Electrodynamics of Moving Particles

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Abstract

A consistent relativistic theory of the classical Maxwell field interacting with classical, charged, point-like particles is proposed. The theory is derived from a classical “soliton-like” model of an extended particle. An approximation procedure for such a model is developed, which leads to an “already renormalized” formula for the total four-momentum of the system composed of fields and particles. Conservation of this quantity leads to a theory which is universal (i.e. does not depend upon a specific model we start with) and which may be regarded as a simple and necessary completion of special relativity. The renormalization method proposed here may be considered as a realization of Einstein’s idea of “deriving equations of motion from field equations”. It is shown that the Dirac’s “3-dots” equation does not describe a fundamental law of physics, but only a specific family of solutions of our theory, corresponding to a specific choice of the field initial data.

1 Introduction

Classical electrodynamics splits into two incompatible parts: 1) Maxwell theory for the evolution of the electromagnetic field with given sources and 2) Lorentz theory of motion of test particles (i.e. point particles, which themselves do not influence the field). Any attempt to describe electromagnetic self-interaction by putting together these two parts leads to a contradiction: due to Maxwell equations the Lorentz force will always be infinite. There have been many attempts to “renormalize” this infinity (e.g. [1]), but none of them is satisfactory. In particular, Dirac’s equation for the radiative friction (see [2]), although very useful in many applications, is based on a non-physical decomposition of the field into the “retarded” and the “radiative” component and does not describe the interaction of particles and fields in terms of locally defined quantities. Moreover, it breaks
the symmetry of electrodynamics with respect to time reversal (the relation between Abraham-Lorentz-Dirac theory and the theory presented in this paper is discussed in Section 8).

It is easy to avoid infinities assuming finite dimensions of the particle. But then an infinite amount of information (Poincaré stresses) is necessary to describe the interior of the particle. Many efforts have been made to replace this infinite number of internal degrees of freedom by a finite number of “collective degrees of freedom” of mechanical nature, i.e. to approximate field theory by mechanics. The easiest way to do this consists in assuming that the particle is rigid (see [3] and [4]), but then the theory is no longer relativistic. Moreover, the result is highly dependent upon the dimension of the particle. Its radius \( r_0 \) plays the role of a “cutoff parameter” and the resulting “renormalized Lorentz force” tends again to infinity when \( r_0 \to 0 \). A similar “cutoff” is provided by the non-linearity parameter in the Born-Infeld theory [5].

In this paper we present a consistent theory of the Maxwell electromagnetic field and point-like particles interacting with each other. Each particle is characterized by its rest mass \( m \) and its electric charge \( e \). Within this framework there is no possibility of describing multipole particles and the reason is not technical but fundamental. Besides the mass, the particle could carry a “multidimensional charge” corresponding to the gauge group of the field it interacts with. Replacing e.g. the electromagnetic field by an SU(2) gauge field, equations of motion for colored particles could be obtained. The gravitational field could be taken as another example. Within the framework of General Relativity, the angular momentum of the particle is one of the “charges” defined by the gauge group of asymptotic Poincaré transformations. The magnetic moment of such a particle is given by its angular momentum and its electric charge. Therefore, particles carrying a magnetic moment could also be described but for this purpose it is necessary to consider also gravitational interactions. In the present paper we limit ourselves to the case of pure electrodynamics. Our particles carry only one (scalar) charge \( e \) and we stress that the theory does not work for particles with non-vanishing higher charge multipoles.

Physically, the theory will be derived as an approximation of an extended-particle model by a suitable approximation procedure, presented in the Appendix. The “particle at rest” is understood as a stable, soliton-like solution of a hypothetical fundamental theory (“super theory”) of interacting electromagnetic and matter fields. We assume that for weak electromagnetic fields and vanishing matter fields the theory coincides with Maxwell electrodynamics. The parameters \( e \) and \( m \) are the total charge and the total mass (including the electromagnetic part) of the non-perturbed soliton (corresponding to the particle at rest). For such a “super theory” an “already renormalized” formula is derived, which enables us to calculate in a good approximation the total four-momentum of a moving particle and its surrounding field, without knowing the internal structure of the particle. Our formula contains only the “mechanical” information about the particle (position, velocity, mass and charge) and the free Maxwell field outside the particle. Within the applicability region of this approximation, our expression does not depend
upon the specific “super theory” we started with or the specific internal structure of the
soliton. The formula is “already renormalized” in the following sense: it produces no
infinities when applied to a point-like object.

We take this formula as a starting point of our mathematically complete theory of
point-like particles. In the simplest case of a single particle, any solution of Maxwell
equations with a δ-like source and any space-like hypersurface Σ generate, according to
formula (7), a tensor density which – when integrated over Σ – produces a four-vector
which we propose to take as the total four-momentum of the physical system composed
of a particle and a field. Then we prove that the above quantity does not change if
we change Σ, provided we do not change the intersection points of Σ with the particle
trajectory. Hence, the four-momentum is defined at each point of the trajectory. For a
generic trajectory and a generic solution of Maxwell equations, the result we get is not
constant along the trajectory. The conservation of the total four-momentum obtained
in this way represents an additional condition which we add to the standard Maxwell
theory. Together with this condition, the theory becomes finite and causal: the initial
data for both particles and fields uniquely determine the future and the past of the system,
although the Lorentz force acting on the particle remains always ill-defined.

Mathematically, the theory is formulated in terms of a problem with moving boundary.
We prove that the conservation law for the total four-momentum is equivalent to a bound-
ary condition for the behaviour of the electromagnetic field on the particle trajectory. We
call this condition “the fundamental equation” of our theory. The field equations of our
theory are therefore precisely the linear Maxwell equations for the electromagnetic field
surrounding point-like sources. The new element which makes the system behave in a
causal way is the above boundary condition, where the particle trajectory plays the role
of the moving boundary.

Our starting point was the following observation, presented in Section 2, concerning
the mathematical structure of linear electrodynamics: Maxwell equations alone enable us
to calculate the acceleration of the particle from the field Cauchy-data, without assuming
any equation of motion.

Section 3 contains our “already renormalized” definition of the total four-momentum
of the system composed of particles and fields.

In Section 4 we present the fundamental equation of our theory. Moreover, we show
that the limit of the theory for e → 0 and m → 0 with their ratio being fixed, coincides
with the Maxwell–Lorentz theory of test particles. However, for any finite e the accel-
eration of the particle can not be equal to the Lorentz force, the latter being always ill
defined.

In Section 5 we prove the mathematical consistency of the “already renormalized”
formula given in Section 3. For this purpose we rewrite the field dynamics in the rest-frame
of the particle. This is an accelerated reference system and not an inertial one. Its use
simplifies considerably all the subsequent calculations.

The fundamental equation is proved as a consequence of the momentum conservation
law in the subsequent Section 6.

The complete many-particles case is discussed in Section 7. In particular, assuming that one particle is very heavy and practically does not move, we may consider its own Coulomb field as an “external field”. This way we are led to an extension of our theory, where the light particle interacts not only with its own “radiation field”, but also with an “external field” or simply a field produced by a heavy external device. This extended theory is also discussed in Section 7.

Although the field equations of the theory are linear, the moving boundary condition makes it relatively difficult. The method developed in Section 5 enables us to replace the problem with moving boundary by a problem with fixed boundary, but the price we pay for it is an effective non-linearity of the resulting dynamical system. It is therefore highly non-trivial to produce exact solutions. There is, however, a one-parameter family of third order ordinary differential equations for the particle trajectory, which result from our theory and which may be useful to construct exact solutions in special physical situations. Each of them is obtained from a specific “Ansatz” for the field surrounding the particle. We analyze this method in Section 8 and show that, in particular, the Abraham-Lorentz-Dirac equation may be obtained this way as a particular example. Therefore, all the solutions of the Abraham-Lorentz-Dirac theory (also the self-accelerating ones) fulfill the dynamical equations of our theory. However, to maintain such a self-acceleration, a non-physical (“fine tuned”) choice of field initial data is necessary, with an infinite amount of field energy prepared far away from the particle. In this way a curious role of Dirac’s “time arrow” has been explained: it is not a fundamental property of the theory, but has been introduced by the choice of initial data. The theory itself remains symmetric with respect to time reversal and the Abraham-Lorentz-Dirac “force of radiative friction” may be changed completely (e.g. may be multiplied by $-1$) if we change field initial data.

Finally, Section 9 contains the non-relativistic limit of the theory.

Although the problems with moving boundaries are always technically more complicated, the conceptual structure of our theory is extremely simple. The author wonders why it was not discovered at least 70 years ago. The only reason for that could be the usual fear of theoretical physicists to deal with boundary problems (see [6]). Even when describing a particle in a finite box, people usually prefer to consider “periodic boundary conditions” than to use any realistic description of the boundary data for the field. Similarly, the ordinary description of the Hamiltonian structure of field theory is based on integration by parts and the dogma of vanishing of all the surface integrals obtained this way (cf. e.g.[7]). As we shall see, in electrodynamics of moving particles the surface integrals do not vanish. On the contrary, they carry the entire useful information about particle dynamics.

The theory has an interesting, highly non-trivial canonical structure. Both the Hamiltonian and the Lagrangian formulations of the dynamics will be given in the next paper.

Since we have not used any quantitative hypothesis about the fundamental field theory
we started with, except the Maxwell equations as a weak field limit, we may claim that our equations of motion have been derived from Maxwell equations only. The method used is a realization of Einstein’s idea of “deriving equations of motion from field equations”. Actually, the present paper is a by-product of the author’s attempts to find a coherent approach to the problem of motion in General Relativity, where the description of the boundary conditions is much more complicated (see [8]). The existing results are already promising and the author hopes to obtain soon the General Relativistic version of this theory, including the possibility to describe the dynamics of spinning particles, interacting not only electromagnetically, but also gravitationally.

2 Boundary conditions for the Maxwell field

Consider the theory of point-like, charged particles. Let $\mathbf{y} = q(t)$ or $y^k = q^k(t)$, $k = 1, 2, 3$; with $t = y^0$, be the coordinate description of a time-like world line $\zeta$ with respect to a laboratory reference frame, i.e. a system $(y^\lambda)$, $\lambda = 0, 1, 2, 3$; of Lorentzian space-time coordinates.

We assume that the field $f_{\mu\nu}$ satisfies the standard vacuum Maxwell equations with point–like sources:

$$\begin{align*}
\partial(\lambda f_{\mu\nu}) &= 0 \\
\partial_m f^{\nu\mu} &= \epsilon u^\nu \delta_\zeta
\end{align*}$$

where $\delta_\zeta$ denotes the $\delta$–distribution concentrated on a smooth world line $\zeta$:

$$\delta_\zeta(y^0, y^k) = \sqrt{1 - (v(y^0))^2} \delta^{(3)}(y^k - q^k(y^0)).$$

Here $v = (v^k)$, with $v^k := \dot{q}^k$, is the corresponding 3–velocity and $v^2$ denotes the square of its 3–dimensional length (we use the Heaviside-Lorentz system of units with $c = 1$).

In the case of many particles, the total current is a sum of contributions corresponding to many disjoint world lines and the value of the charge is assigned to each world line separately.

For a given trajectory, equations (1) define a deterministic theory: initial data for the electromagnetic field uniquely determine its evolution. However, if we want to treat also the particle initial data $(q, v)$ as dynamical variables, the theory based on Maxwell equations alone is no longer deterministic: the particle trajectory can obviously be arbitrarily modified in the future or in the past without changing the initial data.

We would like to warn the reader against the following interpretation of this situation: “The theory is incomplete since equations of motion of the particle are still missing. Maxwell equations enable us to calculate time derivatives of the fields, if we know the fields themselves. To calculate also the time derivative of the particle velocity $\dot{v}$, additional dynamical equations are necessary.”
Such an interpretation is incorrect. *Maxwell equations alone enable one to calculate the acceleration of the particles* in terms of the initial data for the field. Many authors, who were trying to define the “renormalized Lorentz force”, did not realize this fact.

As we shall see, the information about the acceleration is contained in the initial data for the electric field. More precisely, it is given by the asymptotic expansion of the electric field in the neighbourhood of the particles. Such an expansion is, however, meaningless for a general distribution-like solution of (1). In this paper we limit ourselves to the class of solutions for which such an expansion is well defined. This class contains the solutions which differ from the retarded (or advanced) solution for point-like particles by a smooth solution of homogeneous Maxwell equations. We will call these *regular solutions*. Of course, we may always add to such a regular solution a singular (e.g. discontinuous) solution of the homogeneous problem. Such solutions are analogous to non-continuous or delta-like solutions of the elasticity theory or the string theory and have rather limited physical meaning.

Although the relation between field initial data and the particle acceleration is given by a fully relativistic formula, its rest-frame version is especially simple and elegant. For pedagogical reasons, therefore, we shall work in the rest-frame of the particle.

For each point \((t, q(t)) \in \zeta\) we consider the 3-dimensional hyperplane \(\Sigma_t\) orthogonal to \(\zeta\) at the point \((t, q(t)) \in \zeta\), i.e. orthogonal to the four-velocity vector \(U(t) = (u^\mu(t)):\)

\[
(u^\mu) = (u^0, u^k) := \frac{1}{\sqrt{1 - v^2}}(1, v^k).
\] (3)

We shall call \(\Sigma_t\) the “rest frame hyperplane”. Choose on \(\Sigma_t\) any system \((x^i)\) of cartesian coordinates centered at the particle’s position and denote by \(r\) the corresponding radial coordinate. The initial data for the field on \(\Sigma_t\) are given by the electric induction field \(D = (D^i)\) and the magnetic induction field \(B = (B^i)\) fulfilling the conditions \(\text{div} B = 0\) and \(\text{div} D = e \delta^{(3)}_0\). For regular solutions, the radial component \(D^r\) of \(D\) on \(\Sigma_t\) can be expanded in the neighbourhood of the particle into powers of \(r\):

\[
D^r(r) = \frac{1}{4\pi} \left( \frac{e}{r^2} + \frac{\alpha}{r} \right) + \beta + O(r),
\] (4)

where by \(O(r)\) we denote terms vanishing for \(r \to 0\) like \(r\) or faster. For a given value of \(r\) both sides of (4) are functions of the angles (only the \(r^{-2}\) term is angle–independent).

We will show in the sequel that, due to the Maxwell equations (1), the \(r^{-1}\) term is fully determined by the acceleration of the particle:

\[
\alpha = -e a_i \frac{x^i}{\tau}.
\] (5)

Here by \(a = (a^i)\) we denote the acceleration vector. Being orthogonal to the velocity, the acceleration \(\frac{d}{d\tau} u^\mu\) (where \(\tau\) is the proper time along the world line) can be identified with a 3–dimensional vector in the plane \(\Sigma_t\). Such vectors are in one-to-one correspondence with
“dipole-like” functions of angles, i.e. functions of the form \((a_i x^i)/r\). Formula (5) means, in particular, that the quadrupole and higher harmonics of the function \(\alpha\) do vanish.

We stress that formula (5) is not a new element of the theory. Although the author has never seen it written in this explicit form, it is implicitly contained in the formula for Lienard-Wiechert potentials, (e.g. formula (14.14) in Jackson’s book [9] or formula (6-62) in Rohrlich’s book [2]). Choosing the rest frame \((v = 0)\) and calculating the limit of the retarded (or advanced) field in the neighbourhood of the particle one easily proves (5) for the retarded (advanced) solution. Any regular solution of (1) differs from the retarded (advanced) solution by a smooth solution of the free Maxwell equation i.e. the one which does not change the singular part of the expansion (4). Hence, formula (5) is valid universally (in Section 5 we give an independent proof, which does not make use of any specific – retarded or advanced – solutions).

Similar expansion of the magnetic field \(B_r(r)\) does not contain any singular part because the particle is purely scalar. It may be easily shown that the method used here cannot be extended to the particles containing e.g. magnetic moments.

We propose in this paper a new framework, based on formula (5), for electrodynamics of point particles. In this framework equations (1) have to be solved as a boundary problem rather than a distribution equation on the entire space-time \(M\). More precisely, we consider the evolution of the field in the region \(M_\zeta := M - \{\zeta\}\), that means outside of the trajectory. Equation (5) provides the boundary condition on the boundary \(\partial M_\zeta\) of \(M_\zeta\).

There are various ways to set up the boundary value problem in electrodynamics. The best known are Dirichlet and Neumann problems for electrodynamical potentials. Here we found it most useful to assume boundary conditions corresponding to the given double layers of electric and magnetic dipoles. In this way the values \(D_\perp\) and \(B_\perp\) (components of both the electric and the magnetic fields, normal to the boundary) are controlled. It was shown in [10] that under this choice of boundary conditions on \(\partial V\), the hamiltonian generating the evolution of the field in the region \(V\) coincides with the total energy of the field.

The particle trajectory may be considered as a limit of tiny world-tubes. It may be shown that the natural limit of the boundary conditions for \(D_\perp\) and \(B_\perp\) consists in fixing their singular parts at zero. Thus, for an \textit{a priori} given particle trajectory, equation (5) provides the condition for \(D_\perp\) on its boundary. The corresponding condition for the magnetic field follows from the assumption about the vanishing magnetic moment of the particle: \(B_r(r) = O(1)\) (i.e. the singular part of \(B_r\) vanishes). The reader may easily check that, for a given trajectory, the homogeneous Maxwell equations outside of the particles together with the above boundary conditions are equivalent to the non-homogeneous system (1). It will be proved in the next paper that this evolution constitutes an infinite dimensional Hamiltonian system.

Now, let us consider an extended system, where also the particle trajectory is treated as a dynamical object. Here a remarkable phenomenon occurs. Despite the fact that the
time derivatives \((\dot{D}, \dot{B}, \dot{q}, \dot{v})\) of the Cauchy data \((D, B, q, v)\) are uniquely determined by the data themselves, the evolution is not uniquely determined. Indeed, \(\dot{D}\) and \(\dot{B}\) are given by the Maxwell equations, \(\dot{q} = v\) and \(\dot{v}\) is given by equation (5). Nevertheless, the initial value problem is not well posed: keeping the same initial data, particle trajectories can be modified almost at will. This is due to the fact that, after upgrading equation (5) to the level of dynamical equations, we can no longer use it as a boundary condition. Hence, Maxwell theory alone provides sufficiently many dynamical equations of the theory, but a new boundary condition is necessary. The new element of the theory will be precisely this missing boundary condition. It is provided by the conservation law for the renormalized total four-momentum, defined in the next Section.

The situation is similar to that in classical theory of a finite elastic body \(V\) (e.g. an interval of an elastic string). The theory contains sufficiently many dynamical equations, but the evolution is not uniquely determined unless we restrict ourselves to the class of solutions fulfilling the appropriate boundary conditions on the boundary \(\partial V\).

3 Renormalized total four-momentum

Here we define the total four-momentum \(p_\lambda\) of the system composed of the point-like particle and the surrounding Maxwell field.

The field carries the Maxwell energy-momentum tensor-density (the notation is prepared for working in curvilinear coordinates):

\[
T^{\mu\nu} = \sqrt{-g} \left( f^{\mu\lambda} f_{\nu\lambda} - \frac{1}{4} \delta^{\mu}_{\nu} f^{\kappa\lambda} f_{\kappa\lambda} \right). \tag{6}
\]

The tensor has an \(r^{-4}\) singularity on the particle trajectory \(\zeta\), corresponding to the \(r^{-2}\) singularity of the field \(f_{\mu\nu}\).

For a given point \((t, q(t))\) \(\in \zeta\) let us take the solution of Maxwell equations corresponding to the particle uniformly moving along the straight line tangent to \(\zeta\) at \((t, q(t))\) (the solution can be obtained from the static Coulomb field via a Lorentz transformation). By \(T^{\mu}_{(t)\nu}\) we denote the energy-momentum tensor of this field. Both energy-momentum tensors have the same \(r^{-4}\) singularity at \((t, q(t))\). Therefore, their difference has at most an \(r^{-3}\) singularity at this point.

We prove in Section 5 that the following quantity is well defined:

\[
p_\lambda(t) := P \int_{\Sigma} \left( T^{\mu}_{\lambda} - T^{\mu}_{(t)\lambda} \right) d\sigma_{\mu} + mu_\lambda(t), \tag{7}
\]

where \(\Sigma\) is any space-like hypersurface passing through the point \((t, q(t))\), fulfilling standard asymptotic conditions at infinity (see e.g. [11]), \(d\sigma_{\mu}\) is the 3-dimensional volume element on \(\Sigma\) and “\(P\)” denotes the principal value of the singular integral, defined by removing from \(\Sigma\) a sphere \(K(0, r)\) around the particle and then passing to the limit \(r \to 0\). Moreover, we prove that the above quantity does not change its value if we change the
surface $\Sigma$ in such a way, that the intersection point with the trajectory remains the same. Hence, the total four-momentum defined by formula (7) depends only on the point of the trajectory.

It is relatively easy to prove the invariance of (7) upon a choice of $\Sigma$ within a class of hypersurfaces tangent to each other at the intersection with the trajectory. Indeed, due to Maxwell equations both tensors are conserved outside of the particle:

$$\nabla_\mu T^\mu_\lambda = 0 = \nabla_\mu T^{(t)}_\lambda .$$

The difference between the values of the integral corresponding to two different $\Sigma$'s is therefore equal to a boundary (surface) integral. The boundary is composed of two parts: infinity and a piece $V$ of a tiny world tube of radius $r$ around the particle. The integral at infinity vanishes if the field fulfills standard asymptotic conditions at infinity (see again [11]). The 3-dimensional volume of $V$ behaves at least like $r^4$ (2-dimensional volume of the spheres behaves like $r^2$ and the 3-rd dimension of $V$, equal to the time interval between different $\Sigma$'s, behaves also like $r^2$ since they are tangent to each other at $r = 0$). Hence, the integral over $V$ of a function behaving like $r^{-3}$ vanishes in the limit $r \to 0$.

In Section 5 we give a more detailed analysis of the behaviour of the functions $T^\mu_\lambda - T^{(t)}_\lambda$ in the vicinity of $\zeta$, which enables us to extend immediately the above proof to the case of general $\Sigma$'s, not necessarily tangent to each other.

For a generic trajectory $\zeta$ and a generic solution of Maxwell equations (1) the quantity (7) is not conserved, i.e. it depends upon $t$. Guided by the extended particle model (see Appendix), we impose the conservation law for this object as an additional equation of our theory. We shall see that only three among these four equations are independent. As independent conditions we may use e.g. the three-momentum conservation law:

$$\frac{d}{dt} p_j(t) = 0 .$$

We prove in Section 7, that it is equivalent to the fundamental equation of our theory and implies also the energy conservation

$$\frac{d}{dt} p^0(t) = 0 .$$

The above equations, together with the Maxwell equations for the electromagnetic field, define a mathematically consistent, fully deterministic theory. Starting from an appropriate variational principle, we will prove in the next paper that $p_k(t)$ is really a momentum canonically conjugate to the particle position $q^k(t)$ and $p^0(t) = -p_0(t)$ (we take the signature $(-, +, +, +)$ of the space-time metric) is the Hamiltonian of the entire system composed of particles and fields.

Formula (7) is not an *ad hoc* invention. It is derived from an extended particle model by a suitable approximation procedure, which we discuss thoroughly in the Appendix. But the heuristic idea of this derivation is very simple. Observe that, accordingly to (7), the
four-momentum of a uniformly moving particle is equal to \( m u_\lambda \), since we have in this case \( T^\mu_\lambda = T(t)^\mu_\lambda \). This gives the following interpretation of the entire formula: Our point-like particle has to be considered as a mathematical idealization of an extended particle. The particle is always surrounded ("dressed") by the electromagnetic field. For the particle at rest, this field reduces to the static Coulomb field. The number \( m \) represents the total rest-frame energy carried by all the constituents of this physical situation (including also the energy of the Coulomb field, which is finite for an extended particle).

To calculate the total four-momentum in the general situation (e.g. for an accelerated particle) we integrate the total energy-momentum tensor describing all the fields inside and outside of the particle, over a space-like hypersurface \( \Sigma \). We split this integral into the sum of integrals over the exterior and the interior of the particle. Outside of the particle the energy-momentum tensor reduces to the purely electromagnetic tensor (6). We subtract from it the corresponding energy-momentum tensor of the uniformly moving particle and obtain this way the first term of the right-hand side of (7). The remaining internal integral together with the external integral of the energy-momentum tensor corresponding to the uniformly moving particle combine in a good approximation to the value \( m u_\lambda \). The applicability of this approximation is discussed in the Appendix.

Due to the freedom in the choice of \( \Sigma \) in (7), we may always use the rest-frame hyperplane \( \Sigma_t \). Moreover, we may first calculate the rest-frame components of the total four momentum and obtain the laboratory-frame components via a Lorentz transformation. The transformation is given uniquely if we know how the laboratory-frame coordinates \( y^\lambda \) depend upon the rest-frame coordinates \( x^k \) on \( \Sigma_t \), i.e. if we know functions \( y^\lambda = y^\lambda(t,x^k) \). The uniformly moving particle becomes in co-moving frame the particle at rest and its external field reduces to the static Coulomb field

\[
D_0 = \frac{e r}{4\pi r^3} .
\] (11)

It is useful to decompose the rest-frame electric induction field into the sum \( D = D_0 + \mathcal{D} \). The radial component \( \mathcal{D}^r \) of \( \mathcal{D} \) coincides on each sphere \( S(r) := \{ r = \text{const} \} \) with the dipole and higher multipole part of \( \mathcal{D}^r \). We will refer to this part as to the "monopole-free" part of \( \mathcal{D} \), the monopole part being described by \( D_0 \). The monopole free part carries the entire dynamical information about \( \mathcal{D} \) since \( D_0 \) is always the same.

Formula (7) gives the following expressions for the rest-frame components of the total four-momentum:

\[
\mathcal{H}(t) = \mathcal{P}^0(t) := -p_\lambda(t)u^\lambda(t) = m + H(t)
\] (12)

where

\[
H(t) := \frac{1}{2} \int_{\Sigma_t} (\mathcal{D}^2 + B^2) d^3x ,
\] (13)

and

\[
\mathcal{P}_j(t) := p_\lambda(t) \frac{\partial y^\lambda}{\partial x^j}(t,0) = \int_{\Sigma_t} (D^n B^i \epsilon_{nij}) d^3x .
\] (14)
We stress that formula (13) contains *a priori* an $r^{-3}$ term equal to the scalar product $(\mathbf{D}_0 \cdot \mathbf{D})$. It vanishes when integrated over each sphere $S(r)$ since it is a monopole-free function and therefore may be neglected under the principal value integration. With this term removed, the integrals become regular and this is why we may forget about the sign “P” in the above formulae.

4 Fundamental equation. Correspondence principle

Here we present the fundamental equation of our theory, which will be derived in Section 6 from the conservation law (9). It is formulated as a boundary condition for the behaviour of the field in the vicinity of the particle. Again, for the sake of simplicity, we write it down in the particle co-moving frame, i.e. in the form of a relation between different terms of expansion (4). The relation reads:

$$DP(m\alpha + e^2\beta) = 0,$$

(15)

where by DP($f$) we denote the dipole part of the function $f$ on the sphere $S^2$. This equation (together with the vanishing of the singular part of the magnetic field $\mathbf{B}$) provides the missing boundary condition for the behaviour of the field $\mathbf{D}^r$ on the boundary $\partial \mathcal{M}_\zeta$ of the region $\mathcal{M}_\zeta$. Dynamical equations of the theory are, therefore, the Maxwell equations *alone*. They will determine uniquely the evolution of the system composed of particles and fields if we restrict ourselves to solutions fulfilling the boundary condition (15). In this way the system becomes an infinite-dimensional dynamical system. We will show in the next paper that this is even a Hamiltonian system with an interesting canonical structure.

Although the Lorentz force acting on particles is always ill defined, it is contained in our theory as a limiting case, when particles are very light and carry a very small charge. Indeed, passing to the limit with $m \to 0$ and $e \to 0$ (but keeping their ratio constant), we obtain the standard theory of test particles, where the field is not influenced by the particles and the particles are influenced by fields according to the Lorentz force. To prove this feature of our theory we use (5) to rewrite equation (15) in the following, equivalent form:

$$DP(\beta) = \frac{m}{e} \frac{x^i}{r}.$$

(16)

For $e = 0$ equation (1) reduces to the free Maxwell equation and the first two terms of (4) vanish. The field is, therefore, regular at $r = 0$. For regular fields the radial component of $\mathbf{D}$ contains only the dipole part: $DP(\beta) = (D_i(0)x^i)/r = (E_i(0)x^i)/r$ (higher harmonics behave like $O(r)$ and the electric field $\mathbf{E}$ coincides with the electric induction field). Therefore, (16) is equivalent to the equation defining the Lorentz force in the rest-frame:

$$E^i(0) = \frac{m}{e} a^i.$$

(17)
Hence we have proved that our theory fulfills the correspondence principle with the Lorentz theory of test particles. We stress, however, that the interpretation of (16) as the Lorentz force is not possible for a particle with finite charge and mass. Due to the first two (singular) terms in (4), any attempt to define the Lorentz force in this case leads to infinite expressions. Considering the particle as a limit of a small, spherical droplet of matter, the \( r^{-2} \) term can be eliminated (the forces, although big, cancel out because of the spherical symmetry). However, such a “renormalization” fails for the \( r^{-1} \) term which will always be infinite. We see that the Lorentz force cannot be meaningfully defined for point particles not only because of the “proper field”, which increases like \( r^{-2} \), but also because of the infinite contribution of the remaining “radiation field”. Moreover, also the non-singular term of \( \mathbf{D} \) cannot be used to define the Lorentz force in a reasonable way, because its value at \( r = 0 \) can not be uniquely defined (different values of its limit at \( r = 0 \) may be obtained if we approach the origin from different directions – see formulae (50), (51) and the discussion which follows).

We conclude that the Lorentz force is a well defined concept for test particles only and has no meaning for a finite particle. Nevertheless, the dynamics of the system composed of particles and fields, given by Maxwell equations and the boundary condition (15), is well defined and fully deterministic: initial data for particles and fields uniquely determine the entire history of the system.

Thus, we have shown that the evolution of interacting particles and fields is a problem with moving boundary for the Maxwell equations. The particle trajectory plays the role of the boundary. Such problems are often considered in the theory of continuous media (see [12]) and this analogy is a good guiding principle in the analysis of our problem.

5 Behaviour of the Maxwell field in the vicinity of particles

In this Section we prove formula (5) and the consistency of the definition (7). For this purpose we find it most useful to use an accelerated reference frame, always co-moving with the particle. The section has therefore a purely technical character: the same results can be obtained in any reference frame, but the corresponding formulae will be more complicated.

We consider space-time as a sum of disjoint rest frame surfaces \( \Sigma_t \), each of them parameterized by a system \((x^i)\) of cartesian coordinates, centered at the particle position. Choosing e.g. the proper time along the trajectory as the time coordinate \((x^0 = \tau)\) we obtain formally a system of curvilinear coordinates \((x^\alpha) = (x^0, x^i)\) in space-time.

Unfortunately, such a system is regular only in a neighbourhood of \( \zeta \) and not globally, because different \( \Sigma \)'s may intersect. This is already sufficient for the description of local properties of the field in a neighbourhood of \( \zeta \). But we stress that for some purposes coordinates \( x^\alpha \) can also be used globally. Indeed, we can rewrite Maxwell equations as
evolution equations from $\Sigma_\tau$ to $\Sigma_{\tau+d\tau}$ or, more precisely, from one $\Sigma_\tau$ to another. This is a well posed problem and the Maxwell equations rewritten this way contain the entire information about the evolution of the field.

The general, curvilinear (3+1)-decomposition of Maxwell equations reads

\begin{align}
\partial^k D_k &= e^{(3)}_0 \\
\partial^k B_k &= 0 \\
\dot{D}^k &= \partial_l \left( N^l D^k - N^k D^l - N \varepsilon^{lkm} B_m \right) \\
\dot{B}^k &= \partial_l \left( N^l B^k - N^k B^l + N \varepsilon^{lkm} D_m \right),
\end{align}

(18)

where the indices are raised and lowered with the 3-dimensional metric $g^{kl}$ on the space $x^0 = \text{const.}$, $D^k = \sqrt{\text{det} g_{ij}} D^k$, $B^k = \sqrt{\text{det} g_{ij}} B^k$, $N$ is the lapse function and $N^k$ is the shift vector. In our case the 3-dimensional metric is trivial ($g_{kl} = \delta_{kl}$). Hence, \(\sqrt{\text{det} g_{ij}} = 1\). The lapse function is given by the acceleration:

\[ N = 1 + a_i x^i. \]

(19)

The simplest way to fix the remaining $O(3)$-freedom in the choice of the parameterization ($x^i$) consists in Fermi-propagating it from a fixed instant of time $\tau_0$. This implies vanishing of the shift vector ($N^k = 0$). Hence, we have the following system of equations for the dynamical fields $B$ and $\overline{D} = D - D_0$:

\begin{align}
\partial^k D^k &= \partial^k B^k = 0, \\
\check{D}^k &= \dot{D}^k = -\partial_l \left( (1 + a_i x^i) \varepsilon^{lk} m B^m \right), \\
\check{B}^k &= \partial_l \left( (1 + a_i x^i) \varepsilon^{lk} m D^m \right) = \partial_l \left( (1 + a_i x^i) \varepsilon^{lk} m \overline{D}^m \right) + \frac{e}{4\pi r^3} a_m \varepsilon^{lk} m x^m.
\end{align}

(20, 21, 22)

Due to equation (21), the $r^{-1}$ term of $B$ would have produced an $r^{-2}$ term in $\overline{D}$, which has to vanish (otherwise our boundary conditions for the electric field would have not been satisfied). Hence, the singular part of $B$ vanishes i.e. $B^k$ is bounded in the neighbourhood of the particle. This means that the last, inhomogeneous, dipole-like $r^{-2}$-term in (22) has to be compensated by a dipole-like $r^{-1}$-term in $\overline{D}$.

The field $\overline{D}$ can be treated as a sum of the radial field and the field tangent to the spheres. The latter can be further decomposed into the 2-dimensional transversal part (i.e. the 2-dimensional divergence-free part) and the 2-dimensional longitudinal (i.e. the 2-dimensional curl-free part). It is easy to see, that the 2-dimensional transversal part does not give any contribution when inserted into (22). On the other hand, any field
which is 3-dimensional divergence-free and 2-dimensional longitudinal on each \( S(r) \), and which behaves like \( r^{-1} \), has the form

\[
F^k = \frac{1}{8\pi r} \left( \alpha_i \frac{x^i x^k}{r^2} + \alpha^k \right) .
\]  

(23)

Treating \( F^k \) as the singular part of \( \mathbf{D} \) and inserting it into (22), we see that the \( r^{-2} \) terms cancel each other if and only if

\[
\alpha_i = -ea_i .
\]

(24)

Hence, formula (5) has been proved in a new way, without any use of specific (retarded or advanced) potentials.

The formulae (21) and (22) imply that the above field is the only singular part of \( \mathbf{D} \). Hence,

\[
D^k = \frac{e}{4\pi} \left[ \frac{x^k}{r^3} - \frac{1}{2r} \left( \alpha_i \frac{x^i x^k}{r^2} + \alpha^k \right) \right] + \tilde{D}^k
\]

(25)

where \( \tilde{D}^k \) denotes the nonsingular part (i.e. \( \tilde{D}^k \) is bounded in the neighbourhood of the particle).

To prove the mathematical consistency of formula (7) it is sufficient to analyze the behaviour of the fields locally in a neighbourhood of \( \zeta \), where the co-moving system \( (x^\mu) \) is a regular coordinate system.

Let us define the “Coulomb” field \( f_{\mu\nu} \) as the one for which the magnetic field vanishes and the electric field equals \( \mathbf{D}_0 \). We warn the reader against the interpretation of \( f \) as the particle’s “proper field” and the remaining filed \( \mathbf{f}_{\mu\nu} := f_{\mu\nu} - f_{\mu\nu} \) as the “radiation field”. Such a splitting is globally meaningless because different \( \Sigma \)’s do intersect. Moreover, it depends upon the choice of a coordinate system. We stress that there is no possibility of defining the particle’s “proper field” in a consistent way. The decomposition \( f = f + \mathbf{f} \) has purely technical character: the field \( \mathbf{f}_{\mu\nu} \) has no \( r^{-2} \)-component. Its singular part is given by (23) and behaves like \( r^{-1} \).

Suppose now that we choose the rest-frame hyperplane \( \Sigma_r \) in formula (7). Both the energy-momentum tensors \( T \) and \( T_{(t)} \) are bilinear expressions of fields. Hence, the terms bilinear with respect to \( f_{\mu\nu} \) cancel. The terms bilinear in \( \mathbf{f}_{\mu\nu} \) behave like \( r^{-2} \) and produce no difficulties when integrated over a 3-dimensional \( \Sigma \). The only dangerous terms are therefore mixed terms, where \( \mathbf{D}_0 \) encounters the singular part of \( \mathbf{f}_{\mu\nu} \). In the co-moving system the following components of the energy-momentum tensor display the \( r^{-3} \) behaviour:

\[
T^{00} - T_{(t)}^{00} = (\mathbf{D}_0 \cdot \mathbf{D}) + \text{terms behaving like } r^{-2}
\]

and

\[
T^{kl} - T_{(t)}^{kl} = -\mathbf{D}_0^k \mathbf{D}^l - \mathbf{D}^k \mathbf{D}_0^l + g^{kl}(\mathbf{D}_0 \cdot \mathbf{D}) + \text{terms behaving like } r^{-2} .
\]

(26)

(27)
Inserting (11) and (23) into the above formulae we see, that the \( r^{-3} \) terms contain always the odd powers of the coordinates \( x^k \) and consequently vanish when integrated over a sphere \( S(r) \). Therefore, they produce no difficulty under the sign “P” of principal value integration over the rest-frame hyperplane \( \Sigma_r \).

This argument can not be directly used if we integrate over a general \( \Sigma \). To analyze the behaviour of \( T - T(t) \) on such a hypersurface we have to take into account the fact that different points of \( \Sigma \) belong to different rest-frame hyperplanes, i.e. have different values of \( \alpha \) in formula (23). We may expand the dependence of \( \alpha \) upon the points on \( \Sigma \) into a Taylor series. The leading term is the constant \( \alpha \), corresponding to the intersection of \( \Sigma \) with the trajectory. The next terms are at least of first order in \( r \). We conclude that also on a general \( \Sigma \) the \( r^{-3} \) terms are odd and vanish under the principal value integration sign. Using this argument one easily extends the proof of independence of \( p_\lambda \) upon the choice of \( \Sigma \) to the general case.

6 Proof of the fundamental equation

The very reason for using the co-moving coordinates is that this way we transform the problem with moving boundary into the problem with fixed boundary: the particle remains always at the center \( x^k = 0 \). But the price we pay for it is the effective nonlinearity of the system if the trajectory is also treated as a dynamical variable. Indeed, the dynamics is described by equations (20), (21), (22) and (5), equivalent to Maxwell equations. But the acceleration \( a \) which enters into field equations (21) and (22) is no longer given \textit{a priori} but has to be calculated from the field \( \mathbf{D} \), according to equation (5).

This is, however, the simplest description of the system with moving boundary and consequently we will use it in the sequel. In terms of our Fermi-propagated coordinates \((x^\alpha)\) the flat Minkowskian metric reads: \( g_{kl} = \delta_{kl}, g_{k0} = N_k = 0, g_{00} = -N^2 = -(1 + a_i x^i)^2 \). Calculate the corresponding connection. Its only components which do not vanish on the trajectory \( \zeta \) are: \( \Gamma^0_{0k} = a_k \) and \( \Gamma^k_{00} = a^k \). The four-momentum conservation law may be written in our reference system as follows:

\[
\nabla_0 \mathcal{P}^\alpha = \partial_0 \mathcal{P}^\alpha + \Gamma^\alpha_{0\beta} \mathcal{P}^\beta = 0 \tag{28}
\]

where \((\mathcal{P}^\alpha) = (\mathcal{H}, \mathcal{P}^k)\). Hence, the system (9) and (10) is equivalent to the following system of equations for the rest-frame components of the four-momentum:

\[
\dot{\mathcal{H}}(t) = -a_k \mathcal{P}^k \tag{29}
\]

and

\[
\dot{\mathcal{P}}^k(t) = -a^k \mathcal{H} \tag{30}
\]

To calculate the time derivatives of the above quantities we first do it for the integrals \( H(r_o) \) and \( \mathcal{P}(r_o) \) extended over the region \( \{ r > r_o \} \):

\[
\dot{H}(r_o) = \int_{\{ r > r_o \}} (\mathbf{D} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{B}) d^3x = \int_{\{ r > r_o \}} (\mathbf{D} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{B}) d^3x \tag{31}
\]
Here, we used the fact that the time derivative of $D_0$ vanishes and that the scalar product of $D_0$ with $\bar{D}$ vanishes when integrated over any sphere $S(r)$ (the field $D_0$ is angle-independent, whereas $\bar{D}$ contains the dipole and higher harmonics only).

Using equations (21) and (22) we obtain

$$\dot{H}(r_o) = - \int_{\{r>r_o\}} \partial_m (\epsilon_{nk} (1 + a_i x^i) D^n B^k + \epsilon_{nk} D^n B^k a^m) \, d^3 x .$$

(32)

Calculating the limit for $r_o \to 0$ we obtain:

$$\dot{H} = - a^m P_m + \lim_{r_o \to 0} \int_{\{r=r_o\}} \frac{x_m}{r} \left[ \epsilon_{nk} (1 + a_i x^i) D^n B^k \right] d\sigma ,$$

(33)

where $d\sigma$ denotes the surface measure on the sphere $S(r_o)$. The contribution from $D_0$ vanishes since it is parallel to $x^m$. The remaining field $\bar{D}$ behaves like $r^{-1}$ and the volume element $d\sigma$ behaves like $r^2$. Therefore, the surface integral vanishes in the limit, which proves that formula (29) (the energy conservation in the particle rest frame) is always satisfied. This implies that there are only 3 independent equations to satisfy among the 4 conservation laws (9) and (10).

Similar calculations for the momentum give us:

$$\dot{P}_j(r_o) = \int_{\{r>r_o\}} \epsilon_{ni} (\bar{D}^n B^i + D^n \bar{B}^i) \, d^3 x = - \frac{1}{2} a_j \int_{\{r>r_o\}} (D_i D^i + B_i B^i) \, d^3 x$$

$$+ \int_{\{r>r_o\}} \partial_m \left[ \left( D^m D_j + B^m B_j - \frac{1}{2} \delta_j^m (D_i D^i + B_i B^i) \right) (1 + a_k x^k) \right] d^3 x .$$

(34)

and

$$\dot{P}_j = - a_j H - \lim_{r_o \to 0} \left\{ \frac{1}{2} a_j \int_{\{r>r_o\}} |D_0|^2 \, d^3 x \right\}$$

$$+ \int_{\{r=r_o\}} \frac{x_m}{r} \left( D^m D_j + B^m B_j - \frac{1}{2} \delta_j^m (D_i D^i + B_i B^i) \right) (1 + a_k x^k) \, d\sigma .$$

(35)

The contribution of the non-singular field $B$ to the above surface integral vanishes in the limit $r_o \to 0$. Hence:

$$\dot{P}_j = - a_j H - \lim_{r_o \to 0} \left\{ \frac{e^2}{8 \pi r_o} a_j + \int_{\{r=r_o\}} \frac{1}{r} \left( x^m D_m D_j - \frac{1}{2} D_m D^m x_j \right) (1 + a_k x^k) \, d\sigma \right\} .$$

(36)

Inserting (25) into (36) one easily shows that the diverging terms under the limit sign do cancel. The remaining terms give us the following result:

$$\dot{P}_j = - a_j H - \frac{e}{4 \pi} \lim_{r_o \to 0} \int_{\{r=r_o\}} \frac{1}{r^2} \bar{D}_j \, d\sigma .$$

(37)
For a field $\tilde{D}$ continuous in a neighbourhood of the particle, the surface integral would have given in the limit the value $\tilde{D}_j(0)$ multiplied by $4\pi$. Therefore, the last term of the above formula would have produced the dipole part of $\beta$. However, this simple argument cannot be used in our case, because we are not allowed to assume the continuity of $\tilde{D}$ at $x^k = 0$ (we will see in Section 8 that for a non-vanishing acceleration $a$ the field $\tilde{D}$ can not be continuous). What we know is only the regularity outside the center and the behaviour of the radial component of $\tilde{D}$. Unfortunately, the integral contains not only the radial, but also the component tangent to the sphere, which we do not control completely.

Nevertheless, the integral can be computed. Let us observe for this purpose that $\tilde{D}_j$ is a scalar product of $\tilde{D}$ with the constant vector $\partial/\partial x^j$. Hence, it is a sum of two terms: 1) the product of the radial component of the constant field with the radial component $\tilde{D}$ and 2) the 2-dimensional scalar product of the tangent component of the constant field with the tangent component of $\tilde{D}$. Again, we split the latter into the 2-dimensional longitudinal (curl-free) part and the 2-dimensional transversal (divergence-free) part. It is easy to check that the transversal part does not give any contribution to the integral, because the tangent component of the constant field is longitudinal and, therefore, vanishes when multiplied by a transversal field and integrated over the sphere. But, due to equation $\partial_j \tilde{D}^j = 0$, the longitudinal part of the tangent component of $\tilde{D}$ is fully determined by its radial component. Moreover, only the dipole part of the latter enters into the game, because the constant field contains the dipole part only. Using these arguments we may finally express the surface integral in terms of the dipole part of $D^r$. This way we prove that Maxwell equations imply the following formula:

$$\dot{P}_j = -a_j H - e\beta_j = -a_j H + (ma_j - e\beta_j) ,$$

where $\beta_j$ is a component of the dipole part of the function $\beta$, i.e. the vector defined by the decomposition

$$DP(\beta) := \beta_j x^j \frac{x_j}{r} .$$

We see that, for the fields satisfying the Maxwell equations, the fundamental equation

$$ma_j - e\beta_j = 0$$

is equivalent to the rest-frame 3-momentum conservation (30).

7 Many particles. Interaction with heavy external devices

Although up to now we have considered only a single particle interacting with the Maxwell field, our theory is formulated in terms of local quantities and can be immediately extended to the case of many particles: the field outside the particles has to fulfill the
Maxwell equations with the total current being the sum of the contributions from all the trajectories, and the boundary condition (15) has to be satisfied on each trajectory separately.

The above law can be derived from a renormalized four-momentum formula analogous to (7), in which there is a term \( T^{\mu \lambda} \) and a term \( m u^\lambda \) for each particle separately. Any variation of \( \Sigma \) which does not change the intersection points with all the particle trajectories, does not change also the value of \( p^\lambda \). To prove this fact we use the same arguments as in the case of one particle. Hence, the total four-momentum defined this way depends upon the choice of the intersection points only: one for each trajectory.

Assuming the independence of \( p^\lambda \) upon the intersection points one finally gets the fundamental equation separately on each trajectory. Formally, we cannot use the same proof based on the co-moving description of the field dynamics because there is usually no common rest-frame hyperplane for many particles. But the problem is local: each particle interacts with the field in its neighbourhood and does not know anything about the existence of other particles. The variation of \( \Sigma \) may be done locally, in a neighbourhood of one trajectory only. The difference of \( p^\lambda \) between unperturbed \( \Sigma \) and the perturbed one is therefore a local quantity, which does not depend upon the field initial data outside of the perturbation region. Hence, our previous arguments remain valid.

The mathematical description of the many-particle problem as an infinite dimensional hamiltonian system is slightly more complicated. We stress, however, that the rest-frame description, although simplifying considerably the mathematical structure of the theory, is not necessary and the hamiltonian formulation of the theory in an arbitrary reference frame will be given in the next paper. In the present section we would like to consider the limiting case, when one of the two particles is very heavy and practically does not move: its acceleration \( a \) calculated from (5) is of the order of \( 1/m \) which, for sufficiently large \( m \), can be neglected. This means that we do not need to satisfy the boundary condition for the heavy particle (its “infinitesimally small” acceleration fits any value of \( DP(\beta) \) in formula (16) when multiplied by its “infinitely big” mass). The boundary condition has to be satisfied for the light particle only and our co-moving description of the field may be used also in this case.

The heavy particle may be replaced by any heavy (macroscopic) device containing charged bodies, magnets etc. (e.g. an accelerator). Let \( (\phi^{\text{ext}}, f^{\text{ext}}) \) be an exact solution of our “super theory” of interacting matter fields and electromagnetic field (see Appendix), which represents such a device and its own electromagnetic field. We call the field “external”, because it plays such a role for our (relatively light) particle interacting with the device.

Suppose now that there is another solution \( (\phi^{\text{total}}, f^{\text{total}}) \) of the “super theory” which represents such an interaction. We assume that the total strong field region of the new solution consists of the device’s own strong field region (i.e. the strong field region of the previous solution) and the separate region concentrated around the approximate trajectory \( \zeta \) of the light particle. It is important to assume that the trajectory does not
intersect the strong field region of the device.

Let us define the “radiation field” $f$ as a difference between the two solutions:

$$ f_{\mu\nu}^{total} = f_{\mu\nu}^{ext} + f_{\mu\nu}^{\mu}. $$

(41)

The radiation field satisfies the Maxwell equations outside of the matter, because both $f_{\mu\nu}^{total}$ and $f_{\mu\nu}^{ext}$ do. The boundary condition (15) for $f^{total}$ on the trajectory $\zeta$ can be rewritten in terms of the boundary condition for the radiation field:

$$ DP(m \alpha + e^2 \beta) = -e^2 D_i^{ext} x^i, $$

(42)

where $D_i^{ext}$ are the electric induction components of the field $f^{ext}$, corresponding to the actual position of the particle and calculated in the particle rest-frame. This is due to the fact that $f^{ext}$ is a smooth field in the neighbourhood of the particle trajectory. Therefore, $\alpha^{total} = \alpha$ and $\beta^{total} = \beta + (D_i^{ext} x^i)/r$, where $\alpha$ and $\beta$ denote the corresponding quantities for the radiation field $f$.

We see that the dynamical equations for the field $f$ are the same as in the one-particle problem. The influence of the external field is manifested in the non-homogeneous boundary condition (42), which now replaces the homogeneous condition (15).

The right hand side of (42) can be written explicitly in terms of the laboratory-frame components $E^i(t, q)$ and $B^i(t, q)$ of the external field if we know explicitly the Poincaré transformation relating the laboratory frame with the rest frame. Unfortunately, in the case of our Fermi-propagated rest frame which we used for the simplicity of the calculations, the above Poincaré transformation is a result of the propagation along the trajectory of a given Poincaré transformation at the time $\tau_0$ (i.e. from the instant of time when we have “gauged” our reference system). Hence, the transformation depends upon the entire history of the particle between $\tau_0$ and the current time $\tau$. To remove this difficulty we can use another system of co-moving coordinates, for which the Poincaré transformation between the laboratory frame and the rest frame is given uniquely at each point of the trajectory if we know only the velocity of the particle at this point. Such a system can be defined e.g. as follows: as the time variable $t$ we take the laboratory time on the trajectory instead of the proper time. Moreover, we define the axes $\partial/\partial x^k$ simply boosting the laboratory axes $\partial/\partial y^k$ by the same boost transformation which relates the laboratory time-axis $(\partial/\partial y^0)$ with the four-velocity vector $U$. This choice leads to the following transformation formula:

$$ y^0(t, x^l) := t + \frac{1}{\sqrt{1-v^2(t)}} x^l v_l(t), $$

$$ y^k(t, x^k) := q^k(t) + x^k + x^l v_l(t) u^k(t) \varphi(v^2), $$

(43)

where the following function of one real variable has been used:

$$ \varphi(\lambda) := \frac{1}{\lambda} \left( \frac{1}{1-\lambda} - 1 \right). $$

(44)
The function is well defined and regular (even analytic) for $\lambda < 1$ (in the neighbourhood of zero the function behaves like $\varphi(\lambda) = \frac{1}{2} + \frac{3}{4}\lambda + \frac{5}{16}\lambda^2 + o(\lambda^2)$). The reader may easily check that, with respect to our previous coordinate system, the lapse function has been changed by the multiplicative factor and equals now $N = \sqrt{1 - v^2} \left(1 + a_i x^i\right)$. Also the shift vector does not vanish and is purely rotational:

$$N^m = \epsilon^{mk} \omega_k x^l,$$

where

$$\omega_m = \frac{1}{\sqrt{1 - v^2}} \varphi(v^2) v^k v^l \epsilon_{klm}.$$  \hspace{1cm} (45)

It is relatively easy to derive from (43) the corresponding Lorentz transformation relating the rest-frame to the laboratory-frame components of the four-momentum:

$$p^0(t) = \frac{1}{\sqrt{1 - v^2}} \left(\mathcal{H}(t) + v^i \mathcal{P}_i(t)\right)$$

and

$$p_k(t) = \frac{v_k}{\sqrt{1 - v^2}} \mathcal{H}(t) + \left(\delta_k^l + \varphi(v^2)v^lv_k\right) \mathcal{P}_l(t).$$ \hspace{1cm} (48)

The above transformation may be applied to the description of the interaction of our particle with an external field. The reader may easily check that rewriting the right-hand side of (42) in terms of the laboratory-frame components $E_i(t, q), B_i(t, q)$ of the external field $f^{ext}$ and of the velocity $v$ we obtain the following final formula:

$$DP(m\alpha + e^l \beta) =$$

$$= -e^2 \frac{1}{\sqrt{1 - v^2}} \left(\delta_i^l - \varphi(v^2)v^iv_l\sqrt{1 - v^2}\right) E_i(t, q) + \epsilon_{ikm} B^m(t, q) v_k \right| \frac{x^l}{r}.$$ \hspace{1cm} (49)

It will be proved in the next paper that also in this case Maxwell equations for the field $f$ and the trajectory, together with the above non-homogeneous boundary condition, define an infinitely dimensional Hamiltonian system. This means that initial data $(D, B, q, v)$ for the radiation field and for the particle uniquely determine the entire history of the system if the external field is given.

8 Radiative friction as a result of “fine tuning” of the field

In the present section we will describe a particular class of exact solutions of our theory. Each solution of this class can be obtained by solving a specific ordinary differential equation for the trajectory, with the additional term containing third-order derivatives of the
particle position. There is a one-parameter class of such “radiative” terms and the Dirac’s term proposed in [2] corresponds to a specific choice of this parameter. Another possibility could be e.g. an “anti-Dirac” term or any convex combination of these two. Some of the solutions obtained this way realize Dirac’s idea of “radiative friction”. Our theory is symmetric with respect to time reversal and, therefore, there are also “self accelerating” solutions obtained this way. Also solutions without any friction or self-acceleration can be obtained via a similar method.

The physical importance of all these solutions is very limited, because each of them corresponds to a very specific choice of initial data for the field. To maintain specific, purely mechanical behaviour of the particle (e.g. friction or self-acceleration) the field on the entire Cauchy surface Σ has to be chosen in a way perfectly tuned to the prescribed shape of the particle trajectory. In most cases, such a “fine tuning” needs an infinite amount of “infra-red” field energy, since the field initial data do not fall off sufficiently fast at infinity. Generic initial data do not satisfy this “fine tuning” condition and, therefore, do not admit this method of solution.

There are, however, physical situations in which solutions obtained via Dirac’s method are physically meaningful. Such situations correspond to trajectories which are asymptotically free for $t \to -\infty$. If the external field $f^{ext}$ admits such a trajectory for a test particle, then there is a preferable choice of the initial data for the radiation field $f$ at $t \to -\infty$, namely $D = 0 = B$. As we shall see in the sequel, the exact solution of our theory with such initial data can be obtained by solving the Lorentz-Dirac equation for the trajectory and taking as the radiation field $f$ the retarded Lienard-Wiechert solution corresponding to such a trajectory. This method fails in case of closed trajectories (e.g. for the Kepler problem), when there is no natural decomposition of the electromagnetic field into “incoming” and “retarded” fields. Here, the choice of the time-arrow (e.g. friction or self acceleration) corresponds to an arbitrary choice of the fine-tuned, infra-red-divergent initial data. Meaningful physical situations correspond to a local, non-linear interaction of particles and fields and cannot be classified either as an example of the radiative friction or of the radiative acceleration of the particles. To obtain them we need to solve our system of genuine partial differential equations, which does not reduce to any system of ordinary differential equations.

The method we are going to describe is based on the observation that for the retarded field $D_{ret}$ the expansion (4) assumes the following form:

$$D_{ret} = \frac{e}{4\pi} \left\{ \frac{1}{r^2} - \frac{a_i x^i}{r^2} + \frac{3}{8} \left[ 3 \left( \frac{a_i x^i}{r} \right)^2 - a^2 \right] + \frac{2\dot{a}_i x^i}{3r\sqrt{1 - v^2}} \right\} + O(r), \tag{50}$$

(this formula can be easily obtained from formula 6-62 in Rohrlich’s book [2]). Here, the $\sqrt{1 - v^2}$ appears because, in contrast to Rohrlich’s notation, we use the coordinates (43) and, therefore, we denote by dot the derivative with respect to the laboratory time and not to the proper time). A similar formula for the advanced field differs only by the sign
of the last term representing the dipole part of $\beta$:

$$D_{adv}^r = \frac{e}{4\pi} \left\{ \frac{1}{r^2} - \frac{a_i x^i}{r^2} + \frac{3}{8} \left[ 3 \left( \frac{a_i x^i}{r} \right)^2 - a^2 \right] - \frac{2\dot{a}_i x^i}{3r \sqrt{1 - v^2}} \right\} + O(r). \quad (51)$$

The last two terms in both formulae give the total value of $\beta$. The first of the two represents the non-vanishing quadrupole part of $\beta$ (its existence proves that the value $\vec{D}(0)$ of the non-singular part can not be defined). Equation (49) implies, that the retarded field satisfies the equations of our theory if and only if the trajectory satisfies the following third-order ordinary differential equation:

$$ma_i = \frac{e}{\sqrt{1 - v^2}} \left\{ \left( \delta^i_l - \varphi(v^2)v^l v_i \sqrt{1 - v^2} \right) E_l(t, q) + \epsilon_{ikm} B^m(t, q) v^k + \frac{e}{6\pi} \dot{a}_i \right\} . \quad (52)$$

The above equation has been written with respect to the particle rest frame and in terms of the 3-dimensional objects $v$, $a$ and $\dot{a}$. It is obviously equivalent to the Lorentz-Dirac equation written in the 4-dimensional notation:

$$ma^\mu = eF_{\mu\nu}^{ext} v_\nu + \frac{e^2}{6\pi} (\dot{a}^\mu - a^\nu a_\nu v^\mu) \quad (53)$$

(with respect e.g. to the corresponding formula in Rohrlich’s book there is an additional factor $4\pi$ due to the fact that we use Heaviside-Lorentz system of units). Finding a trajectory, which satisfies this equation and calculating the corresponding retarded field we find an exact solution of our theory. Similarly, we may use advanced fields. Due to equation (51) the corresponding equation for the trajectory will differ from the above one by the sign of the last term.

A more general “Ansatz” is a combination of advanced and retarded solutions:

$$f := \frac{1 + \lambda}{2} f_{ret} + \frac{1 - \lambda}{2} f_{adv} , \quad (54)$$

where $\lambda$ is a real parameter. The field provides an exact solution of our theory if and only if the trajectory satisfies the “Lorentz-Dirac type” equation

$$ma^\mu = eF_{\mu\nu}^{ext} v_\nu + \frac{e^2}{6\pi} (\dot{a}^\mu - a^\nu a_\nu v^\mu) . \quad (55)$$

Dirac’s case $\lambda = 1$ corresponds to the retarded fields, the “anti-Dirac” case $\lambda = -1$ to advanced fields. The specific case $\lambda = 0$ corresponds to the pure Lorentz equation with external forces:

$$ma^\mu = eF_{\mu\nu}^{ext} v_\nu . \quad (56)$$

The above method is useless if the trajectory is bounded, because then the retarded and advanced fields behave like $1/r$ for $r \to \infty$ and the field energy, necessary to maintain such a behaviour of the particle, is infinite.
We stress that the Lorentz-Dirac equation describes only one possible behaviour of the particle interacting with the field, corresponding to a specific choice of the initial data. We conclude, that it does not describe a fundamental law of nature but only a special family of solutions of the complete theory described in the present paper.

We stress also, that there is even no universal choice of the sign of the parameter $\lambda$ in equation (55), which would always correspond to friction rather then to self-acceleration. The reader may easily check, that for the free particle (i.e. $f_{\text{ext}} = 0$) positive $\lambda$ lead to self acceleration, whereas for the Kepler problem they lead to radiative friction. Of course, the modulus of $\lambda$ is also arbitrary ($\lambda = 5$ is as legitimate as Dirac’s $\lambda = 1$).

But the main drawback of the Lorentz-Dirac theory consists in the contradiction between the fundamental role of the Lorentz force and the fact that there is no way to define it consistently. This paradox may be avoided only if one treats the interaction between particles and fields as a problem with moving boundary. Here, the notion of the Lorentz force is meaningful for test particles only.

9 Non-relativistic approximation

As we already mentioned in Section 5, field equations (21) and (22) together with (5) form a non-linear system of equations. If the fields are weak we may linearize the theory, leaving only the zero order and the first order terms in fields. Remembering that, due to (5), the acceleration $\mathbf{a}$ is the first order term in the fields, we obtain from (21) and (22) the following equations:

$$
\dot{D}^{n} = \epsilon^{nm}_{k} \frac{\partial}{\partial x^{m}} B^{k}
$$

and

$$
\dot{B}^{n} = - \epsilon^{nm}_{k} \left( \frac{\partial}{\partial x^{m}} D^{k} + \frac{e}{4\pi r^{3}} a_{m} x^{k} \right).
$$

We will look for a special solution which has the following form of the electric field:

$$
\overline{D}^{k} := \frac{1}{4\pi} \left\{ \phi_{i}(r,t) \frac{x^{i} x^{k}}{r^{3}} + f^{k}(r,t) - f_{i}(r,t) \frac{x^{i} x^{k}}{r^{2}} \right\}
$$

where $\phi_{i}$ and $f_{i}$ are vector-valued functions of $r$ and $t$. The reader may easily check that the constraint equation $\partial_{k} \overline{D}^{k} = 0$ implies the following relation between functions $f$ and $\phi$:

$$
f_{i} = \frac{1}{2r} (r \phi_{i})',
$$

where by ‘$r$’ we denote radial derivative $\partial/\partial r$. The corresponding Ansatz for the magnetic field reads

$$
B^{k} = \frac{1}{4\pi} \epsilon^{kn}_{m} \chi_{n} \frac{x^{m}}{r}.
$$
Equations (57) and (58) give
\[ \dot{\phi}_i = 2\chi_i \] (62)
and
\[ \dot{\chi}_i = -\frac{e}{r^2} a_i - \frac{\phi_i}{r^2} + \frac{1}{2r}(r\phi_i)'' \] (63)
whereas (5) gives
\[ \phi_i(0) = -ea_i . \] (64)
Combining these equations we finally have the following second order partial differential ("string-type") equation for the vector-valued function \( \phi_i \) of two independent variables \( (t,r) \):
\[ \ddot{\phi} = -2\frac{\phi - \phi(0)}{r^2} + \frac{1}{r}(r\phi)'' . \] (65)
The acceleration is given by the boundary value \( \phi(0) \) according to formula (64). The solution of (65) has to fulfill the non-relativistic approximation of the boundary condition (49), i.e. the equation
\[ \frac{m}{e}\phi_i(t,0) + \frac{e}{4\pi} \phi'_i(t,0) = -e \left\{ E_i(t,q) + E_i^{\text{hom}}(t,0) + \epsilon_{ikm} B^m(t,q)v^k \right\} , \] (66)
where by \( E_i^{\text{hom}} \) we denote any regular solution of the homogeneous part of equations (57) and (58). Equation (65) can be solved using an integral representation of the function \( \phi \) as a combination of Bessel functions.

**Appendix: Extended particle model**

Here we derive the “already renormalized” formula (7) for the total four-momentum of a system composed of a point-like particle and the Maxwell field surrounding the particle from an extended particle model.

Consider a general field theory describing the electromagnetic field interacting with a hypothetical multi-component matter field \( \phi = (\phi^K) \). We assume that the dynamical equations of this “super theory” may be derived from a gauge-invariant variational principle
\[ \frac{\delta L}{\delta A} = 0 \] (67)
\[ \frac{\delta L}{\delta \phi} = 0 \] (68)
where $A = (A_\mu)$ is the electromagnetic potential, i.e. $f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. As an example one may consider the complex (charged) scalar field or the classical spinorial Dirac field. For our purposes, however, no further assumptions about the geometrical character of the field $\phi$ are necessary. In particular, the space $\Phi$, where the matter field $\phi$ takes its values, may have a non-trivial topological structure.

Noether’s theorem together with the Belinfante-Rosenfeld theorem imply that, due to field equations, the symmetric total energy-momentum tensor density $T_{\mu\nu}$ is conserved:

$$\nabla_\mu T^\mu_{\nu} = 0 .$$

This enables us to assign to each Poincaré symmetry field $X$ on space-time $\mathcal{M}$ the corresponding generator

$$\mathcal{P}(X) := \int_\Sigma T^\mu_{\nu} \ X^\nu d\sigma_\mu ,$$

where $d\sigma_\mu$ is the 3-dimensional volume element on the hypersurface $\Sigma$. Under standard asymptotic conditions imposed at infinity on the fields and $\Sigma$ (see e.g. [11]), the value of the generator is conserved and does not depend upon the choice of the integration surface $\Sigma$. Choosing e.g. constant fields we obtain this way components of the total four-momentum.

We assume that for vanishing matter fields and sufficiently weak electromagnetic field the theory can be well approximated by the linear Maxwell theory. This means that the Lagrangian reduces in this regime to the standard expression:

$$L = L_{Maxwell} = -\frac{1}{4} \sqrt{-g} \ f^{\mu\nu} f_{\mu\nu}$$

and the energy-momentum tensor can be well approximated by the Maxwell energy-momentum tensor (6).

We assume also that our theory admits a stable, static, localized solution which we call the “particle at rest” solution. The soliton may be of topological nature, but for our purposes no assumption about its internal structure is necessary: the particle will be always viewed from outside. By stability we mean that the solution corresponds to the local minimum of the energy $p^0$, treated as a functional defined on the phase space of complete Cauchy data (the data are functions depending on the space coordinates $(y^k)$, $k = 1, 2, 3$). By “localizability” we mean that there is a radius $R > 0$ such that, outside of the sphere $K(0, R)$, the matter field is so weak that it may be practically neglected, and the electromagnetic field is so weak that its dynamics is practically linear. The solution is characterized by its total mass $m = p^0$ and its total electric charge $e$ (localized in the strong field region):

$$e := \int_{\{y^0 = \text{const}\}} (\partial_\mu F^{0\mu}) \ dy^1 dy^2 dy^3 = \int_{r < R} (\partial_k F^{0k}) \ dy^1 dy^2 dy^3 ,$$

25
where by $\mathcal{F}^{\mu\lambda}$ we denote the momentum canonically conjugate to the electromagnetic potential:

$$\mathcal{F}^{\mu\lambda} := \frac{\partial L}{\partial A_{\mu\lambda}} = 2 \frac{\partial L}{\partial f_{\lambda\mu}}$$

(73)

and

$$A_{\mu\lambda} := \partial_{[\lambda} A_{\mu]}.$$  

(74)

We stress that $m$ represents the ("already renormalized") total mass, including also the entire energy contained in the static Coulomb field surrounding the particle at rest. In this approach questions like "how big the bare mass of the particle is and which part of the mass is provided by the purely electromagnetic energy?" are meaningless. In the strong field region the energy density may be highly non-linear and there is probably no way to divide it consistently into such two components.

Lorentz transformations applied to the "particle at rest" solution give us other solutions, having the property that the "strong field region" of each of them is concentrated in a neighbourhood of a straight line. We call them "uniformly moving particle" solutions.

In the present paper we assume that there is also a bigger class of solutions of the "super theory", for which the corresponding strong field region is concentrated in a neighbourhood of a time–like world line (called "an approximate trajectory of the particle"), not necessarily being a straight line. Our goal is to show that such a trajectory cannot be arbitrary, but must fulfill certain dynamical conditions, which we will formulate.

A solution of the field equations of our "super theory" will be called a "moving particle" solution if there is a world line $\zeta$ such that

- for each point $(t, q(t)) \in \zeta$ the matter field on the rest frame hyperplane $\Sigma_t$ practically vanishes for $r > R$ (here $R$ is the same as for the "particle at rest" solution) and the electromagnetic field is sufficiently weak to be well approximated in this region by the linear Maxwell field,

- the curvature of $\zeta$ (acceleration) is small with respect to $1/R$,

- the dimension $R$ of the strong field region is small with respect to the characteristic length $l_0$ of the surrounding electromagnetic field, i.e. $R \ll l_0$ (the definition of $l_0$ will be discussed in the sequel),

- within the strong field region ($r < R$) the field configuration does not differ substantially from the non-perturbed soliton. More precisely, we assume that the amount of the total four-momentum contained in this region may be well approximated by the corresponding value for the "particle at rest" solution.
From the point of view of the phase space of Cauchy data on the surface $\Sigma_t$, a “moving particle” solution can be treated as a perturbation of the “particle at rest” solution and the evolution of the fields inside the particle is assumed to be adiabatic. The last assumption is, therefore, compatible with the postulate of stability of the soliton, which implies vanishing of the variation of energy.

We are going to show that for a “moving particle” solution it is possible to approximate the total four-momentum by formula (7) containing the Maxwell field outside of the particle and the particle parameters $m$, $u^\mu$ and $e$ only. The approximation will be satisfactory if the radius $R$ of the particle is very little with respect to the characteristic length $l_0$ of the field surrounding the particle (the definition of $l_0$ is given in the sequel).

It is sufficient to derive our approximation for the rest-frame components of the four-momentum separately. Consider therefore the rest-frame energy $\mathcal{H}$, given by the formula

$$-\mathcal{H}(t) = \mathcal{P}_0(t) := \mathcal{P}(U(t)) ,$$

and the rest-frame momentum

$$\mathcal{P}_i(t) := \mathcal{P} \left( \frac{\partial}{\partial x^i}(t) \right) .$$

Using the freedom in the choice of $\Sigma$ in (70), we may choose the rest frame hyperplane and express the above quantities in terms of the Cauchy data on $\Sigma_t$. Let us choose any radius $r_o$ which is big enough compared with the diameter $R$ of the strong field region, but very small with respect to the characteristic length $l_0$ of the field, i.e. $R \ll r_o \ll l_0$. The integral over the entire $\Sigma_t$ can be calculated as a sum of two integrals: 1) the integral over the interior $K(0,r_o)$ of the sphere $\{r = r_o\}$, and 2) the remaining external integral. In the external region the energy-momentum tensor of our theory coincides with the Maxwell expressions

$$T^{00} = \frac{1}{2}(D^2 + B^2)$$

and

$$T^0_j = D^b B^i \epsilon_{bij} .$$

Using the decomposition $D = D_0 + \bar{D}$, we obtain the following result for the external part of $\mathcal{H}(t) = \mathcal{P}^0(t)$:

$$\int_{\{r > r_o\}} T^{00} = \frac{1}{2} \int_{\{r > r_o\}} D_0^2 d^3x + \frac{1}{2} \int_{\{r > r_o\}} (D^2 + B^2) d^3x$$

(the mixed term, containing the scalar product $(D_0 \cdot \bar{D})$, vanishes when integrated over each sphere $S(r)$ since it is a monopole-free function).
The internal part of $H(t)$ can be well approximated by the corresponding value for the unperturbed soliton. Together with the first integral of (79) it gives the total rest mass $m$ of the particle (the contribution of higher multipoles possibly contained in the electrostatic field surrounding the particle at rest is of the order $o(R/r_o)$ and may be neglected because of the condition $R \ll r_o$).

To analyze the second integral of (79) we need to give precise meaning to the characteristic length $l_0$ of the electromagnetic field. By a characteristic length of a field $f$ we usually mean the quantity of the order of $|f|/|\nabla f|$. This definition cannot be directly applied to the electric field $D$, which behaves like $r^{-2}$ in the vicinity of the particle. Using, however, the formula
\[
\frac{1}{2} \int_{|r| > r_o} (D^2 + B^2) d^3x = \frac{1}{2} \int_{r_o}^\infty dr \int_{|r| = r_o} \left((rD)^2 + (rB)^2\right) d\Omega ,
\]
(80)
where by $d\Omega$ we denote the 2-dimensional surface element on the unit sphere $S(1)$, we can see by the argument that follows that only the characteristic lengths of the fields $rD$ and $rB$ (regular for $r \to 0$) is necessary. Consider, therefore, $l_0$ be such, that the relative change of $rD$ and $rB$ corresponding to the distance $l_0$ is of the order 1. In this situation the integral (80) practically does not depend upon the choice of $r_o$ within the range $R \ll r_o \ll l_0$: the relative error will be of the order of $O(r_o/l_0) \ll 1$. In particular, the integral (80) can be extended to the entire space $\Sigma_t$ and the relative error will be of the same order (by $D := D - D_0$ we always denote the “monopole-free” part of $D$. In the interior of the strong field region the monopole part $D_0$ of $D$ is regular for $r \to 0$ and formula (11) is no longer valid).

Hence, we found the following approximation for the total rest-frame energy:
\[
H(t) \approx m + \frac{1}{2} \int_{\Sigma_t} (D^2 + B^2) d^3x .
\]
(81)

Similarly, we calculate the approximate value of the rest-frame momentum $P_j$. The internal integral for the particle at rest vanishes and the external one can be extended to the entire $\Sigma_t$ due to the fact that the expression (78) diverges at most like $r^{-2}$. Hence,
\[
P_j(t) \approx \int_{\Sigma_t} (D^n B^i \epsilon_{nij}) d^3x .
\]
(82)

We see, that replacing the approximation sign "$\approx$" by the equality sign, formulae (81) and (82) reproduce formulae (12) and (14), i.e. formula (7) written in the particle rest-frame. The principal value sign $P$ can be removed, since we have removed from the left hand side of (81) the only term behaving like $r^{-3}$, namely the mixed term $(D_0 \cdot D)$. The remaining quantities in both (81) and (82) have only an $r^{-2}$ singularity and, therefore, are directly integrable.

We stress that both the expressions (81) and (82) do not depend upon the internal structure of the particle or the particular choice of $r_o$ within the range: $R \ll r_o \ll l_0$. The
approximation is no longer valid if the particle becomes exposed to a short-wave radiation (i.e. when \( l_0 \) is of the order of \( R \)), which may strongly influence its internal state. Such physical situations are beyond the applicability range of the theory developed in the present paper. To describe them, the particle can no longer be treated as a point-like object and its internal dynamics has to be taken into account.

For an exact solution of the “super theory” the total four-momentum is conserved, i.e. does not depend on time. Consider now a point-like particle moving along a world line \( \zeta \) and a solution \( f_{\mu\nu} \) of the Maxwell equations with corresponding point-like sources concentrated on \( \zeta \). We may ask whether \( \zeta \) can be an “approximate trajectory” and \( f_{\mu\nu} \) the corresponding approximate external field for a hypothetical exact “moving particle” solution \((f, \phi)\) of the “super theory.” We see, that the conservation law (9) is indeed a necessary condition for the existence of such a solution.

Decomposing the momentum (14) into a sum

\[
\mathcal{P}_j(t) = P_j + \frac{e}{4\pi} L_j ,
\]

where

\[
P_j(t) := \int_{\Sigma_t} (\mathcal{D}^n B^i \epsilon_{nij}) d^3 x ,
\]

and

\[
L_j(t) := \int_{\Sigma_t} \frac{1}{r^3} (x^n B^i \epsilon_{nij}) d^3 x ,
\]

we see that, unfortunately, formula (7) does not guarantee \textit{a priori} the time-like character of the total four-momentum. Although both vectors defined in the co-moving reference frame by \((m, 0)\) and \((H, P_j)\) are time-like, the remaining vector \( L_j \) may produce difficulties. Formula (85) shows that the field \( B \) which is strong far away from the particle does not produce difficulties. (In this case also the term \((B^2)\) in (13) is big and the total four-momentum remains time-like.) Difficulties occur if the field \( B \) is very strong in the vicinity of the particles. A careful analysis shows that strong fields within the radius \( r \) become dangerous if \( r \) is of the order of \( e^2/m \). In most applications this is far away from the applicability region of our theory. In all situations when the initial state consists e.g. of an almost uniformly moving particle and the radiation field far away from it, the total four-momentum has always time-like character.

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