# Variational priciples and Hamiltonian formulation of spherical shell dynamics

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A general approach to the hamiltonian description of thin shells of matter in General Relativity is duscussed. The system composed of an ideal fluid self-gravitating spherical shell is then analyzed and its lagrangian and hamiltonian functions are derived from first principles. For this purpose the standard Hilbert action is modified by an appropriate surface term at spatial infinity. Known results for the spherical dust shell are then recovered as a special case.

## 1. Introduction

Thin matter shells were introduced by Werner Israel[1] as the simplest model to study gravitational collapse. The dynamics of a thin shell of matter is obtained considering Einstein's equations concentrated on an hypersurface which tailors together two different manifolds. The simplest case is that of a spherical dust shell in vacuum whose dynamics was already exhaustively discussed in the pioneering work by Israel.[2] Spherical shells with more general equations of state have been also investigated.[3]

The formulation of shell dynamics within the context of canonical gravity however was developed only recently.[4] In the spherically symmetric case (which means tailoring of an internal Minkowski geometry to an external Schwarzschild) this leads to a simple Hamiltonian system which has only one degree of freedom.[5] Nevertheless this Hamiltonian, as evaluated from the standard Hilbert action, does not coincide with the total energy of the system for an observer at spatial infinity. The solution to this problem is obtained when an appropriate boundary term at spatial infinity is introduced to improve the Hilbert action.[6]

## 2. Tailoring, curvature tensor and variational principle

The history of a dynamical 2-dimensional matter shell is described by the tailoring of two different vacuum space times, namely $\mathcal{M}^+$, the exterior, and $\mathcal{M}^-$, the interior, along a common hypersurface $\Sigma$. The hypersurface $\Sigma$ is therefore assumed to carry the matter content of the shell which will be described by constitutive equation $m(\nu)$ depending on the specific volume $\nu$ of the fluid (or, equivalently, on its local density).[7] The function $m$ contains both the rest frame energy density, the dust case will therefore be $m = m_0$, and the interaction energy of the fluid particles.

Restriction to spherical symmetry suggests that the internal geometry must be that of Minkowski, while the external is Schwarzschild with fixed mass parameter $M$. The dynamical evolution of the shell will be described by a function $\psi(t)$, where $t$ is the Schwarzschild time and dotted quantities represent derivatives with respect to $t$.[8] The matching conditions of the two geometries across $\Sigma$ will give the constraint equation:

$$\frac{\sinh\mu}{\cosh\mu - \sqrt{1-\frac{2M}{\psi}}} = \frac{\dot{\psi}}{1-\frac{2M}{\psi}} \ . \tag{1}$$

Where $\mu$ is the hyperbolic angle between the surfaces $\{t = const.\}$ on the Schwarzschild side and the surfaces $\{\tau = const.\}$ on the Minkowski side and can be thought of as an implicit function of $\psi$ and $\dot{\psi}$.

With the use of the theory of distribution the entire dynamics of the gravitational field interacting with the shell may be obtained performing the variation of an appropriate Hilbert action consisting in a singular part concentrated on the shell and a regular part outside the shell.[9] However in this manner the variation of the standard Hilbert action leads to an Hamiltonian which fails to represent the ADM (Arnowitt-Deser-Misner) mass at infinity for the system. This is due to the fact that the mass parameter $M$ in the Schwarzschild metric represents the total mass of the system for an observer at spatial infinity and therefore having it fixed a priori (before performing the variation of the Hilbert action) does not lead to a true Hamiltonian variational principle.

Analysis of boundary terms arising in the variational principle[10] suggests that a more general family of external fields, namely a Schwarzschild-like geometry with a variable mass parameter $M(t)$, might be considered, provided that the standard Hilbert action is substituted with an improved one $\mathcal{A}_{tot}$ consisting in a regular part outside the shell, a singular part, concentrated on $\Sigma$ and a boundary part, evaluated on a world tube external to the shell and whose radius will be shifted to infinity.[6] In this manner the total Lagrangian $\mathrm{L}_{tot}$ in the variational principle will be a function on the dynamical variables $\psi$, $\dot{\psi}$, $M$ and $\dot{M}$. The improved Hilbert action takes the form:

$$\mathcal{A}_{tot} = \int_{t_1}^{t_2} \mathrm{L}_{tot}(\psi, \dot{\psi}, M, \dot{M})dt + F(t_2) - F(t_1) \tag{2}$$

where the boundary terms $F(t_i)$ $(i = 1, 2)$ may be neglected and

$$\mathrm{L}_{tot} = m(\nu)\sqrt{\left(1 - \frac{2M}{\psi}\right) - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}} + M + \frac{2M\cosh\mu}{\cosh\mu - \sqrt{1 - \frac{2M}{\psi}}} - 2\psi + \psi\dot{\psi}\mu.$$

It is immediately evident that $\mathrm{L}_{tot}$ does not depend on $\dot{M}$ and therefore the fact that $M$ must be constant comes now as a consequence of the equations of motion rather than as an imposed prerequisite for the system.

## 3. Hamiltonian

Evaluating the equation of motion for the variable $M$ it is possible to solve explicitly for $M$ thus obtaining:

$$M(\mu, \nu) = \sqrt{\frac{\nu}{2}}\left\{1 - \left(\cosh\mu - \sqrt{\frac{m(\nu)^2}{2\nu} + \sinh^2\mu}\right)^2\right\} \tag{3}$$

where units were chosen such as the total amount of homogeneous fluid contained in the shell equals $8\pi$ therefore giving the relation $\nu = \frac{1}{2}\psi^2$.

Equation (3) can be substituted in $\mathrm{L}_{tot}$ to give:

$$\mathrm{L}_{tot} = \psi\dot{\psi}\mu - M\ . \tag{4}$$

Now from the usual Legendre transformations it is easy to obtain the Hamiltonian function as:

$$\mathcal{H}(\mu, \nu) = M(\mu, \nu)\ . \tag{5}$$

Furthermore the hyperbolic angle $\mu$ can be seen as the momentum canonically conjugated to the proper volume $\nu$.

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