The Hamiltonian formulation for the dynamics of a multishell self-gravitating system

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(Received 21 February 2010; accepted 21 April 2010; published online 19 July 2010)

Hamiltonian function describing a system composed of \( n \) gravitating shells in general relativity is derived from general considerations and its dynamics is presented. The results appear to be promising for the description of colliding system of astrophysical and cosmological interest. © 2010 American Institute of Physics. [doi:10.1063/1.3431030]

I. INTRODUCTION

The thin matter shell is a three-dimensional hypersurface which tailors together two smooth portions of a four-dimensional space-time. Thin shells were first introduced by Werner Israel (cf. Ref. 1) as a toy model to investigate gravitational collapse (see Ref. 2) and since then have found applications in many different fields of general relativity, ranging from cosmological models to quantum gravity (see Ref. 3).

Tailoring two manifolds is understood in the sequel as the coincidence of the three geometries induced on the shell \( \Sigma \) from each side. The induced metric on the shell is, therefore, assumed to be continuous. Nonetheless, its first derivatives (and, consequently, the connection coefficients) may exhibit discontinuities across \( \Sigma \). The resulting Einstein equations contain derivatives of these discontinuities and are, therefore, singular on the shell. The jump of the extrinsic curvature of \( \Sigma \) across the shell is interpreted as representing the singular matter distribution concentrated on the hypersurface. It was shown by one of us how the deltak like singularities of the curvature may be described with help of theory of distributions (cf. Ref. 4).

A timelike spherical shell is described as the history of a two-dimensional sphere that tailors together two static spherical manifolds (for an analogous treatment of lightlike shells, see Ref. 5). The dynamics of such a system may be derived from first principles once an appropriate Lagrangian function is defined. As was discussed in Ref. 6, this Lagrangian is given as a sum of three parts: (1) the gravitational Lagrangian that accounts for the Ricci part of the action, (2) the matter Lagrangian that describes the properties of the matter fields composing the shell, and (3) the boundary Lagrangian that takes into account the boundary contributions to the action. The Hamiltonian of the system is then recovered via the usual Legendre transformations. It turns out that its numerical value is equal to the total energy of the system (i.e., its Arnowitt-Deser-Misner mass).

The system composed of only one gravitating shell has just one degree of freedom (represented, for example, by the radial position of the shell), and its entire symplectic structure can be reduced with respect to both the Hamiltonian (scalar) and the momentum (vector) constraints imposed by the Einstein equations. As a result, the dynamics is governed by a uniquely defined Hamiltonian, being a function of two parameters: the configuration variable and the corresponding canonical momentum (as proven already in Ref. 4, the canonical momentum is given by the value of the hyperbolic angle between the surfaces \( \{t=\text{const.}\} \) on both sides of the shell).

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The physical interest of such a mathematical models arises from a series of problems of great importance in astrophysics and cosmology, such as the formation of black holes and naked singularities following complete gravitational collapse, the description of moving clouds ejected in supernova explosions, phase transitions in the early universe, and possible sources for dark energy and dark matter (see, e.g., Ref. 2). Of course, all these models refer to a single object, while considerations for two or more shells could have interesting applications in open problems where the analytical treatment is nearly impossible, such as the collapse of a multilayered object or the merging of compact sources of the gravitational field in general relativity. In this field, however, only very few results exist and the dynamics of a system of \( n \) gravitating shells and its collisions has been discussed only in the simplest case of a dust matter and under other simplifying assumptions (cf. Ref. 7).

With the aim of developing mixed numerical-analytical approach to the description of such scenarios, we derive here the Hamiltonian structure of a multishell self-gravitating system. Furthermore, the study of such a model, besides its interest from a physical point of view, reveals an intriguing mathematical structure. The Hamiltonian governing the dynamics of the system is given explicitly as a function of the \( n \) positions and \( n \) momenta and its numerical value turns out to be equal to the total ADM mass of the system.

The paper is organized as follows. In Sec. II we derive the Hamiltonian for the system composed of a single shell separating two Schwarzschild space-times. Although mathematically this is just a generalization of the result obtained by us in Ref. 6, physically it represents a variety of new different situations that were not included in the previous model. Particular emphasis is here devoted to the relation between the time frames on both sides and the total energy as measured in each time frame.

In Sec. III we discuss how to extend the result to the system composed of \( n \) gravitating shells and derive the correct value of the Hamiltonian from general principles. Finally, we derive the equations of motion describing the dynamics of the system.

II. ONE SHELL SEPARATING TWO SPHERICALLY SYMMETRIC MANIFOLDS

The Hamiltonian for a thin matter shell separating the spherical Schwarzschild exterior (with mass parameter \( M \)) from a flat Minkowski interior was derived from first principles in Ref. 6. The result was given as a function \( H \) of two parameters: (1) the configuration variable \( \nu \), representing the proper volume of the shell and (2) the momentum \( \mu \), representing the hyperbolic angle between the surfaces of constant time on both sides. The value of \( H \) was proven to be

\[
H = \sqrt{\frac{\nu}{2}} \left( 1 - \left( \cosh \mu - \sqrt{\frac{m(\nu)^2}{2\nu} + \sinh^2 \mu} \right)^2 \right),
\]

where \( m(\nu) \) represents the total (mechanical) rest mass (energy) of the matter composing the shell. The mass contains also the contribution from the deformation energy (hydro- or elastomechanical, see Ref. 8), responsible for the interaction between the particles of the shell. This is why \( m \) depends on the actual volume \( \nu \) of the shell (the particular case \( m=\text{const} \) describes the dust matter, composed of noninteracting particles, cf. Ref. 4). The arbitrary function \( m(\nu) \) plays role of the constitutive equation for the matter fields concentrated on the shell. As expected, \( H \) represents exactly the total mass of the “matter+gravity” system, \( M \), as seen by a faraway observer. We stress that the energy conservation is not postulated \textit{a priori} in our approach, but is implied by the energy conservation of an autonomous Hamiltonian system [i.e., by the time independence of the function (2.1)].

The above result may be extended to the scenario representing two static spherical manifolds separated by a thin surface of matter. This is due to the trivial structure of the Cauchy surfaces in the case of a system composed of only one shell. In Appendices A and C we give the detailed discussion of the appropriate tailoring procedure. The result of this discussion may be summarized as follows. Assuming the simplest spherically symmetric configuration of both manifolds, we consider two Schwarzschild solutions,
\[ ds_i^2 = -\left( 1 - \frac{2M_i}{r_i} \right) dt_i^2 + \left( 1 - \frac{2M_i}{r_i} \right)^{-1} dr_i^2 + r_i^2 d\Omega^2, \]  

(2.2)

with \( i=r,l \) (here, \( r \) stands for “right” and \( l \) for “left”). Using Schwarzschild coordinates, we cover all the configurations with the shell located outside of the horizon (bifurcation point) at \( r_i = \phi(t_i) \) (with \( \psi \geq 2M_i \) and \( r_i = \phi(t_i) \) (with \( \phi \geq 2M_i \)). The radial coordinates range from the shell position to infinity: \( \phi(t_i) \leq r_i \leq +\infty \) and \( \phi(t_i) \leq r_i \leq +\infty \).

To include also the shell configurations lying “below” the horizon (or the bifurcation point), we can slightly improve this description, using the isotropic coordinates (see, e.g., Ref. 9, for the discussion of the time problem). The Schwarzschild metric given in isotropic coordinates equals

\[ ds_i^2 = -\left( 1 - \frac{2\rho_i}{M_i} \right)^2 dt_i^2 + \left( 1 + \frac{M_i}{2\rho_i} \right)^4 (d\rho_i^2 + \rho_i^2 d\Omega^2), \]  

(2.3)

where again \( i=r,l \) and

\[ r_i = \rho_i \left( 1 + \frac{M_i}{2\rho_i} \right)^2. \]

In isotropic coordinates the radial variable \( \rho \) covers the entire half-line \((0, \infty)\), with \( \rho \to 0 \) representing the other spatial “end” of space-time, and the horizon is located at \( \rho = M/2 \).

Of course the analysis which we present in the sequel makes no sense when the shell is situated at the bifurcation surface (i.e., at the horizon). In fact, the bifurcation surface acts in a repulsive way on the shell and, consequently, the shell will never cross the horizon. On the other hand, there is no obstruction against positioning the shell inside the horizon. Hence, provided that we do not include the horizon in any of the two portions of space-time considered, with the use of these coordinates, we have the following three possible scenarios: In the first one the shell separates two portions of space-time outside of the horizon, with the radial coordinates ranging from the shell to infinity. In the second one, it separates two portions both lying inside the horizon with the coordinates ranging from zero to the shell position, while in the third scenario the shell divides one outside (with radial coordinate from the shell to infinity) from one inside (with the coordinate ranging from the origin to the shell). We stress, however, that this improvement is irrelevant as far as the derivation of the Hamiltonian function is concerned, the latter being the function of gauge invariant, reduced variables (the “true degrees of freedom” of the system). Once its analytic form is derived at the external region, it extends naturally to the entire phase space of the system.

Having in mind the above remarks, we shall derive the form of the Hamiltonian in the simplest environment of Schwarzschild coordinates. Note that the entire picture is clearly symmetrical with respect to the exchange of the left side with the right side. The symmetry is broken once we decide to describe the Hamiltonian evolution of the system with respect to a specific time frame. Choosing, e.g., the right-hand Schwarzschild time \( t_r \) as the time parameter of the evolution, we obtain the right-hand mass \( M_r \) as the total energy (Hamiltonian) of the system, whereas the left-hand mass parameter \( M_l \) is a constant which must be assigned \textit{a priori}.

If we choose to describe the system in the time frame of the manifold on the right-hand side, the matching conditions across the surface will lead the left-hand side coordinates \( \{ x_i^l \} \) to be functions of the right-hand side coordinates \( \{ x_i^r \} \). As already discussed in Ref. 10, it is possible to choose the coordinate system in such a way that the induced metric is continuous across the shell, whereas its derivatives might present discontinuities across \( \Sigma \). This choice represents only a gauge fixing that simplifies calculations and does not affect the coordinate-independent results.

In the new coordinate system, continuous across \( \Sigma \), the time coordinate \( t_l \) on shell becomes a function of the time coordinate \( t_r \).
so the matching condition for the space-space components of the 3-metric on the shell implies

\[ \phi(f(t_r)) = \psi(t_r). \]  

At this point it is useful to introduce the hyperbolic angle \( \mu \) between surfaces \( t_l = \text{const} \) on the left-hand Schwarzschild side and surfaces \( t_r = \text{const} \) on the right-hand side (see Fig. 1). It was proven in Ref. 6 that the matching condition for the left-hand and the right-hand metric on \( \Sigma \) may by written in the form of a constraint equation, which gives \( \mu \) as an implicit function of \( \psi \) and \( \dot{\psi} \), where by “dot” we denote the derivative with respect to \( t_r \). This implicit equation can be taken as a definition of the quantity \( \mu \),

\[ \frac{\sinh \mu}{\cosh \mu - \sqrt{1 - \frac{2M_l}{\psi}}} = \frac{\dot{\psi}}{1 - \frac{2M_r}{\psi}}. \]

Now, the matching condition (2.4) relating the left-hand time with the right-hand time on the shell may be written explicitly as

\[ j = \frac{dt_l}{dt_r} = \frac{1 - \frac{2M_l}{\psi}}{\cosh \mu - \sqrt{1 - \frac{2M_l}{\psi}}}. \]

Using the technique developed in Ref. 6, where an important step consists in admitting a slightly more general geometry for \( M_r \), with the Schwarzschild mass \( M_r \) taken to be \textit{a priori} time dependent (cf. Fig. 1), we are now able to evaluate the Hamiltonian function for such a system (see Appendices A and C, for the detailed calculations). The fact that \( M_r \) is actually time independent will follow \textit{a posteriori} as a consequence of the variational principle and of the equations of motion.
Expressed as a function of the canonically conjugated variables \((\psi, p_\psi)\), where \(p_\psi := \psi \mu\) is the momentum canonically conjugated to \(\psi\), the Hamiltonian results in
\[
H(\psi, p_\psi) = \frac{1}{2} \psi^2 \left( 1 - \sqrt{1 - \frac{2M_l}{\psi}} \cosh \mu + \sqrt{\frac{m^2}{\psi^2} + \left( 1 - \frac{2M_l}{\psi} \right) \sinh^2 \mu} \right)^2 ,
\] (2.8)

where \(\mu\) must be seen as \(\mu = p_\psi / \psi\). As expected, the Hamiltonian turns out to be numerically equal to the total mass of the system, measured by a far away observer on the “right” side,
\[
H = M_r.
\] (2.9)

Since the time coordinate remains unchanged when we pass to isotropic coordinates, the above result is valid for any coordinate system used to parametrize our space-time, provided we use the Schwarzschild time \(t_r\) as the time variable.

The symplectic structure of the theory is given in terms of the conjugated variables \((\psi, p_\psi)\) or, equivalently, in terms of variables \((\mu, \nu)\),
\[
dp_\psi \wedge d\psi = d(\psi \mu) \wedge d\psi = d\mu \wedge d\left( \frac{1}{2} \psi^2 \right) = d\mu \wedge d\nu ,
\] (2.10)

where, as already mentioned, \(\nu := \frac{1}{2} \psi^2\) represents the proper volume of the shell.

Evolution equations written in the Hamiltonian form are, therefore, the following:
\[
\dot{\nu} = \frac{\partial H}{\partial \mu} ,
\] (2.11)
\[
\dot{\mu} = - \frac{\partial H}{\partial \nu} ,
\] (2.12)

of which the first one is equivalent to the constraint equation (2.6), while the second one contains the whole dynamics of the shell.

The Hamiltonian (2.8) exhibits the following, fundamental property, which is responsible for the “left versus right” symmetry of the system.

**Theorem II.1:** The derivative of the total energy \(H = M_r\) with respect to the mass parameter \(M_l\) is equal to the ratio between the two Schwarzschild times on the shell,
\[
\frac{\partial M_r}{\partial M_l} = \frac{f}{\tilde{f}} .
\] (2.13)

**Proof:** First of all, we may rewrite formula (2.8) for the quantity \(M_r = H\) in the following way:
\[
\sqrt{1 - \frac{2M_l}{\psi}} = - \sqrt{\frac{m^2}{\psi^2} + \left( 1 - \frac{2M_l}{\psi} \right) \sinh^2 \mu + \sqrt{1 - \frac{2M_l}{\psi}} \cosh \mu} .
\] (2.14)

Now, derivative of (2.8) with respect to \(M_l\) may be easily calculated and simplified with use of (2.14). This way we finally obtain
\[
\frac{\partial M_r}{\partial M_l} = \left( \frac{\sinh^2 \mu}{\sqrt{1 - \frac{2M_l}{\psi} \cosh \mu}} - \frac{\cosh \mu}{\sqrt{1 - \frac{2M_l}{\psi}}} \right) \sqrt{1 - \frac{2M_l}{\psi}} ,
\] (2.15)

which, due to (2.7), is equal to \(\tilde{f}\).

As already mentioned, the above Hamiltonian is evaluated with respect to the specific time coordinate, namely, \(t_r\). A different choice of the time variable would lead to a numerically different Hamiltonian, being a function of a different set of canonical variables. In particular, we can use the left-hand time \(t_l\) as the time variable. It is easy to show that the role of the two Schwarzschild mass
parameters $M$, and $M_f$ must then be interchanged. In particular, due to (2.13), the new Hamiltonian equations of motion, with $H=M_f$ and $M_t$ treated as a fixed parameter, are equivalent to the old ones. This proves that our procedure is symmetric with respect to the exchange of the left side of the space-time with the right side.

Finally, note that $\dot{f}$, which measures the variation in the time measurement between the two coordinate systems, is positive provided the shell separates two portions of Schwarzschild located on the same side of the horizon (also inside the horizon if we make use of isotropic coordinates) but it is negative in case the shell separates a portion of Schwarzschild located outside the horizon from one located inside (since time flows in opposite directions in this case).

III. HAMILTONIAN FOR A SYSTEM COMPOSED OF N SHELLS

The above description of the dynamics of a single shell was derived from first principles. Indeed, we were able to calculate the total “matter+gravity” action and then to reduce it with respect to the gauge freedom. For this purpose, the smooth Cauchy surfaces $\Sigma_t$ were constructed in the following way: first we take the $\{t_t=\text{const}\}$ surfaces outside the shell; next, we prolong them across the shell in a smooth way. As a consequence, our Cauchy surfaces are different from the surfaces $\{t_t=\text{const}\}$ inside the shell.

This procedure cannot be immediately generalized to the case of a multishell system because already the second shell would be parametrized by a gauge-depending time variable and our techniques would fail. On the other hand, the Cauchy surface composed piecewise of subsequent $\{t_t=\text{const}\}$ surfaces is not smooth and, a priori, the description of the singular curvature in terms of the theory of distributions does not work. However, an appropriate generalization of this techniques, admitting nonsmooth Cauchy surfaces for Einstein equations, could be used in this case. This theory is slightly more complicated than the one used in the present paper and, at the moment, it is still not entirely developed.

To construct the Hamiltonian description for multiple shells we then proceed as follows: we begin with a single shell $\Sigma_n$ and denote its exterior “right-hand side” Schwarzschild component by $M_n$. Now, we add a second surface $\Sigma_{n-1}$ so that the left-hand side manifold $M_t$ results divided into two parts, which now will be called $M_{n-1}$ and $M_{n-2}$.

$$M = M_{n-2} \cup M_{n-1} \cup M_n.$$ 

This procedure can be iterated. As a result, we obtain the complete system composed of $n$ gravitating shells (Fig. 2), described by the $2n$ canonical variables $(q_i, p_i)$, or equivalently $(\mu_i, \nu_i)$ ($i = 1, \ldots, n$). The hyperbolic angles $\mu_i$ will be given implicitly by equations analogous to Eq. (2.6),
\[
\frac{\sinh \mu_i}{\sqrt{1 - \frac{2M_i}{\psi_i} \frac{1 - \frac{2M_{i-1}}{\psi_{i-1}}}{1 - \frac{2M_i}{\psi_i}}}} = -\frac{\psi_i}{\psi_{i-1}},
\]  
(3.1)

\[
cosh \mu_i = \sqrt{1 - \frac{2M_i}{\psi_i} \frac{1 - \frac{2M_{i-1}}{\psi_{i-1}}}{1 - \frac{2M_i}{\psi_i}}},
\]

where \(\psi_{i+1}/\psi_i\) denotes the derivative of \(\psi_i\) with respect to the Schwarzschild time \(t_i\) defined in \(\mathcal{M}_i\). Moreover, we have on each shell \(\Sigma_i\) the matching condition analogous to Eq. (2.7).

\[
\frac{dt_{i-1}}{dt_i} = \frac{\frac{2M_{i-1}}{\psi_i} - \frac{2M_i}{\psi_{i-1}}}{\frac{1 - \frac{2M_{i-1}}{\psi_i}}{1 - \frac{2M_i}{\psi_{i-1}}} - \frac{2M_i}{\psi_i}}.
\]  
(3.2)

It is obvious that equations of motion of the last shell \(\Sigma_n\) are generated by the Hamiltonian function (2.8) because the dynamics is local, i.e., the shell “does not know” about other shells. The Hamiltonian is given as a function of the variables \((\mu_n, \nu_n)\) and of the parameter \(M_{n-1}\) analogously to Eq. (2.8). Now, the same reasoning is true for the next shell \(\Sigma_{n-1}\). This means that the mass parameter \(M_{n-1}\) can be treated as a function of the configuration variables \((\mu_n, \nu_n)\) and of the mass parameter \(M_{n-2}\). This way we obtain a sequence of nested functions \(M_i(\mu_i, \nu_i; M_{i-1})\),

\[
M_i = \sqrt{\frac{\nu_i}{2} \left( 1 - \sqrt{1 - \frac{2M_{i-1}}{\sqrt{2\nu_i}} \cosh \mu_i - \frac{m_i(v_i)^2}{2\nu_i} + \left( 1 - \frac{2M_{i-1}}{\sqrt{2\nu_i}} \sinh^2 \mu_i \right)^2 \right)},
\]  
(3.3)

with \(M_0\) being a fixed, constant parameter.

We can now show that \(H = M_n\) is indeed the true Hamiltonian function of the system.

**Theorem III.1:** Let \(H\) be the function obtained as the end result of successive compositions of all the functions (3.3) and let it be considered as a function of the variables \((\mu_i, \nu_i)\), \(i=1, \ldots, n\) and of the parameter \(M_0\). Then, \(H\) is the Hamiltonian function for the system composed of \(n\) gravitating shells and labeled by the external Schwarzschild time coordinate \(t_n\).

**Proof:** Equations of motion for the \(i\)th shell as evaluated with respect to the time coordinate \(t_n\) are

\[
\dot{\mu}_i = -\frac{\partial H}{\partial \nu_i},
\]  
(3.4)

\[
\dot{\nu}_i = \frac{\partial H}{\partial \mu_i}.
\]  
(3.5)

Now consider Eq. (III.4) (the same analysis applies also to the second Hamilton equation). We have

\[
\dot{\mu}_i = \frac{d\mu_i}{dt_i} \frac{dt_i}{dt_n}
\]

and
But, in virtue of lemma (2.1) iterated \( n \) times, we have \( \partial H / \partial M_i = dt_i / dt_n \). So we conclude that the local dynamics of the \( i \)th shell,

\[
\frac{d \mu_i}{d t_i} = - \frac{\partial M_i}{\partial v_i},
\]

\[
\frac{d v_i}{d t_i} = \frac{\partial M_i}{\partial \mu_i},
\]

is well described by Hamilton equations for the single \( i \)th shell and that is true for every \( i \). Therefore, the Hamiltonian for the whole system is correct.

\[\blacksquare\]

**IV. DISCUSSION**

The aim of the present paper was to show how to derive from first principles the Hamiltonian function that correctly reproduces the ADM mass for a system of \( n \) gravitating massive shells composed of a generic fluid. The dynamics of such a system can be analyzed via the equations of motions once a set of constitutive equations \( m_i(\nu) \) for the matter composing the shells is provided. This result appears to be promising for the classification of the dynamics of multiple shells system with different equations of state because of the powerful (analytical and numerical) tools allowed by the Hamiltonian mechanics and because it is not restricted by a specific choice of the properties of the matter. Scenarios describing collisions (see Ref. 7) and shell crossing (see Ref. 11) are most likely to be expected but also stable configurations might appear for a suitable choice of the matter constituting the shells.

The whole picture was derived from general considerations. The method bypasses the fact that a rigorous variational treatment would require a mathematical framework able to describe singular tensor densities and nonsmooth Cauchy surfaces. Further work in this direction is in progress.

**ACKNOWLEDGMENTS**

This work was developed thanks to the support of the Institute of Mathematics of the Polish Academy of Science and, especially, Professor Janusz Grabowski, to whom the authors wish to express their gratitude.

**APPENDIX A: TAILORING**

Consider the system composed of only one gravitating shell \( \Sigma \subset \mathcal{M} \) separating two Schwarzschild manifolds \( \mathcal{M} = \mathcal{M}_l \cup \mathcal{M}_r \). The dynamics of the matter shell is then described by the history of the boundary hypersurface \( \Sigma = \partial \mathcal{M}_l = - \partial \mathcal{M}_r \) that tailors together the two manifolds. The dynamics must obey Einstein equations. For purposes of the shell theory, the tensor-density version, \( \mathcal{G} \), is especially useful. Here, Einstein’s tensor density is defined as \( \mathcal{G}_\mu^\nu = 8 \pi T_\mu^\nu \). The Riemann tensor \( R \) contains derivatives of the Christoffel coefficients \( \Gamma \) (discontinuous across the shell) and, whence, must be understood in the sense of distributions. We can see that therefore \( \mathcal{G} \) splits into two parts: (1) a regular part outside of the shell and (2) a singular part, proportional to the Dirac delta, concentrated on the shell,

\[
\text{sing} \mathcal{G}_\mu^\nu = G_\mu^\nu \delta_2.
\]
It was proven in Ref. 10 that $G$ is a three-dimensional tensor-density living on $\Sigma$, whose components transversal with respect to $\Sigma$ vanish identically. Moreover, its components tangent to $\Sigma$ are given by the jump of the extrinsic curvature $Q^a_b$ of $\Sigma$ between the left-hand and the right-hand side,

$$G^a_b = [Q^a_b]$$  \hspace{1cm} (A2)

(and $a, b$ label coordinates on the shell).

The stress-energy tensor vanishes outside the shell. Also here the transversal-to-shell components vanish, and we have

$$T^a_b = \text{sing} \ T^a_b = \text{T}^a_b \delta_x.$$

Therefore, Einstein equations can be written in terms of a regular part evaluated outside the shell and a singular part relating three-dimensional objects living on $\Sigma$,

$$G^a_b = 8 \pi T^a_b.$$  \hspace{1cm} (A3)

The regular part of these equations is equivalent to the vacuum Einstein equations outside of $\Sigma$, whereas the singular part describes the mechanical equations of motion for the matter on the shell, implied by its constitutive equation.

**APPENDIX B: VARIATIONAL PRINCIPLE**

As it is well known different variational principles lead to (essentially, unique) Hamiltonian description of the system (see Refs. 12 and 13).

We will consider the total Hilbert action as composed of three different parts: (1) the gravitational part being the integral of the scalar curvature over the four-dimensional domain $D$ that encloses the shell, (2) the matter part concentrated on the hypersurface and carrying the information about the matter content of the shell, and (3) the boundary part,

$$A^{\text{grav}} = \int_D L^{\text{grav}},$$

$$A^{\text{mat}} = \int_{D \cap \Sigma} L^{\text{mat}},$$

$$A^{\text{boundary}} = \int_{\partial D} L^{\text{boundary}},$$

where $D$ is the finite worldtube containing the shell. The tube is chosen to be finite because boundary manipulations are easier to perform here. However, at the end of the procedure, the boundary $\partial D$ of $D$ will be shifted to infinity, see Fig. 3. The presence of a boundary term in the action arises from the analysis of the standard boundary phenomena in Hilbert variational principle. Indeed, it was proven in Ref. 12 that via the variation of the standard Hilbert action $A = A^{\text{grav}} + A^{\text{mat}}$ (gravitational part plus the matter part), we obtain

$$\delta A = \int_D \frac{\delta L}{\delta g_{\mu \nu}} d_{g_{\mu \nu}} + \int_{D \cap \Sigma} \frac{\delta L}{\delta z} dz - \frac{1}{16 \pi} \int_{\partial D} g_{ab} \delta Q^{ab},$$

where "$z$" stands for the matter fields. The usual volume terms of the right-hand side give rise to Euler–Lagrange equations. In the last term, indices $a, b, = 0, 1, 2$ label coordinates on the boundary $\partial D$, whereas $Q^{ab} := \sqrt{\det \hat{g}} (L_{\hat{g}}^{ab} - L^{ab})$ denotes the “ADM version” of the extrinsic curvature $L_{ab}$ of $\partial D$ (by $\hat{g}_{ab}$ we denote the inverse of the 3-metric $g_{ab}$).
Different Hamiltonians for a gravitational system correspond then to different choices of the boundary control mode which annihilate the boundary term. Here, for the spherical shell system, the control of the extrinsic curvature $Q^{ab}$ could, in principle, be used: $\delta Q^{ab}=0$, but this choice does not lead to the correct Hamiltonian. So an appropriate Legendre transformation at the boundary is necessary. For this purpose consider a spatially bounded world tube,

$$D=\{t'_r \leq t, t' \leq R_l \leq R_r \leq t_r\}$$

that contains the shell $\Sigma$, the radii of the tube $R_r$ and $R_l$ are such that $\psi(t_r) \leq R_r$ and $\phi(t_l(t_r)) \leq R_l$ for $t' \leq t_r \leq t'_l$ and will be shifted to infinity at the end of the procedure. The boundary of $D$ is composed of two spatial cylinders $C_j:=\{x \in \partial D | r_j=R_j\}$ (with $j=r,l$) and two covers (that we assume including also the corners) $K^i:=\{x \in \partial D | t_i=t'_i\}$ (with $i=I,F$, see Fig. 3), so that

$$\partial D= C_r \cup C_l \cup K^r \cup K^F.$$

Since we are limiting ourselves to spherically symmetric field configurations, we can use coordinates $(x^a)=(x^0,x^A)$, $A,B=1,2$, compatible with this symmetry, where $x^0$ is the time variable and $(x^A)=(\theta,\varphi)$ are angular coordinates. The symmetry implies $g_{0a}=0$ and $Q^{0a}=0$.

If we keep fixed the initial and final configurations the integrals on the covers vanish and we obtain

$$\int_{\partial D} g_{ab} \delta Q^{ab} = \int_{C_l \cup C_r} (g_{00} \delta Q^{00} - Q^{AB} \delta g_{AB} + \delta g_{AB} Q^{AB})) .$$

(B5)

It is now useful to move the last term (being a total variation) to the left-hand side of Eq. (B4) and to treat it as a part of the action (see Ref. 12, for a detailed explanation). So, we define

$$A_{\text{boundary}} = \frac{1}{16 \pi} \int_{C_l \cup C_r} g_{AB} Q^{AB} .$$

(B6)

Writing the total Hilbert action as the sum of the three parts,

$$A_{\text{tot}} = A_{\text{grav}} + A_{\text{mat}} + A_{\text{boundary}},$$

we can now evaluate its variation as
This variational principle is designed for controlling only the time-time component $Q^{00}$ of the external curvature of the boundary, whereas the control of its space-space components $Q^{AB}$ has been replaced by control of the space-space components $g_{AB}$ of boundary metric. Indeed, such a control annihilates the boundary term provided that

$$g_{00}Q^{00} - Q^{AB}g_{AB} = 0.$$  

This is due to the fact that both control variables $Q^{00}$ and $g_{AB}$ do not depend on $M_r$ and assume the standard value. We can therefore consider a different initial configuration that still annihilates the boundary term. Namely, we consider a space-time $\mathcal{M} = \mathcal{M}_l \cup \mathcal{M}_r$ obtained by the tailoring of a Schwarzschild space-time $\mathcal{M}_l$ (with fixed parameter $M_l$) with a Schwarzschild-like space-time $\mathcal{M}_r$ (see also Fig. 1) given by

$$ds_r^2 = -\left(1 - \frac{2M_l(t)}{r}ight)dt_r^2 + \frac{dr_r^2}{\left(1 - \frac{2M_l(t)}{r}ight)} + r_r^2d\Omega^2.$$  

The new Lagrangian that is obtained depends on configuration variables $\psi, \dot{\psi}$ and the mass parameter $M_r(t)$ treated as a new configuration variable. As will be shown in the sequel, the Lagrangian nonetheless does not depend on $\dot{M}_r$. This means that the momentum canonically conjugate to the variable $M_r$ vanishes and the theory may be reduced with respect to this constraint: variation in the action with respect to $M_r$ gives its value in terms of $\psi$ and $\dot{\psi}$. Further analysis is standard.

We stress that leaving the mass parameter $M_r$ not fixed a priori is an important step in our construction. Indeed, fixed mass means also fixed energy. A procedure based on such an assumption would lead rather to a Maupertuis-like variational principle, whereas our approach gives directly the Hamilton-like variational principle where the fact that the mass is constant is obtained a posteriori as a consequence of energy conservation.

APPENDIX C: LAGRANGIAN AND HAMILTONIAN

The gravitational Lagrangian is divided into a regular part outside the shell and a singular part concentrated on the hypersurface which is proportional to the jump of the extrinsic curvature across the shell,

$$\mathcal{L}^{\text{sing}} = -\frac{1}{16\pi}\int_{\mathcal{D}\cap\Sigma} [Q].$$  

It is easy to see how the terms containing the derivative $\dot{M}_r$ arising from the regular part of the Lagrangian annihilate the corresponding terms arising from the singular part so that the gravitational part of the Hilbert action does not depend on $\dot{M}_r$ and can be written as

$$\mathcal{A}^{\text{grav}} = \int_{\mathcal{D}} \mathcal{L}^{\text{grav}} = \int_{t_i}^{t_f} \mathcal{L}^{\text{grav}}dt_r,$$  

with
\[ L^{\text{grav}} = \psi \dot{\psi} \mu - \frac{1}{2} M_r - \psi \left( 1 - \frac{2 M_r}{\dot{\psi}} \right) + \frac{1}{2} M \ddot{f} - \psi \left( 1 - \frac{2 M_f}{\psi} \right) f. \] (C3)

The matter Lagrangian describes the rest-frame energy density of the matter. Taking the matter model to be described by a constitutive equation \( m(\nu) \) where the rest frame energy of the fluid is given as a function of the specific volume of the shell \( \nu \) [the simplest situation being that of the dust case which is obtained when the particles of the fluid do not have any interaction energy, namely, when \( m(\nu) = \bar{m} = \text{const} \)]. For a homogeneous fluid the specific volume equals the total volume of the shell \( 4\pi \rho^2 \) divided by the total amount of fluid.

The matter part of the action is then simply given by

\[ A^{\text{mat}} = \int_{t_i}^{t_f} m(\nu) dt_r = \int_{t_i}^{t_f} m(\nu) \sqrt{\left( 1 - \frac{2 M_r}{\dot{\psi}} \right) - \frac{\dot{\psi}^2}{\left( 1 - \frac{2 M_r}{\dot{\psi}} \right)}} dt_r. \] (C4)

Finally, it is easy to prove that the boundary part of the action, once the boundary world tube \( D \) is shifted to infinity, is given by

\[ A^{\text{boundary}} = -\frac{1}{2} \int_{t_i}^{t_f} M_r dt_r + \frac{1}{2} \int_{t_i}^{t_f} M_f dt_f = -\frac{1}{2} \int_{t_i}^{t_f} (M_r - M_f) dt_f. \] (C5)

Adding together all the parts of the Lagrangian we obtain the total action as

\[ A^{\text{tot}} = \int_{t_i}^{t_f} L^{\text{tot}}(\psi, \dot{\psi}, M_r) dt_r, \] (C6)

with \( L^{\text{tot}} \) given by

\[ L^{\text{tot}} = m(\nu) \sqrt{\left( 1 - \frac{2 M_r}{\dot{\psi}} \right) - \frac{\dot{\psi}^2}{\left( 1 - \frac{2 M_r}{\dot{\psi}} \right)}} + \psi \dot{\psi} \mu - M_r - \psi \left( 1 - \frac{2 M_r}{\dot{\psi}} \right) - \psi \left( 1 - \frac{2 M_f}{\psi} \right) f. \] (C7)

The Euler-Lagrange equation for the variable \( M_r \) implies:

\[ \frac{dL^{\text{tot}}}{dM_r} = \frac{dL^{\text{tot}}}{dM_r} = 0, \] (C8)

from which we obtain Eq. (2.8) as an explicit expression for \( M_r \) in terms of the canonical variables \( \psi, \dot{\psi} \) and the parameter \( M_f \).

Finally, the total action for the system, once reduced with respect to Eq. (2.8), takes the form

\[ A^{\text{tot}} = \int_{t_i}^{t_f} (\psi \dot{\psi} \mu - M_r) dt_r. \] (C9)

Following the same procedure to evaluate the action in the left-hand time frame we would obtain

\[ A^{\text{tot}} = \int_{t_i}^{t_f} (\psi \dot{\psi} \mu - M_f) dt_f. \] (C10)
The relation between the two time frames can therefore be described also by changing the time coordinate in Eq. (9) from $t_r$ to $t_l$ and evaluating the new total action, equivalent to Eq. (10), as

$$A_{\text{tot}} = \int_{t_r}^{t_l} \left( \psi \gamma + \frac{\mathcal{L}_N}{f} \right) dt_r.$$  

(C11)