Comment on the “arrival time” in quantum mechanics

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Contrary to claims contained in papers by Grot-Rovelli-Tate and by Delgado-Muga, the “time operator” which I have constructed (1974) in an axiomatic way, is a self-adjoint operator living in a usual Hilbert space of (non-relativistic or relativistic) quantum mechanics.

In 1974 I have solved the following problem in the standard quantum mechanics (see [1]). Consider a 2-dimensional plane in physical space (e. g. a plane $P_z = \{z = \zeta\}$, where $\zeta$ is a fixed constant) and its measurable subsets (call them “windows”). Given a window $D \subset P_z$ and the time interval $I := [t_1, t_2]$, is it possible to define consistently the probability $\frac{Q_{I \times D}(\psi)}{Q_i}$ that a freely moving particle described by a quantum state $\psi$ crosses the window $D$ within the time interval $I$?

By consistency I meant a set of obvious axioms. Some of them were implied by the structure of quantum mechanics: 1) the probability should be given as a 3/2-linear form of the state $\psi$: $Q_{I \times D} = \frac{\langle \psi|\hat{q}_{I \times D}|\psi \rangle}{\langle \psi|\psi \rangle}$, where the operator $\hat{q}_{I \times D}$ must be a projector; 2) probabilities should sum up for disjoint 3-dimensional regions $I \times D$ (we may call them “space-time windows”) and 3) must be normalized. Others were implied by the Galilei invariance of the non-relativistic quantum mechanics and by the Poincaré invariance in the relativistic case (both cases were considered).

It was proved that these axioms define uniquely an operator $\hat{t}_z$ which, together with the position operators $\hat{x}$ and $\hat{y}$, give the projector $\hat{q}_{I \times D}$ as an integral of their common spectral measure $d\mathcal{E}(t, x, y)$ over the space-time window $I \times D$.

In a recent publication [2] the authors write:

Kijowski obtained a probability distribution, but not on the usual Hilbert space: thus the interpretation of the wave function in terms of familiar quantities is obscure.

Moreover, in paper [3] the authors write:

Our result turns out to be similar to those previously obtained by Kijowski. However, the approach by Kijowski was based on the definition of a non-conventional wave function which evolves on a family of $x = \text{const}$ planes (instead of evolving in time according to the Schrödinger equation), and whose relation to the conventional wave function is unclear.

These comments are entirely incorrect. I stress that my construction and the uniqueness proof, were performed in the framework of the absolutely conventional quantum mechanics. The “time operator” $\hat{t}_z$ on the plane $P_z$ was uniquely obtained from the axioms.

The main technical device which I have used to simplify the mathematical aspect of the theory, was a new representation of wave functions, which is little known, although it is perfectly equivalent to both the position and the momentum representations. This new representation was obtained from the wave function $\psi(p_x, p_y, p_z)$ in the momentum representation (Fourier transform of the wave function $\psi(x, y, z)$ in the position representation). The new representation is obtained by replacing the variable $p_z$ by the “signed energy” variable:

$$s := E_z \cdot \text{sgn}(p_z),$$

(1)

where $E_z$ is the amount of energy carried by the $z$-th degree of freedom (in non-relativistic case it is simply $E_z = \frac{p_z^2}{2m}$; in relativistic case it is equal to the difference between the actual energy and the energy corresponding to $p_z = 0$). The symbol “sgn” stands for “the sign of” and enables us to distinguish between the “right movers” and the “left movers” carrying the same energy. This way, any quantum state may be represented by a square-integrable function of the three variables:

$$\tilde{\psi}(s, p_x, p_y) := \sqrt{\frac{m}{|p_z(s, p_x, p_y)|}} |\psi(p_x, p_y, p_z(s, p_x, p_y))|,$$

(2)

(the square root factor arises because, geometrically, the wave function is a half-density and must follow the corresponding transformation law when passing to a new coordinate system). The transformation from the square integrable functions $\psi$ to square integrable functions $\tilde{\psi}$ is unitary:

$$|\psi|^2 = \int |\psi|^2 d^3x = \int |\tilde{\psi}|^2 d^3p$$

$$= \int |\tilde{\varphi}(s, p_x, p_y)|^2 ds dp_x dp_y. \quad (3)$$

In this representation the formula for the “time operator” is the simplest possible: for $\zeta = 0$ it turns out to be a “momentum canonically conjugate” to the parameter $s$:

$$\hat{t}_0 \tilde{\varphi} = -i\hbar \frac{\partial}{\partial s} \tilde{\varphi}, \quad (4)$$

(see page 373 in [1]) and for other values of $\zeta$ it may be obtained from the above operator by a standard translation in the direction of $z$-th axis. (Actually, I have used...
in [1] the parameter $s$ defined in a slightly different way, namely: $s := E \cdot \text{sgn}(p_z)$. With respect to (1), the complete energy $E$ replaces here the quantity $E_z$. Because both definitions of $s$ differ only by a constant depending on $p_x$ and $p_y$, formula (4) gives the same result in both descriptions. In the present paper I have chosen the variable $s$ defined by formula (1) because its range is equal to the real axis — without any “hole” in the middle — and the entire representation looks more similar e. g. to the standard formula for the position operator in the momentum representation.)

Still another representation of the quantum state is very useful, because it gives the common eigenvector expansion of the three commuting operators: $(\hat{t}_\xi, \hat{x}, \hat{y})$. This new representation uses the inverse Fourier transform $\varphi(t, x, y)$ of the wave functions $\tilde{\varphi}(s, p_x, p_y)$. Again, the transformation from $\psi$ to the space of square-integrable wave functions $\varphi$ is unitary and the probability in question is simply given by the integral over the space-time window:

$$Q_{I \times D}(\psi) = \int_{I \times D} |\varphi(t, x, y)|^2 dtdxdy .$$  \hspace{1cm} (5)$$

The importance of this representation consists in the fact that it gives the generalized eigenfunction expansion of the quantum state $\psi$ with respect to the operator $\hat{t}_\xi$. Indeed, its eigenfunctions are simply Dirac delta functions in the variable $t$.

The transition from a plane $P_\xi$ to another plane $P_\zeta$ was also studied within this representation. There is absolutely nothing non-conventional in the fact, that such transition operators form a group and that the generator of this group is nothing but the momentum operator $\hat{p}_z$. Hence, such a transition may be formulated as an “evolution on a family of $z =$ const planes” (as mentioned by Delgado and Muga) with $\hat{p}_z$ being the “Hamiltonian” of such an “evolution”. Here, the only non-conventional aspect was the use of the above “$\varphi$”-representation. To obtain the explicit formula for $\hat{p}_z$ in this representation, we had to express it in terms of momenta $(\hat{s}, \hat{p}_x, \hat{p}_y)$, canonically conjugate to the “positions” $(t, x, y)$. I am afraid that the “non-conventionality” of my paper, claimed by the above authors, was based on a misunderstanding of this simple fact.

On the other hand, any quantum state (no matter whether described in my representation as a function $\varphi$ or in the position representation as the conventional wave function $\psi$) undergoes the standard “chronological evolution” from time $t_1$ to $t_2$, described by the Schrödinger equation. There is absolutely no contradiction between these two different “evolutions”. They are used to answer different physical questions.

By the way: formula (4) may be recalculated from one representation to another (e. g. to the momentum representation or to the position representation). This immediately implies the following formula, relating the above “time operator” with the position operator $\hat{z}$ and the momentum operator $\hat{p}_z$:

$$\hat{t}_\xi \psi = - \text{sgn}(p_z) \frac{1}{2} \times \left\{ (\hat{z} - \zeta)(\hat{p}_z)^{-1} + (\hat{p}_z)^{-1}(\hat{z} - \xi) \right\} \psi ,$$

valid for sufficiently regular wave functions $\psi$ (such that all the symbols used have an unambiguous meaning). For a beam prepared in such a way, that it contains “right movers” exclusively (i. e. $p_z > 0$) this formula may be considered as an analog of the corresponding classical formula for the arrival time defined by the plane $P_\xi$, expressed in terms of classical observables $z$ and $p_z$:

$$t_\xi = - m \frac{z - \xi}{p_z} .$$  \hspace{1cm} (7)$$

(In formula (6) we obtain the symmetric order for the product of non-commuting operators). For a beam containing “left movers” exclusively, the arrival time arises here with an opposite sign. Formula (6) was not given in [1], but it is a one-line consequence of (4). The reason that I do not like such formulae is that it is mathematically “dangerous”: without specifying precisely the domain of the operator it is a priori not even a self-adjoint operator.

The paper [4] proves, indeed, how dangerous might be playing with such formulae: this author considers the symmetric version of the classical arrival-time (7), i. e. an operator defined in such a way, that the sign of $p_z$ in front of (6) has been deleted (cf. also [5]). Using rather complicated arguments he proves that this operator is not self-adjoint. No wonder: in our $\tilde{\varphi}(s, p_x, p_y)$ — representation this operator is equal to

$$\hat{T} := - i \text{sgn}(s) \hbar \frac{\partial}{\partial s}$$

and one sees immediately that such an operator has no self-adjoint extension because its deficiency indices are different (no arguments based on the Pauli theorem are necessary!). On the other hand, the axiomatic approach proposed in [1] leads uniquely to a perfectly self-adjoint operator (4) (the same techniques was later used in [6]) in order to prove the uniqueness of the Newton – Wigner position operator in relativistic quantum mechanics).

Both papers [2] and [3] confirm the main theorem proved in [1]: the above construction is unique. Indeed, formulae (26) – (33) or (66) of [3] are identical with the definitions proposed in [1]. On the other hand, [2] constructs an approximation (in a certain sense) of the “classical” formula by self-adjoint operators. Again, our “$\tilde{\varphi}$” representation simplifies considerably this construction: it is sufficient to smooth out the singular vector field (8) within the interval $s \in [-\epsilon, \epsilon]$ and define the operator $\hat{T}_\xi$ as a Lie derivative of $\tilde{\varphi}$ with respect to this smooth field. Keeping in mind the fact, that the wave function is not a scalar but a half-density, we immediately obtain formula (37) of [2].

Although, there is no room to modify the mathematically unique definition of the probability $Q_{I \times D}(\psi)$ within
the standard mathematical framework of Quantum Mechanics (spectral measures, self-adjoint operators etc.), the problem is still open from the physical point of view. Indeed, the probabilistic interpretation of the quantum mechanics is based on “short” measurements of the type “What is a probability that the particle will be found within the three-dimensional volume $V$ precisely at a given time instant $t$?” How to relate them with “long” measurements, corresponding to time intervals $I$ rather than to time instants $t$ is by no means obvious. From this point of view the result of papers [2] – [4] are very interesting.

I want to stress that the classification: “nonconventional wave function . . . whose relation to the conventional wave function is unclear” could only be conceived by somebody who did not read my paper [1]. Indeed, the composition of the paper is following. In Section 1 the Heisenberg energy-time uncertainty principle is discussed. In Section 2 I propose to understand the quantity $\Delta t$ appearing in this principle as the “average deviation of the time of passing through the plane $Q$” (or “arrival time” in the modern language – see page 363). Moreover, I analyze the physical origin of this uncertainty. Section 3 contains the proof that the three naive ways of defining the above “arrival time” in conventional wave mechanics are incorrect. In Section 4 I show how the arrival time can be defined axiomatically in classical statistical mechanics. In Section 5 I prove that the same set of axioms may be taken as the definition of the arrival time in conventional quantum mechanics. At this point the time operator is already defined without any nonconventional tools. The remaining part of the paper is devoted to the analysis of the properties of the quantum mechanical observable $t$ defined this way. In Section 6 I prove that is possesses the correct classical limit. And finally in Section 7 I show how to diagonalize it (or to find its complete set of eigenvectors). For this technical purpose the nonconventional representations $\tilde{\varphi}$ and $\varphi$ are introduced. There is no doubts that these are new representations of a conventional wave function: in the first formula of this Section (page 370) I define the wave function in this new representation starting from the wave function in the conventional momentum representation. As a conclusion I claim that the complete construction of my “arrival time” in the context of the absolutely conventional quantum mechanics is contained in first six Sections of paper [1].

In the remaining part of the paper I also analyze the relation between “arrival times” corresponding to different planes (different values of the variable $\zeta$). The authors of [3] do not like this analysis. They complain that the wave function in the $\varphi$ representation “evolves on a family of $x = \text{const}$ planes instead of evolving in time according to the Schrödinger equation”. I stress, however, that this “nonconventional evolution” does not contradict the Schrödinger evolution because both “evolutions” describe different aspect of quantum mechanics.