A Gauge-invariant Theory of Motion of Charged Test Particles

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Abstract
A new method of deriving equations of motion from field equations is proposed. It is applied to classical electrodynamics. As a result, we obtain a new, gauge-independent, second order Lagrangian for the motion of classical, charged test particles. It implies also a new, gauge-independent Hamiltonian formulation of the theory.

1 Introduction
The motion of classical, charged test particles, in the classical Maxwell field is derived usually from the gauge-dependent Lagrangian function

\[ L = L_{\text{particle}} + L_{\text{int}} = -\sqrt{1 - v^2} \left( m - eu^\mu A_\mu(t, \mathbf{q}) \right), \]

where \( u^\mu \) denotes the (normalized) four-velocity vector

\[ (u^\mu) = (u^0, u^k) := \frac{1}{\sqrt{1 - v^2}}(1, v^k), \]

and \( v^k := \dot{q}^k \) (we use the Heaviside-Lorentz system of units with the velocity of light \( c = 1 \)).

Since the Lorentz force \( eu^\mu f_{\mu\nu} \) derived from this Lagrangian is perfectly gauge invariant, it is not clear, why we have to use the gauge-dependent interaction term \( eu^\mu A_\mu \), with no direct physical interpretation. Moreover, in this approach, the equations of motion are not uniquely implied by the field equations. As an example, non-linear forces of the type \( u_\mu f^{\mu\lambda} u^\kappa \nabla_\lambda f_{\kappa\nu} \) cannot be a priori excluded.

In the present paper we show that the same Lorentz force may be derived from a gauge-invariant, second order Lagrangian \( \mathcal{L} \):

\[ \mathcal{L} = L_{\text{particle}} + \mathcal{L}_{\text{int}} = -\sqrt{1 - v^2} \left( m - a^\mu u^\nu M_{\mu\nu}^{\text{int}}(t, \mathbf{q}, \mathbf{v}) \right), \]

where \( a^\mu := u^\nu \nabla_\nu u^\mu \) is the particle’s acceleration. The skew-symmetric tensor \( M_{\mu\nu}^{\text{int}}(t, \mathbf{q}, \mathbf{v}) \) is equal to the amount of the angular-momentum of the field, which is acquired by our physical
system, when the Coulomb field accompanying the particle moving with constant velocity \( \mathbf{v} \) through the space-time point \((t, \mathbf{q})\), is added to the background (external) field. More precisely: the total energy-momentum tensor corresponding to the sum of the background field \( f_{\mu\nu} \) and the above Coulomb field decomposes in a natural way into a sum of 1) terms quadratic in the background field, 2) terms quadratic in the Coulomb field 3) mixed terms. The quantity \( M^{int}_{\mu\nu} \) is equal to this part of the total angular-momentum \( M_{\mu\nu} \), which we obtain integrating only the mixed terms of the energy-momentum tensor. It is proportional to the particle’s electric charge \( e \) contained in the particle’s Coulomb field. The interaction Lagrangian \( \mathcal{L}_{int} \) can thus be expressed in terms of the following quantity

\[
Q_\mu := \frac{1}{e} u^\nu M^{int}_{\mu\nu} ,
\]

which is obviously orthogonal to the four-velocity \( u^\mu \). Therefore, it has only three independent components. In the particle’s rest frame we have \( Q_0 = 0 \) and the three-vector \( eQ_k \) equals to the amount of the static moment acquired by the system when the particle’s own field is added to the background (external) field. Hence, we have from (3) and (4):

\[
\mathcal{L}_{int} = \sqrt{1 - v^2} ea^\mu Q_\mu(t, \mathbf{q}, \mathbf{v}) .
\]

We prove in Section 2 that the new interaction Lagrangian (5) differs from the old one by (gauge-dependent) boundary corrections only. Hence, both Lagrangians generate the same physical theory (although the new Lagrangian is of second differential order, its dependence upon the second derivatives is linear; this implies that the corresponding Euler-Lagrange equations of motion are of the second order). The relation between \( L \) and \( \mathcal{L} \) is, therefore, analogous to the one well known in General Relativity: the gauge-invariant, second order Hilbert Lagrangian for Einstein equations may be obtained starting from the first order, gauge-dependent Lagrangian and supplementing it by an appropriate boundary term.

At this point, our result can be summarized as follows: a physical interpretation of the interaction term \( eu^\mu A_\mu \) has been found. Up to boundary terms, it is equal to the interaction-angular-momentum term \( a^\mu u^\nu M^{int}_{\mu\nu} \). The question arises whether such an identity is a pure coincidence or is a result of a universal law of physics.

In this paper we try to convince the reader that the second conjecture is correct. In fact, we propose a new method of deriving equations of motion from field equations. The method is based on an analysis of the geometric structure of generators of the Poincaré group, related with any special-relativistic, lagrangian field theory. This analysis leads us to a simple theorem, which we call “variational principle for an observer” (see Section 3). Applying this observation to the specific case of classical electrodynamics, we show how to derive, in principle, equations of motion from field equations.

This derivation is based on a following heuristic idea: a point-like particle has to be understood as an approximation of an extended particle, i.e., an exact, stable, soliton-like solution of a hypothetical theory of matter fields interacting with electromagnetic field. To prove that the Maxwell theory (describing the free field outside of “strong-field-core of the solitons”) imply affirmatively equations of motion for the solitons themselves, we need several qualitative assumptions of heuristic nature, about their stability. Under these assumptions the gauge-invariant Lagrangian describing the motion of point particles is unambiguously derived from the general invariance principles of the theory.

The mathematical status of the above derivation is, therefore, similar to the Einstein – Infeld – Hoffmann derivation of equations of motion from field equations in General Relativity. It does not depend upon a particular model which we take for the description of
the gravitating body under consideration (e. g. a hydrodynamical or an elastomechanical model). The derivation is valid for any stable body and enables us to describe (in a good approximation) its motion in a model-independent way, as a geodesic motion of a point-particle. Hence, even if we have at the moment no realistic mathematical theory describing the interior of a star and fulfilling all the necessary assumptions, we can expect that the above equations of motion are valid for relatively stable objects. The present paper shows that a similar argumentation is possible also in electrodynamics: Lorentz force acting on test particles does not need to be postulated as an independent physical law, but is implied by the geometry of Maxwell field, provided one accepts the existence of the hypothetical fundamental theory of matter fields, admitting sufficiently stable soliton-like solutions.

The above result follows immediately from the consistent theory of interacting particles and fields (cf. [1], [2]), called Electrodynamics of Moving Particles. All the formulae of the present paper can be derived directly from the above theory in the test particle limit (i.e. \(m \to 0, e \to 0\) with the ratio \(e/m\) being fixed). The present paper, however, does not rely on this, much more general, context. The consistent theory of test particles is constructed independently.

It turns out that the above theory possesses also very interesting, perfectly gauge-invariant Hamiltonian structure. Standard Hamiltonian formalism, based on the gauge-dependent Lagrangian (1), leads to the gauge-dependent Hamiltonian

\[
H(t, q, p) = \sqrt{m^2 + (p + eA(t, q))^2 + eA_0(t, q)},
\]

where the gauge-dependent quantity

\[
p_k := p_k^{\text{kin}} - eA_k(t, q) = mu_k - eA_k(t, q)
\]

plays role of the momentum canonically conjugate to the particle’s position \(q^k\).

As was observed by Souriau (see [5]), the same theory may be described in a gauge-invariant way. For this purpose the electromagnetic field has to be taken as a deformation of the free-particle contact structure:

\[
\Omega = dp_k^{\text{kin}} \wedge dq^k.
\]

The 2-form \(\Omega\) is defined on the “mass-shell” of the kinetic momentum, i. e. on the surface \((p_{\text{kin}})^2 = -m^2\) in the cotangent bundle \(T^*M\) over the space-time \(M\) (we use the Minkowskian metric with the signature \((-+,+,+))\). The form contains the entire information about dynamics of free particles: the admissible trajectories are those, whose tangent vectors belong to the degeneracy distribution of \(\Omega\). Souriau noticed that replacing (8) by its deformation

\[
\Omega = \Omega - e f_{\mu\nu} dq^\mu \wedge dq^\nu,
\]

where \(e\) is the particle’s charge, we obtain this way the motion of the particle in a given electromagnetic field \(f_{\mu\nu}\).

The new approach proposed in the present paper leads directly to a perfectly gauge-invariant Hamiltonian. It has a clear physical interpretation being equal to the sum of two terms: 1) kinetic energy \(mu_0\) and 2) “interaction energy” equal to the amount of field energy acquired by our physical system, when the particle’s Coulomb field is added to the background field.
When formulated in terms of contact geometry, our approach leads uniquely to a new form $\Omega_N$:

$$
\Omega_N := \Omega - e h_{\mu\nu} dq^\mu \wedge dq^\nu, 
$$

(10)

where

$$
h_{\mu\nu} := 2(f_{\mu\nu} - u_{[\mu} f_{\nu]} \lambda u^\lambda)
$$

(11)

(brackets denote antisymmetrization). It is easy to see that both $\Omega_S$ and $\Omega_N$, although different, have the same degeneracy vectors, because $h$ and $f$ give the same value on the velocity vector $u^\nu$:

$$
u^\nu h_{\mu\nu} = \nu^\nu f_{\mu\nu}.
$$

(12)

Hence, both define the same equations of motion. We stress, however, that our $\Omega_N$ is uniquely obtained from the gauge-invariant Lagrangian (3) via the Legendre transformation.

The paper is organized as follows. Section 2 contains a direct proof that our new Lagrangian differs from the standard, gauge-dependent one by boundary terms only. Section 3 contains our basic geometric observation concerning any relativistic, Lagrangian field theory, which makes our approach possible. Using it, we give in Sections 4 and 5 the derivation of our variational principle. In particular, the renormalization procedure defined in Section 5 depends upon the stability assumptions (which may be relatively difficult to check for any specific mathematical model). In Section 6 we sketch briefly the (relatively little known) Hamiltonian formulation of theories arising from the second order Lagrangian. In Section 7 we prove explicitly that the Euler-Lagrange equations derived from $L$ are equivalent to the Lorentz equations of motion. Finally, Section 8 contains the gauge-invariant Hamiltonian structure of the theory.

2 Equivalence between the two variational principles

The easiest way to prove the equivalence consists in rewriting the field $f_{\mu\nu}$ in terms of the electric and the magnetic induction, using a special accelerated reference system, adapted to the particle’s trajectory $\zeta$, which we define in the sequel. The system will be also very useful to formulate our “variational principle for the observer” in the next Section.

We begin with any laboratory coordinate system $(y^\mu) = (y^0, y^k)$ in Minkowski space-time $M$ and parameterize the trajectory $\zeta$ with the laboratory time $t = y^0$. Let $y^k = q(t)$ be the corresponding coordinate description of $\zeta$. At each point of the trajectory we choose an orthonormal tetrad $(e_\mu)$, such that its element $e_0$ is tangent to $\zeta$, i.e. is equal to the four-velocity vector $u$. Take now the unique boost transformation relating the laboratory time axis $\partial/\partial y^0$ with the observer’s time axis $e_0$. We define the vector $e_k$ by transforming the corresponding $\partial/\partial y^k$ – axis of the laboratory frame by the same boost.

Finally, we parameterize space-time points by four coordinates $(t, x^k)$:

$$
y(t, x) := (t, q(t)) + x^k e_k(t).
$$

(13)

Expressing explicitly the tetrad in terms of the particle’s velocity, it is easy to obtain (cf. [2]) the following relation between our curvilinear coordinates $(t, x^k)$ and the laboratory
(Lorentzian) coordinates:

\[ g^0(t, x^l) := t + \frac{1}{\sqrt{1 - \mathbf{v}^2(t)}} \ x^l v_l(t), \]
\[ g^k(t, x^l) := q^k(t) + (\delta^k_l + \varphi(\mathbf{v}^2) v^k v_l) x^l, \] 

where we denote \( \varphi(z) := \frac{1}{z^2} \left( \frac{1}{\sqrt{1 - z^2}} - 1 \right) = \frac{1}{\sqrt{1 - z^2}} \).

The above formula may be used as a starting point of the entire proof. To rewrite field equations with respect to this system we need to calculate the components of the flat Minkowskian metric in our new coordinates. We easily get \( g_{kl} = \delta_{kl} \) for the space-space components. This is an obvious consequence of formula (13), because for a given value \( t \) parameters \( (x^k) \) provide a system of cartesian coordinates on the 3-dimensional hyperplane \( \Sigma_t \), orthogonal to \( \zeta \) at the point \( (t, q(t)) \).

The remaining information about the metric is carried by the lapse function and the shift vector, which again may be easily calculated from formula (14):

\[ N = \frac{1}{\sqrt{-g^{00}}} = \sqrt{1 - \mathbf{v}^2 \ (1 + a_i x^i)} , \]
\[ N_m = g_{0m} = \sqrt{1 - \mathbf{v}^2} \epsilon_{mkli} \omega^l x^i \].

Here, by \( a^i \) we denote the rest-frame components of the particle’s acceleration. They are given by formula:

\[ \frac{d}{d\tau} e_0 = a^i e_i , \]

where \( \tau \) is the proper time along \( \zeta \). A straightforward calculation gives us the following value:

\[ a^i = \frac{1}{1 - \mathbf{v}^2} \left( \delta^i_k + \varphi(\mathbf{v}^2) v^i v_k \right) \dot{v}^k \] 

where \( \dot{v}^k \) is the acceleration in the laboratory frame.

Moreover, at each point of \( \zeta \) we define the rotation vector \( (\omega^j) \) of the tetrad by the following formula:

\[ \frac{d}{d\tau} e_i = a_i e_0 - \epsilon_{ij}^k \omega^j e_k \] .

Again, straightforward calculation leads to the following expression (see e. g. [2]):

\[ \omega_m = \frac{1}{\sqrt{1 - \mathbf{v}^2}} \varphi(\mathbf{v}^2) v^k \dot{v}^l \epsilon_{klm} . \]

The transformation (14) is not invertible. Coordinates \( (x^k) \) are regular parameters on each \( \Sigma_t \). But hyperplanes \( \Sigma_t \) corresponding to different values of \( t \) may intersect. Hence, the same physical event may correspond to different values of coordinates \( (t, x^k) \).

Nevertheless, we may describe the free Maxwell field in terms of our parameters \( (t, x^k) \). In particular, we have:

\[ \dot{A}_k - \partial_k A_0 =: f_{0k} = -N D_k + \epsilon_{mklt} N^m B^t , \]

5
where $D^k$ and $B^k$ are the electric and the magnetic field on each $\Sigma_t$, $N$ and $N^m$ are given by (15) and (16) (for the description of the Maxwell field with respect to an accelerated reference system see e. g. [3]).

Let us multiply (21) by $\frac{x^k}{r^3}$ and integrate this scalar product over hyperplane $\Sigma_t$ with respect to the measure $dx^3$. We first integrate over the exterior of the sphere $S(r_0)$. Observe, that in this region the following identity holds:

$$\frac{x^k}{r^3} \partial_k A_0 = \partial_k \frac{x^k A_0}{r^3}. \quad (22)$$

Moreover, we have:

$$\frac{x^k}{r^3} \epsilon_{mkl} N^m B^l = -\sqrt{1 - \mathbf{v}^2} \partial_l \left( B^l \frac{\omega_k x^k}{r} \right). \quad (23)$$

Hence, after integration, we obtain in the limit $r_0 \to 0$:

$$\int_{\Sigma_t} \frac{x^k}{r^3} \dot{A}_k d^3x + 4\pi A_0(t,0) = -\int_{\Sigma_t} \frac{x^k}{r^3} N D_k d^3x \quad (24)$$

($A_0(t,0)$ is the only surface term which survives in the limit, due to the standard asymptotic behaviour of the field). Observe that the constant part of the lapse function (15) does not produce any contribution to the right-hand side of the above formula, because the flux of the field $D_k$ through any sphere $S(r)$ vanishes due to the Gauss law. Hence, we may replace “$N$” by “$\sqrt{1 - \mathbf{v}^2} a_i x_i$” under the integral and obtain

$$\int_{\Sigma_t} \frac{x^k}{r^3} N D_k d^3x = \frac{4\pi}{e} \sqrt{1 - \mathbf{v}^2} a_k \int_{\Sigma_t} x^k D^n D_n d^3x = -\frac{4\pi}{e} a^k M_{k0}^{int} = -\frac{4\pi}{e} a^\mu M_{\mu0}^{int}, \quad (25)$$

where

$$D^n := \frac{e}{4\pi} \frac{x^n}{r^3} \quad (26)$$

is the Coulomb field on $\Sigma_t$, corresponding to the charge $e$.

The lower index “0” in our particular system comes from the vector $\frac{\partial}{\partial t}$ which is proportional to the particle’s velocity $u^\mu$, where the proportionality coefficient $\sqrt{1 - \mathbf{v}^2}$ is due to the ratio between the proper time and the laboratory time on the trajectory. This means that $A_0(t,0)$, calculated in our particular coordinate system, is equal to $\sqrt{1 - \mathbf{v}^2} u^\mu A_\mu(t, q(t))$ in any other coordinate system. The same is true for $M_{\mu0}^{int}$. We have, therefore:

$$\frac{e}{4\pi} \int_{\Sigma_t} \frac{x^k}{r^3} \dot{A}_k d^3x + \sqrt{1 - \mathbf{v}^2} e u^\mu A_\mu(t, q(t)) = \sqrt{1 - \mathbf{v}^2} a^\mu u^\nu M_{\mu\nu}^{int}. \quad (27)$$

Integrating this identity over a time interval $[t_1, t_2]$ we finally obtain

$$\int_{t_1}^{t_2} L_{int} = \int_{t_1}^{t_2} L_{int} - \frac{e}{4\pi} \left( \int_{\Sigma_{t_2}} \frac{x^k}{r^3} A_k d^3x - \int_{\Sigma_{t_1}} \frac{x^k}{r^3} A_k d^3x \right). \quad (28)$$

We will see in the sequel that the value of $M_{\mu\nu}^{int}$ does not depend upon the choice of a particular hypersurface $\Sigma_t$ which we have used for integration. Any other $\Sigma$, which intersects the trajectory at the same point and is flat at infinity will give the same result.
3 Variational principle for an observer

Consider any relativistic-invariant, Lagrangian field theory (in this paper we will consider mainly Maxwell electrodynamics, but the construction given in the present Section may be applied to any scalar, spinor, tensor or even more general field theory). Choose any non-inertial observer, moving along a time-like trajectory $\zeta$. We want to describe the field evolution with respect to the observer’s rest frame. For this purpose we choose the space-time parameterization defined in the previous Section.

The field evolution with respect to the above non-inertial reference frame is a superposition of the following three transformations:

- time-translation in the direction of the local time-axis of the observer,
- boost in the direction of the acceleration $a^k$ of the observer,
- purely spatial O(3)-rotation $\omega^m$.

It is, therefore, obvious that the field-theoretical generator of this evolution is equal to

$$H = \sqrt{1 - v^2} \left( \mathcal{E} + a^k R_k - \omega^m S_m \right),$$

(29)

where $\mathcal{E}$ is the rest-frame field energy, $R_k$ is the rest-frame static moment and $S_m$ is the rest-frame angular momentum. The factor $\sqrt{1 - v^2}$ in front of the generator is necessary, because the time $t = x^0$, which we used to parameterize the observer’s trajectory, is not the proper time along $\zeta$ but the laboratory time. For any point $(t, q(t)) \in \zeta$ the values of the Poincaré generators $\mathcal{E}, R_k$ and $S_m$ are given as integrals of appropriate components of the field energy-momentum tensor over any space-like Cauchy surface $\Sigma$ which intersects $\zeta$ precisely at $(t, q(t))$ (due to Noether’s theorem, the integrals are independent upon the choice of such a surface). These values are, therefore, equal to the components of the total four-momentum $p_\mu$ and the total angular momentum $M_{\mu\nu}$ of the field, calculated in the observer’s co-moving frame $e_\mu$, i.e. $\mathcal{E} = -p_\mu w^\mu$, $R_\mu = w^\nu M_{\nu\mu}$ and $R_\mu w^\mu = 0$.

Given a field configuration, we are going to use the quantity $H$ as a second order Lagrangian for the observer’s trajectory. For this purpose let us first choose a “reference trajectory” $\zeta_0$. Next, for each point $(t, q(t)) \in \zeta_0$ let us calculate the corresponding “reference values” of the generators $\mathcal{E}(t), R_k(t), S_m(t)$ and insert them into $H$. Finally, consider the function obtained this way as a Lagrangian depending on a generic trajectory $\zeta$ via its velocity $v$ and acceleration $a$.

**Theorem**

Euler-Lagrange equations derived from the above Lagrangian are automatically satisfied by the trajectory $\zeta = \zeta_0$.

This theorem was derived in [2] in a much more general framework. Within this framework, it was an obvious consequence of the invariance of the theory with respect to the choice of an observer. More precisely, the function “$-H$” was proved to be a Routhian function playing the role of the Lagrangian with respect to the observer’s degrees of freedom and the Hamiltonian (with opposite sign) with respect to the field degrees of freedom. For purposes of the present paper we do not need, however, this larger context. The Theorem may be checked by simple inspection: Euler-Lagrange equations derived from the second order Lagrangian (29) are automatically satisfied as a consequence of the field energy-momentum and angular-momentum conservation (this direct proof was also given in [2]).
4 Adding a test particle to the field

¿From now on we limit ourselves to the case of electrodynamics. This means that the field energy-momentum and angular-momentum are defined as appropriate integrals of the components of the Maxwell energy-momentum tensor

\[ T^\mu_\nu = f^{\mu\lambda} f^\nu_\lambda - \frac{1}{4} \delta^\mu_\nu f^{\kappa\lambda} f^\kappa_\lambda . \] (30)

Suppose now that to a given background field \( f_{\mu\nu} \) we add a test particle carrying an electric charge \( e \). Denote by \( f(y,u)_{\mu\nu} \) the (boosted Coulomb) field accompanying the particle moving with constant four-velocity \( u \), which passes through the space-time point \( y \). Being bi-linear in fields, the energy-momentum tensor \( T_{\text{total}} \) of the total field

\[ f_{\mu\nu} := f_{\mu\nu} + f(y,u)_{\mu\nu} \] (31)

may be decomposed into three terms: the energy-momentum tensor of the background field \( T_{\text{field}} \), the Coulomb energy-momentum tensor \( T_{\text{particle}} \), which is composed of terms quadratic in \( f(y,u)_{\mu\nu} \) and the “interaction tensor” \( T_{\text{int}} \), containing mixed terms:

\[ T_{\text{total}} = T_{\text{field}} + T_{\text{particle}} + T_{\text{int}} \] (32)

Let us try to calculate the generator (29) for such a composed field configuration. For this purpose we have to integrate appropriate components of \( T_{\text{total}} \) over any \( \Sigma \) which passes through \( y = (t, q(t)) \). Integrating \( T_{\text{field}} \) and \( T_{\text{int}} \) we obtain the corresponding generator for the background field, which we call \( H_{\text{field}} \), and the “interaction generator” \( H_{\text{int}} \). Because the left-hand side and the first two terms of the right-hand side of (32) are conserved (outside of the particle’s trajectory), we conclude that also \( T_{\text{int}} \) is conserved. This implies, that the above integrals are invariant with respect to changes of \( \Sigma \), provided the intersection point with the trajectory does not change (see [1] for more details).

Unfortunately, the Coulomb tensor \( T_{\text{particle}} \) has an \( r^{-4} \) singularity at \( y \) and cannot be integrated. According to the renormalization procedure defined in [2] and sketched briefly in the next Section, we replace its integrals by the corresponding components of the total four-momentum of the particle: \( p^\lambda_{\text{particle}} = m u^\lambda \) and the total angular momentum: \( M^{\mu\nu}_{\text{particle}} = 0 \). The renormalized particle generator is, therefore, defined as follows:

\[ H_{\text{particle}}_{\text{ren}} = m \sqrt{1 - v^2} . \] (33)

This way we obtain the total (already renormalized) generator as a sum of three terms

\[ H_{\text{ren}} = H_{\text{field}} + H_{\text{int}} + H_{\text{particle}}_{\text{ren}} \] (34)

where the first term is quadratic and the second term is linear with respect to the background field \( f_{\mu\nu} \).

Let us observe that the only non-vanishing term in \( H_{\text{int}} \) comes from the static moment term \( R \) in (29), because the mixed terms in both the energy \( E \) and the angular momentum \( S \) vanish when integrated over any \( \Sigma \). The easiest way to prove this fact consists in choosing the hypersurface \( \Sigma_i \) which is orthogonal to the velocity \( u \) at \( (t, q(t)) \), i.e. the rest-frame surface (our integrals do not depend upon the choice of a hypersurface). On this surface, the Coulomb field \( f(y,u)_{\mu\nu} \) is spherically symmetric and carries, therefore, only the monopole component. On the other hand, the monopole component of the background field \( f_{\mu\nu} \) vanishes as
a consequence of the homogeneous Maxwell equations (no charges!). The mixed term in the energy integral is, therefore, a product of a monopole-free functions and the pure monopole. Hence, it vanishes after integration. A similar argument applies to the angular momentum $S$.

Finally, we have

$$H^\text{total}_{\text{ren}} = H^\text{field} - \sqrt{1 - v^2} \, a^\mu u^\nu M^\text{int}_{\mu\nu}(t, \mathbf{q}, \mathbf{v}) + \sqrt{1 - v^2} \, m ,$$

where the interaction term is defined as the following integral

$$M^\text{int}_{\mu\nu}(y) := \int_{\Sigma} \left\{ (x^\mu - y^\mu) T^\text{int}_{\nu\lambda}(x) - (x^\nu - y^\nu) T^\text{int}_{\mu\lambda}(x) \right\} d\Sigma^\lambda(x),$$

and $\Sigma$ is any hypersurface which intersects the trajectory at the point $y = (t, \mathbf{q}(t))$.

In particular, using the particle’s rest-frame and integrating over the rest-frame hypersurface $\Sigma_t$ we have

$$M^\text{int}_{k\dot{0}} = -\sqrt{1 - v^2} \int_{\Sigma_t} x_k D^n D_n \, d^3x ,$$

where $D^n$ is given by formula (26).

5 Renormalization. Derivation of equations of motion from field equations

According to the approach developed in [1] and [2] we treat the moving particle as a solution of a hypothetical “fundamental theory of matter fields interacting with electromagnetic field”. We assume that such a theory is a relativistic, Lagrangian (possibly highly non-linear) field theory. Moreover, we assume linear Maxwell theory as a limiting case of the above theory, corresponding to sufficiently weak electromagnetic fields and vanishing matter fields.

We will suppose that the particles, whose interaction with the electromagnetic field we are going to analyze, are simply global solutions of the above field theory. Each solution of this type is characterized by a tiny “strong field region”, concentrated in the vicinity of a time-like trajectory $\zeta$, which we may call an approximate trajectory of the extended particle. Outside of the strong field region the matter fields vanish (or almost vanish in the sense, that the following approximation remains valid) and the electromagnetic field is sufficiently weak to be described by Maxwell equations.

To be more precise, we imagine the “particle at rest” as a stable, static, soliton-like solution of our hypothetical “super theory”. The solution is characterized by two parameters: its total charge $e$ and its total energy $m$. The energy is not concentrated within the interior of the particle but contains also part of the energy carried by its “Coulomb tail”. This means that $m$ is an already renormalized mass, (or dressed mass), including the energy of the field surrounding the particle. Within this framework questions like “how big the bare mass of the particle is and which part of the mass is provided by the purely electromagnetic energy?” are meaningless. In the strong field region (i. e. inside the particle) the energy density may be highly non-linear and there is probably no way to divide it consistently into two such components.
Due to relativistic invariance of the theory, there is a 6 parameter family of the “uniformly moving particle” solutions obtained from our soliton via Poincaré transformations.

An arbitrarily moving particle is understood as a “perturbed soliton”. This means that it is again an exact solution of the same “super theory”, with its strong-field-region concentrated in the vicinity of a time-like world line ζ, which is no longer a straight line, as it was for “uniformly moving particles”. Let us choose an observer who follows this “approximate trajectory” ζ. We know that he automatically satisfies the Euler-Lagrange equations derived from the second order Lagrangian (29), where E, R^k and S^m are the quantities calculated for the complete non-linear theory.

Suppose now, that the particle may be treated as a test particle. This means, that the total field outside of the particle does not differ considerably from a background field f, satisfying homogeneous Maxwell equations. Using this hypothesis we may approximate, for each point (t, q(t)), the exact value of (29) by the value of (35). Indeed, we may decompose the total energy-momentum tensor of the complete non-linear field as follows:

\[ T^\mu_\lambda = (T^\mu_\lambda - T^\mu_\lambda') + T^\mu_\lambda' . \]  

Here, by T^\mu_\lambda we denote the total energy-momentum of the “super theory”, corresponding to the “uniformly moving particle” solution, which matches on Σ the position and the velocity of our particle. Stability of the soliton means that the “moving particle solution” does not differ considerably from the “uniformly moving particle solution” inside of the particles. Hence, the contribution of the first term (T - T') to the integrals \( E, R^k \) and \( S^m \) may be neglected “inside the particle”, i.e. we may replace it under integration by the purely Maxwellian quantity

\[ (T_{\text{total}} - T_{\text{particle}}) = T_{\text{field}} + T_{\text{int}} . \]  

As a result, we obtain the first two terms of (35). On the other hand, integrating the last term T in (38) we obtain without any approximation the corresponding value of the four-momentum of the particle. This way we reproduce the last term of (35).

Replacing (29) by its approximate value (35) and using the Theorem we conclude that the trajectory ζ has to fulfill Euler-Lagrange equations derived from (35). Applying the Theorem to the linear Maxwell theory we conclude, that the term \( H_{\text{field}} \) produces Euler-Lagrange equations which are automatically fulfilled by ζ. Hence, we may drop out this term, leaving only the remaining two terms. They finally give us our formula (3) for the Lagrangian of the test particle (the sign has to be changed because as a Lagrangian we should have taken “−\( H \)” instead of “\( H \)” – see remark at the end of Section 3).

### 6 Canonical formalism for a 2-nd order Lagrangian theory

Consider a theory described by the 2-nd order lagrangian \( L = L(q^i, \dot{q}^i, \ddot{q}^i) \) (to simplify the notation we skip the index “i” corresponding to different degrees of freedom \( q^i \); extension of this approach to higher order Lagrangians is straightforward). Introducing auxiliary variables \( v = \dot{q} \) we can treat our theory as a 1-st order one with lagrangian constraints \( \phi := \dot{q} - v = 0 \) on the space of lagrangian variables \( (q, \dot{q}, v, \dot{v}) \). Dynamics is generated by the following relation:

\[ d L(q, v, \dot{v}) = \frac{d}{dt}(p dq + \pi dv) = \dot{p} dq + p \dot{dq} + \pi dv + \pi \dot{dv} . \]  

10
where \((p, \pi)\) are momenta canonically conjugate to \(q\) and \(v\) respectively. Because \(L\) is defined only on the constraint submanifold, its derivative \(dL\) is not uniquely defined and has to be understood as a collection of all the covectors which are compatible with the derivative of the function along constraints. This means that the left hand side is defined up to \(\mu(\dot{q} - v)\), where \(\mu\) are Lagrange multipliers corresponding to constraints \(\phi = 0\). We conclude that \(p = \lambda\) is an arbitrary covector and (40) is equivalent to the system of dynamical equations:

\[
\begin{align*}
\pi &= \frac{\partial L}{\partial \dot{v}}, \\
\dot{p} &= \frac{\partial L}{\partial q}, \\
\dot{\pi} &= \frac{\partial L}{\partial v} - p.
\end{align*}
\] (41)

The last equation implies the definition of the canonical momentum \(p\):

\[
p = \frac{\partial L}{\partial v} - \dot{\pi} = \frac{\partial L}{\partial v} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right).
\] (42)

We conclude, that equation

\[
\dot{p} = \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) - \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{v}} \right).
\] (43)

is equivalent, indeed, to the Euler-Lagrange equation:

\[
\frac{\delta L}{\delta \dot{q}} = \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{v}} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right) + \frac{\partial L}{\partial q} = 0.
\] (44)

The Hamiltonian description (see e. g. [4]) is obtained from the Legendre transformation applied to (40):

\[
-H = \dot{p} dq - \dot{q} dp + \dot{\pi} dv - \dot{v} d\pi,
\] (45)

where \(H(q, p, v, \pi) = p v + \pi \dot{v} - L(q, v, \dot{v})\). In this formula we have to insert \(\dot{v} = \dot{v}(q, v, \pi)\), calculated from equation \(\pi = \frac{\partial L}{\partial \dot{v}}\). Let us observe that \(H\) is linear with respect to the momentum \(p\). This is a characteristic feature of the 2-nd order theory.

Euler-Lagrange equations (44) are of 4-th order. The corresponding 4 Hamiltonian equations have, therefore, to describe the evolution of \(q\) and its derivatives up to third order. Due to Hamiltonian equations implied by relation (45), the information about successive derivatives of \(q\) is carried by \((v, \pi, \dot{p})\):

- \(v\) describes \(\dot{q}\)

\[
\dot{q} = \frac{\partial H}{\partial p} \equiv v
\] (46)

hence, the constraint \(\phi = 0\) is reproduced due to linearity of \(H\) with respect to \(p\),

- \(\pi\) contains information about \(\ddot{q}\):

\[
\ddot{v} = \frac{\partial H}{\partial \pi},
\] (47)
\[ \dot{\pi} = -\frac{\partial H}{\partial v} = \frac{\partial L}{\partial v} - p , \] (48)

- the true dynamical equation equals

\[ \dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} . \] (49)

### 7 Equations of motion from the variational principle

In this Section we explicitly derive the particle’s equations of motion from the gauge-invariant Lagrangian (3). The Euler-Lagrange equations for a second order Lagrangian theory are given by

\[ \dot{p}_k = \frac{\partial L}{\partial q^k} , \] (50)

where, as we have seen in the previous Section, the momentum \( p_k \) canonically conjugate to the particle’s position \( q^k \) is defined as:

\[ p_k := \frac{\partial L}{\partial v^k} - \pi_k \] (51)

and

\[ \pi_k := \frac{\partial L}{\partial \dot{v}^k} = \frac{1}{\sqrt{1 - \nu^2}} \ u^\nu M^\text{int}_{k\nu}(t, q, v) . \] (52)

Now,

\[ u^\nu M^\text{int}_{k\nu} = u^0 M^\text{int}_{k0} + u^i M^\text{int}_{k0} = -u^0 r^\text{int}_k + u^i \epsilon^m_{kl} s^\text{int}_m , \] (53)

where \( r^\text{int}_k \) and \( s^\text{int}_m \) are the static momentum and the angular momentum of the interaction tensor (see 32).

In particular, \( r^\text{int} \) and \( s^\text{int} \) may be written in terms of the laboratory-frame components of the electric and magnetic fields as follows:

\[ r^\text{int}_k(t, q, v) = \int_{\Sigma} d^3 x \ (x_k - q_k)(DD_0 + BB_0) , \] (54)

\[ s^\text{int}_m(t, q, v) = \epsilon_{mij} \int_{\Sigma} d^3 x \ (x^i - q^i)(D \times B_0 + D_0 \times B)^j , \] (55)

where \( D \) and \( B \) are components of the external field \( f \), whereas \( D_0 \) and \( B_0 \) are components of \( f(y,u) \), i.e.:

\[ D_0(x; q, v) = \frac{e}{4\pi|x - q|^3} \frac{1 - v^2}{\left(1 - v^2 + \left(\frac{v(x - q)}{|x - q|}\right)^2\right)^{3/2}} (x - q) , \] (56)

\[ B_0(x; q, v) = v \times D_0(x; q, v) . \] (57)
It may be easily seen that quantities \( r_k^{\text{int}} \) and \( s_m^{\text{int}} \) are not independent. They fulfill the following condition:

\[
s_k^{\text{int}} = -\epsilon_{kl} m^l r_m^{\text{int}}.
\]  
(58)

To prove this relation let us observe that in the particle’s rest-frame (see Section 2 for the definition) the angular momentum corresponding to \( T^l \) vanishes. When translated to the language of laboratory frame, this is precisely equivalent to the above relation.

Inserting (58) into (53) we finally get

\[
\pi_k = -\left(\delta_k^l + \frac{v^l v_k}{1 - \mathbf{v}^2}\right) r_l^{\text{int}}.
\]  
(59)

The quantity \( r_k^{\text{int}} \) depends upon time via the particle’s position \( \mathbf{q} \) and velocity \( \mathbf{v} \), contained in formulae (56) – (57) for the particle’s Coulomb field, but also through the time dependence of the external fields \( (\mathbf{D}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x})) \).

Now, we are ready to compute \( p_k \) from (51):

\[
p_k = \frac{m v_k}{\sqrt{1 - \mathbf{v}^2}} + \dot{v}^l \partial \pi_k \partial v^l - \left(\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} + \dot{v}^l \frac{\partial \pi_k}{\partial v^l}\right)
\]  
(60)

Observe, that the momentum \( p_k \) depends upon time, particle’s position and velocity but also on particle’s acceleration. However, using (54) one shows after straightforward calculations that the term proportional to \( \dot{v}^l \) vanishes, i.e.

\[
\frac{\partial \pi_k}{\partial v^l} - \frac{\partial \pi_l}{\partial v^k} = 0.
\]  
(61)

Moreover, it turns out that (see Appendix):

\[
\frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} = -p_k^{\text{int}},
\]  
(62)

where

\[
p_k^{\text{int}}(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma d^3x \left(\mathbf{D} \times \mathbf{B}_0 + \mathbf{D}_0 \times \mathbf{B}\right)_k.
\]  
(63)

We see that \( p_k^{\text{int}} \) is the spatial part of the “interaction momentum”:

\[
p_{\mu}^{\text{int}}(t, \mathbf{q}, \mathbf{v}) = \int_\Sigma T_{\mu \nu}^{\text{int}} d\Sigma^\nu,
\]  
(64)

where \( \Sigma \) is any hypersurface intersecting the particle’s trajectory at the point \( (t, \mathbf{q}(t)) \). The above integral is well defined (cf. [1]) and it is invariant with respect to changes of \( \Sigma \), provided the intersection point with the trajectory does not change. We know from Section 4 that \( p_{\mu}^{\text{int}} \) is orthogonal to the particle’s four-velocity, i.e. \( p_{\mu}^{\text{int}} u^\mu = 0 \).

Finally, the momentum canonically conjugate to the particle’s position equals:

\[
p_k = p_k^{\text{kin}} + p_k^{\text{int}}(t, \mathbf{q}, \mathbf{v}).
\]  
(65)

It is a sum of two terms: kinetic momentum \( p_k^{\text{kin}} \) and the amount of momentum which is acquired by our system, when the particle’s Coulomb field is added to the background
We stress, that contrary to the standard formulation based on (1), the canonical momentum \( p_k \) is gauge-invariant.

Now, Euler-Lagrange equations (50) read

\[
\frac{dp^\text{kin}_k}{dt} + \frac{dp^\text{int}_k}{dt} = \frac{\partial \mathcal{L}}{\partial q^k},
\]

or in a more transparent way:

\[
d \left( \frac{mv_k}{\sqrt{1-v^2}} \right) = -\left( \frac{\partial p^\text{int}_k}{\partial t} + v^l \frac{\partial p^\text{int}_k}{\partial q^l} \right) - v^l \left( \frac{\partial p^\text{int}_k}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} \right).
\]

Again, using definitions of \( \pi_l \) and \( p^\text{int}_k \) one shows that the term proportional to the particle’s acceleration vanishes, i.e.

\[
\frac{\partial p^\text{int}_k}{\partial v^l} - \frac{\partial \pi_l}{\partial q^k} = 0.
\]

The last step in our derivation is to calculate \( \frac{\partial p^\text{int}_k}{\partial t} + v^l \frac{\partial p^\text{int}_k}{\partial q^l} \). In the Appendix we show that this term gives exactly the Lorentz force, i.e.

\[
\frac{\partial p^\text{int}_k}{\partial t} + v^l \frac{\partial p^\text{int}_k}{\partial q^l} = -e \sqrt{1-v^2} u^\nu f_{k\nu}(t,\mathbf{q}) = -e(E_k(t,\mathbf{q}) + \epsilon_{klm} v^l B^m(t,\mathbf{q})).
\]

This way we proved that the Euler-Lagrange equations (50) for the variational problem based on \( \mathcal{L} \) are equivalent to the Lorentz equations for the motion of charged particles:

\[
\frac{d}{dt} \left( \frac{mv_k}{\sqrt{1-v^2}} \right) = e(E_k(t,\mathbf{q}) + \epsilon_{klm} v^l B^m(t,\mathbf{q})).
\]

### 8 Hamiltonian formulation

Usually by the Hamiltonian formulation of the theory we understand the phase space of Hamiltonian variables \( \mathcal{P} = (q, p) \) endowed with the symplectic 2-form \( \omega = dp \wedge dq \) and the Hamilton function \( H \) (Hamiltonian) defined on \( \mathcal{P} \). This function is interpreted as an energy of the system. However, for time-dependent systems this framework is usually replaced by (a slightly more natural) formulation in terms of a contact form. For this purpose one considers the evolution space \( \mathcal{P} \times \mathbb{R} \) endowed with the contact 2-form (i.e. closed 2-form of maximal rank):

\[
\omega_H := dp \wedge dq - dH \wedge dt.
\]

In analytical mechanics this form, or rather its “potential” \( pdq - H dt \), is called the Poincaré-Cartan invariant. Obviously, \( \omega_H \) is degenerate on \( \mathcal{P} \times \mathbb{R} \) and the one-dimensional characteristic bundle of \( \omega_H \) consists of the integral curves of the system in \( \mathcal{P} \times \mathbb{R} \). This kind of description may be called the “Heisenberg picture” of classical mechanics. In this picture states are not points in \( \mathcal{P} \) but “particle’s histories” in \( \mathcal{P} \times \mathbb{R} \) (see [4]).

Let us construct the Hamiltonian structure for the theory based on the second order Lagrangian \( \mathcal{L} \). Let \( \mathcal{P} \) denote the space of Hamiltonian variables, i.e. \( (q, p, v, \pi) \), where \( p \) and \( \pi \) stand for the momenta canonically conjugate to \( q \) and \( v \) respectively. Since our
system is manifestly time-dependent (via the time dependence of the external field) we pass to the evolution space endowed with the contact 2-form

\[ \Omega = dp_k \wedge dq^k + d\pi_k \wedge dv^k - dH \wedge dt, \tag{72} \]

where $H$ denotes the time-dependent particle’s Hamiltonian.

To find $H$ on $P \times \mathbb{R}$ one has to perform the (time-dependent) Legendre transformation $(q, \dot{q}, v, \dot{v}) \rightarrow (q, p, v, \pi)$, i.e. one has to calculate $\dot{q}$ and $\dot{v}$ in terms of Hamiltonian variables from formulae:

\[ p_k = \frac{\partial L}{\partial \dot{q}^k} - \pi_k, \quad \pi_k = \frac{\partial L}{\partial \dot{v}^k}. \tag{73} \]

This transformation is singular due to linear dependence of $L$ on $\dot{v}$ and gives rise to the time-dependent constraints, given by equations (52) and (65). The constraints can be easily solved i.e. momenta $p_k$ and $\pi_k$ can be uniquely parameterized by the particle’s position $q^k$, velocity $v^k$ and the time $t$. Let $P^*$ denote the constrained submanifold of the evolution space $P \times \mathbb{R}$ parametrized by $(q, p^{\text{kin}}, t)$. The reduced Hamiltonian on $P^*$ reads (we denote the reduced Hamiltonian by the same letter):

\[ H(t, q, v) = p_k v^k + \pi_k v^k - L = \frac{m}{\sqrt{1 - v^2}} + v^k p^{\text{int}}_k(t, q, v). \tag{74} \]

Due to identity $u^\mu p^{\text{int}}_\mu = 0$ we have

\[ H(t, q, v) = \frac{m}{\sqrt{1 - v^2}} - p^0_0(t, q, v) \tag{75} \]

and, therefore, the particle’s Hamiltonian equals to the “$-p_0$” component of the following, perfectly gauge-invariant, four-vector

\[ p_\mu := p^{\text{kin}}_\mu + p^{\text{int}}_\mu(t, q, v) = mu_\mu + p^{\text{int}}_\mu(t, q, v). \tag{76} \]

Using the laboratory-frame components of the external electromagnetic field we get:

\[ p^0_0(t, q, v) = -\int d^3x (DD_0 + BB_0). \tag{77} \]

Now, let us reduce the contact 2-form (72) on $P^*$. Calculating $p_k = p_k(q, p^{\text{kin}}, t)$ and $\pi_k = \pi_k(q, p^{\text{kin}}, t)$ from (52) – (65) and inserting them into (72) one obtains after a simple algebra:

\[ \Omega_N = dp^{\text{kin}}_\mu \wedge dq^\mu - e h_{\mu\nu} dq^\mu \wedge dq^\nu, \tag{78} \]

where $q^0 \equiv t$ and $h_{\mu\nu}$ is the following 4-dimensional tensor:

\[ e h_{\mu\nu}(t, q, v) := \frac{\partial p^{\text{int}}_\mu}{\partial q^\nu} - \frac{\partial p^{\text{int}}_\nu}{\partial q^\mu}. \tag{79} \]

Using techniques presented in the Appendix one easily gets

\[ \frac{\partial p^{\text{int}}_\nu}{\partial q^\mu} = e \Pi^\lambda_\mu f_{\lambda\nu}, \tag{80} \]
where
\[ \Pi^\lambda_\mu := \delta^\lambda_\mu + u_\mu u^\lambda \] (81)
is the projection on the hyperplane orthogonal to \( u^\mu \) (i.e. to the particle’s rest-frame hyperplane). Therefore
\[ h_{\mu\nu} = \Pi^\lambda_\mu f_{\lambda\nu} - \Pi^\lambda_\nu f_{\lambda\mu} = 2(f_{\mu\nu} - u_\mu f_{\nu\lambda} u^\lambda) . \] (82)

where \( a_{[\alpha \beta]} := \frac{1}{2} (a_\alpha b_\beta - a_\beta b_\alpha) \). The form \( \Omega_N \) is defined on a submanifold of cotangent bundle \( T^*M \) defined by the particle’s “mass shell” \( (p_{\text{kin}})^2 = -m^2 \).

Observe, that the 2-form (78) has the same structure as the Souriau’s 2-form (9). They differ by the “curvature” 2-forms \( f \) and \( h \) only. However, the difference \( "h - f" \) vanishes identically along the particle’s trajectories due to the fact that both \( f_{\mu\nu} \) and \( h_{\mu\nu} \) have the same projections in the direction of \( u^\mu \) (see formula (12)). We conclude that the characteristic bundle of \( \Omega_N \) and \( \Omega_S \) are the same and they are described by the following equations:
\[ \dot{q}^i = v^i , \] (83)
\[ \dot{v}^i = \sqrt{1 - v^2} \frac{e}{m} (g^{kl} - v^k v^l) (E_l + \epsilon_{lij} B^j) , \] (84)
which are equivalent to the Lorentz equations (70).

We have two different contact structures which have the same characteristic bundles. Therefore, from the physical point of view, these forms are completely equivalent.

Appendix
To calculate \( \frac{\partial \pi_k}{\partial t} + v^l \frac{\partial \pi_k}{\partial q^l} \) let us observe, that
\[ \frac{\partial}{\partial t} + v^l \frac{\partial}{\partial q^l} = \sqrt{1 - v^2} \frac{\partial}{\partial q^\mu} \equiv \left( \frac{\partial}{\partial t} \right)_u , \] (A.1)
where \( \left( \frac{\partial}{\partial t} \right)_u \) denotes the time derivative with respect to an observer moving along the particle’s trajectory \( \zeta \). We shall use the frame associated with moving observer and defined in Section 2 (cf. also [2]).

Let \( \mathcal{D}_k \) and \( \mathcal{B}_k \) denote the rest-frame components of the electric and magnetic field. Observe, that the particle’s Coulomb field is given by
\[ \mathcal{D}_0(x) = \frac{e x}{4\pi r^3} , \quad \mathcal{B}_0(x) = 0 , \] (A.2)
where \( r := |x| \). Now, \( r^i_{\text{int}} \) may be expressed in terms of \( \mathcal{D}_k \) and \( \mathcal{B}_k \) as follows:
\[ r^i_{\text{int}} = \frac{1}{\sqrt{1 - v^2}} (\delta^i_k - \sqrt{1 - v^2} \varphi (v^2) v^i v_k) \frac{e}{4\pi} \int_{\Sigma_t} \frac{x_i x_i}{r^3} \mathcal{D}^i d^3x . \] (A.3)
The matrix \( (\delta^i_k - \sqrt{1 - v^2} \varphi (v^2) v^i v_k) \) comes from the boost transformation. Using the Maxwell equation in the particle’s rest frame:
\[ \left( \frac{\partial}{\partial t} \right)_u \mathcal{D} = \sqrt{1 - v^2} \text{ curl } \mathcal{B} , \] (A.4)
\[ \left( \frac{\partial}{\partial t} \right)_u \mathcal{B} = -\sqrt{1 - v^2} \text{ curl } \mathcal{D} , \] (A.5)

16
Due to the Lorentz system of units, Therefore, using (A.5) one obtains:

\[
\left( \frac{\partial}{\partial t} \right)_u \int_{\Sigma_t} \frac{x^i x_i}{r^3} D^j d^3x = \sqrt{1 - v^2} \int_{\Sigma_t} \frac{x^i x_i}{r^3} \epsilon^{ijm} \partial_j B_m d^3x
\]

\[
= \sqrt{1 - v^2} \int_{\Sigma_t} \partial_j \left( \frac{x^i x_i}{r^3} \epsilon^{ijm} B_m \right) d^3x + \sqrt{1 - v^2} \int_{\Sigma_t} \epsilon^{lm} \frac{x^i}{r^3} B_m d^3x . \tag{A.6}
\]

Using the Gauss theorem we obtain:

\[
\int_{\Sigma_t} \partial_j \left( \frac{x^i x_i}{r^3} \epsilon^{ijm} B_m \right) d^3x = \int_{\partial \Sigma_t} \frac{x^i x_i}{r^3} \epsilon^{ijm} B_m d\sigma \equiv 0 , \tag{A.7}
\]

where \( d\sigma \) denotes the surface measure on \( \partial \Sigma_t \). The term \( \frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{lm} \frac{x^i}{r^3} B_m d^3x \) equals to the momentum acquired by our system in the particle’s rest-frame when the Coulomb field \( D_0 \) is added to the external field. Therefore, this term may be expressed in terms of the laboratory-frame components of \( p^{int}_\mu \):

\[
\frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{lm} \frac{x^i}{r^3} B_m d^3x = \frac{v_l}{\sqrt{1 - v^2}} p_{0l}^{int} + (\delta_l^k + \varphi(v^2) v^l v_k) p_k^{int} . \tag{A.8}
\]

Using the Gauss theorem again we get:

\[
\int_{\Sigma_t} \partial_j \left( \frac{x^i x_i}{r^3} \epsilon^{ijm} B_m \right) d^3x = \int_{\partial \Sigma_t} \frac{x^i x_i}{r^3} \epsilon^{ijm} B_m d\sigma \equiv 0 . \tag{A.7}
\]

where \( d\sigma \) denotes the surface measure on \( \partial \Sigma_t \). The term \( \frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{lm} \frac{x^i}{r^3} B_m d^3x \) equals to the momentum acquired by our system in the particle’s rest-frame when the Coulomb field \( D_0 \) is added to the external field. Therefore, this term may be expressed in terms of the laboratory-frame components of \( p^{int}_\mu \):

\[
\frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{lm} \frac{x^i}{r^3} B_m d^3x = \frac{v_l}{\sqrt{1 - v^2}} p_{0l}^{int} + (\delta_l^k + \varphi(v^2) v^l v_k) p_k^{int} . \tag{A.8}
\]

Finally

\[
\left( \frac{\partial}{\partial t} \right)_u \pi_k = - \left( \delta_k^l + \frac{v^l v_k}{1 - v^2} \right) \left( \frac{\partial}{\partial t} \right)_u p_l^{int}
\]

\[
= - \left( \delta_k^l + \frac{v^l v_k}{1 - v^2} \right) \left( \delta_l^j - \sqrt{1 - v^2} \varphi(v^2) v^i v_j \right) \left( \frac{v_i}{\sqrt{1 - v^2}} p_{0l}^{int} + (\delta_l^j + \varphi(v^2) v^i v_j) p_l^{int} \right)
\]

\[
= - \frac{v_k}{1 - v^2} p_{0l}^{int} - p_l^{int} - \frac{v_k}{1 - v^2} v^l p_l^{int} = - p_l^{int} \tag{A.9}
\]

due to \( u^\mu p^{int}_\mu = 0 \). This ends the proof of (62).

Now, to proof (69) let us express \( p_k^{int} \) in terms of \( D_k \) and \( B_k \). One gets:

\[
p_k^{int} = (\delta_k^l + \varphi(v^2) v^l v_k) \frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{ijl} \frac{x^i}{r^3} B^j d^3x . \tag{A.10}
\]

Therefore, using (A.5) one obtains:

\[
\left( \frac{\partial}{\partial t} \right)_u \left( \frac{e}{4\pi} \int_{\Sigma_t} \epsilon^{ijl} \frac{x^i}{r^3} B^j d^3x \right) = \sqrt{1 - v^2} \frac{e}{4\pi} \int_{\partial \Sigma_t} \frac{1}{r^2} D_l d\sigma
\]

\[
= - \sqrt{1 - v^2} \frac{e}{4\pi} \lim_{r_0 \to 0} \int_{S(r_0)} \frac{1}{r^2} D_l d\sigma = - \sqrt{1 - v^2} e D_l(t, 0) , \tag{A.11}
\]

where \( d\sigma \) denotes the surface element on the sphere \( S(r_0) \) (we choose as two pieces of a boundary \( \partial \Sigma_t \) a sphere at infinity and a sphere \( S(r_0) \)). Using the fact that in the Heaviside-Lorentz system of units \( D_k = \mathcal{E}_k \) and expressing \( \mathcal{E}_k \) in terms of the laboratory-frame fields \( E_k \) and \( B^m \):

\[
\mathcal{E}_k(t, 0) = \frac{1}{\sqrt{1 - v^2}} \left[ (\delta_k^l - \sqrt{1 - v^2} \varphi(v^2) v^l v_k) E_l(t, q) + \epsilon_{klm} v^l B^m(t, q) \right] \tag{A.12}
\]

we finally obtain (69).
References


