically. This is due to the fact that the "would be" control parameters on the right hand side: $(\mathbf{Q}, \mathbf{Q}_A, g_{AB})$, cannot be freely controlled because they are subject to constraints (cf. [18]).

Hence, formula (39) plays role of the *homogeneous* Hamiltonian formula in mechanics:

Such a formula arises in the so called "homogeneous formulation" of mechanics of point particles. Consider spacetime coordinates (q^μ), $\mu = 0, 1, 2, 3$; of a particle and a Lagrangian function which is homogeneous in velocities $\dot{q}^\mu$. As an example we can take theory of a charged particle moving in electromagnetic potential $A = A_\mu \mathrm{d}q^\mu$. The action integral equals:

$$W = \int L\mathrm{d}t = -m \int \mathrm{d}s + e \int A_\mu \mathrm{d}q^\mu \ . \tag{52}$$

where t is any parameter along the particle's trajectory and $\mathrm{d}s$ denotes the spacetime interval (by m and e we denote the mass and the charge of the particle, respectively). Rewriting explicitly the above integral in terms of the Lagrangian variables $(q^\mu, \dot{q}^\mu)$ we conclude that the Lagrangian function

$$L(q^\mu, \dot{q}^\mu) = -m\sqrt{g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu} + eA_\mu(q^\lambda)\, \dot{q}^\mu \ , \tag{53}$$

is, indeed, homogeneous with respect to the velocities $(\dot{q}^\mu)$. This implies that the "super-Hamiltonian function"

$$\mathbf{H} = p_\mu \dot{q}^\mu - L = \frac{\partial L}{\partial \dot{q}^\mu}\dot{q}^\mu - L \ ,$$

vanishes identically. Moreover, the Legendre transformation (16) gives the following equation:

$$0 = \dot{p}_\mu \, \delta q^\mu - \dot{q}^\mu \, \delta p_\mu \ , \tag{54}$$

$$0 = p_0 \dot{q}^0 + \big(p_k \dot{q}^k - L\big) = p_0 \dot{q}^0 + H(p_k, q^k, q^0) \ , \tag{55}$$

where by $H = p_k \dot{q}^k - L$ we denote the true energy of the system.

In this formulation, the parameter t along the trajectory is a pure gauge quantity and has no physical meaning. The theory is invariant with respect to re-parameterizations of the trajectories. This is a consequence of the fact that the "control parameters" (p_μ, q^μ) in generating formula (54) are not free, but subject to constraint (55). To derive the standard (3+1)-formulation of mechanics we must fix a gauge, e.g. putting:

$$t \equiv q^0 \qquad \Longrightarrow \qquad \dot{q}^0 = 1 \quad \text{and} \quad \delta q^0 = 0 \ . \tag{56}$$

Formula (55) implies that $p_0 = -H$ and, consequently,

$$0 = \dot{p}_0\, \delta q^0 - \dot{q}^0\, \delta p_0 + \dot{p}_k\, \delta q^k - \dot{q}^k\, \delta p_k = \delta H + \dot{p}_k\, \delta q^k - \dot{q}^k\, \delta p_k\ . \tag{57}$$

We recover this way the "true" Hamiltonian formula:

$$-\delta H(p_k, q^k, t) = \dot{p}_k\, \delta q^k - \dot{q}^k\, \delta p_k\ . \tag{58}$$

7 Examples of the gravitational boundary control and corresponding Hamiltonians

To obtain from (39) a "true Hamiltonian" of gravity, like (58) was obtained from (54), we have to perform another Legendre transformation between the present "control parameters" $(\mathbf{Q},\ Q_A, g_{AB})$ (false because constrained!) and the "response parameters" $(n, n^A, \mathbf{Q}^{AB})$ at the boundary. The new control parameters should not be constrained. We claim that any serious search for the "true gravitational energy" should be based on the analysis of the boundary problem for Einstein equations. A "well posedness" of the problem and, possibly, the positivity of the resulting Hamiltonian function (cf. [19]) could provide a decisive argument in favor of such a choice (see also [13] for further discussion).

In the present paper we give three different examples of such a control mode.

Example 1. Metric control mode is obtained when we control the complete 3-metric of the tube T, i.e. we choose (n, n^A, g_{AB}) as control parameters. Due to

$$n\delta\mathbf{Q} - n^A\delta\mathbf{Q}_A = \delta\left(n\mathbf{Q} - n^A\mathbf{Q}_A\right) - \mathbf{Q}\delta n + \mathbf{Q}_A\delta n^A$$

we obtain

$$\begin{aligned} -\delta\,\widetilde{\mathcal{E}} \ &= \ \frac{1}{16\pi}\int_V \dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl} + \frac{1}{8\pi}\int_{\partial V} \dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda \\ &+ \ \frac{1}{16\pi}\int_{\partial V} \left(-2\mathbf{Q}\delta n + 2\mathbf{Q}_A\delta n^A + \mathbf{Q}^{AB}\delta g_{AB}\right)\ . \end{aligned} \tag{59}$$

with the Hamiltonian given by:

$$\widetilde{\mathcal{E}} = \tfrac{1}{16\pi}\int_{\partial V} \left(n\mathbf{Q} - n^A\mathbf{Q}_A\right) + E_0\ . \tag{60}$$

The additive constant E_0 (always free in the Hamiltonian formalism) may be fixed in such a way that $\widetilde{H} = 0$ for the empty Minkowski spacetime.

It can be easily shown that $\mathbf{Q}_A = \lambda \ell_A$, where ℓ_A denotes the torsion of the 2-surface ∂V:

$$\ell_A := (\nabla_A \mathbf{N} \,|\, \mathbf{M}) \ . \tag{61}$$

Moreover, $\mathbf{Q} = \lambda k(\mathbf{M})$, where $k(\mathbf{M}) =$ denotes the 2-dimensional trace of the external curvature $k_{AB}(\mathbf{M})$ of ∂V in the direction of $\mathbf{M}$:

$$k_{AB}(\mathbf{M}) := (\nabla_A \partial_B | \mathbf{M}) \ . \tag{62}$$

In particular, putting $n = 1$ and $n^A = 0$ and controlling in formula (59) also the two dimensional metric g_{AB} on the boundary, we may define the total energy of the "gravity + matter" system (cf. [15]):

$$E = \tfrac{1}{8\pi} \int_{\partial V} \lambda k(\mathbf{M}) + E_0 \ . \tag{63}$$

Although the above formula resembles the Brown-York definition of the gravitational energy (see [8]), we stress that the latter contains the curvature of $\partial V \subset V$, considered as a submanifold embedded in V, i.e. calculated with respect to $\mathbf{m}$. The Brown-York formula reads, therefore:

$$E = \tfrac{1}{8\pi} \int_{\partial V} \lambda k(\mathbf{m}) + E_0 \ , \tag{64}$$

and is not gauge-invariant with respect to gauge transformations visualized on (Fig. 6). Putting $n = 0$ in (60), and taking vector n^A equal to the generator of translations or rotations, we may define in a similar way the total momentum and the angular momentum of the system.

Analyzing the linearization of gravitational energy in the asymptotic region, i. e. when gravitational field on ∂V is very weak, it is easy to show that, similarly as it was done for Maxwell equations, the whole dynamics of the field reduces to two (decoupled) wave equations (cf. [16]). Summing up the "Dirichlet energies" (1) for both degrees of freedom we obtain unambiguously the field energy in the weak field region. This energy is positive!

Unfortunately, the Hamiltonian function (63) in the purely metric control mode *does not* reproduce the above expression for the linearized version of the theory and, therefore, cannot be treated as the true gravitational field. In [18] it was called the "free energy".

Example 2. In paper [18] yet another (mixed, not purely metric) control mode on the boundary was considered. It was based on the control of the quantities (Q^{0a}, g_{AB}), $a = 0, 1, 2$. It turns out that the resulting Hamiltonian function provides the best approximation of energy in the weak field regime, which was obtained in paper [16] *via* field linearization. In the particular spherically symmetric case this quantity reduces to the Hawking mass.

Example 3. Yet another control mode was proposed in paper [4]. It is based on the control of the ratio between the curvature $k(\mathbf{M})$ and the lapse n. Namely, we propose to control the following quantity:

$$b := \frac{k(\mathbf{M})}{n} \ .$$

But the following identity holds:

$$2n\delta\mathbf{Q} = 2n\delta(\lambda k) = \delta(\lambda nk) - \lambda k^2 \delta(\frac{n}{k}) + nk\delta\lambda \ .$$

The last two terms vanish because we control b and λ. Whence, we obtain the following formula for the Hamiltonian function:

$$\mathcal{E} = \tfrac{1}{8\pi} \int_{\partial V} n\lambda k(\mathbf{M}) + E_0 = \tfrac{1}{8\pi} \int_{\partial V} \lambda \frac{k^2}{b}(\mathbf{M}) + E_0 \ . \tag{65}$$

Choosing "standards" b and E_0, we obtain the quantity which has nice properties (see [4] for further discussion).

8 Concluding remarks

In our opinion, identity (39) has to be taken into account whenever a quasi local definition of the gravitational energy is proposed. It would be difficult to accept a definition which is not related to a specific boundary data control. In this context, it seems natural to control the two-metric g_{AB} of the boundary ∂V. Indeed, admitting the change of the metric would also mean that we accept the energy acquisition due to the change of the shape of V. In special relativity the generator of such a transformation would never be called "an energy". If we accept this rule, the only freedom which remains is the choice between metric parameters (n, n^A) and the curvature parameters $(\mathbf{Q}, \mathbf{Q}_A)$. Here, the leading principle for any reasonable definition should be the situation in linear gravity: the theory reduces to two decoupled wave equations and, therefore, its energy is unambiguously defined *via* formula (1).

ACKNOWLEDGMENTS

This research was supported by Polish Ministry of Science and Higher Education, grant Nr N N201 372736.

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