Gauge Theories with Fermions in Terms of Gauge Invariants

J. Kijowski¹, G. Rudolph² and M. Rudolph²

¹ Center for Theor. Phys., Polish Academy of Sciences
   al. Lotników 32/46, 02 - 668 Warsaw, Poland
² Institut für Theoretische Physik, Univ. Leipzig
   Augustusplatz 10/11, 04109 Leipzig, Germany

1 Introduction

We present some results, which are part of our programme of analyzing gauge theories in terms of physical observables (gauge invariants). For applications of these ideas to theories of gauge fields interacting with fermionic matter fields we refer to [1] – [4]. In the present contribution we show that the classical Dirac-Maxwell system can be reformulated in a spin rotation covariant way in terms of invariants. Next, we demonstrate that the functional integral of QED can be reduced to an integral over physical degrees of freedom described by an analogous set of invariants. The latter invariants are, however, Grassmann algebra valued. Finally, we outline an application. We show how to calculate the anomaly in the 2-dimensional Schwinger model within our formulation.

2 The Classical Dirac–Maxwell–System

A field configuration of this model consists of a $U(1)$-gauge potential $(A_\mu)$ and a four component spinor field $(\psi^a)$, where $a, b, \ldots = 1, 2, \dot{1}, \dot{2}$ denote bispinor indices and $\mu, \nu, \ldots = 0, 1, 2, 3$ spacetime indices. A bispinor will be represented by a pair of Weyl spinors:

$$\psi^a = \begin{pmatrix} \phi^K \\ \varphi_K \end{pmatrix} \equiv \begin{pmatrix} \phi^1 \\ \phi^2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix},$$

where $K, L, \ldots = 1, 2$ denote ordinary spinor indices. The components of $(\psi^a_\lambda)$ are considered as commuting quantities.

The Lagrangian of the classical Dirac–Maxwell–system is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m \overline{\psi}^a \beta_{ab} \psi^b - \text{Im} \left\{ \overline{\psi}^a \beta_{ab} (\gamma^\mu)_c^b D_\mu \psi^c \right\},$$

(2)
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (3)
\]
\[
D_\mu \psi^a = \partial_\mu \psi^a + ieA_\mu \psi^a \quad (4)
\]
are the electromagnetic field strength and the covariant derivative. The bar denotes complex conjugation, \(\beta_{ab}\) denotes the Hermitian metric in bispinor space and \((\gamma_\mu)^b_c\) are the Dirac matrices. For the representation used for these quantities see [1].

Let us define the following gauge invariant quantities:
\[
l_\mu := \frac{1}{2} \bar{\phi} K \phi L \sigma_{KL}, \quad r_\mu := \frac{1}{2} \bar{\varphi} K \varphi L \sigma_{KL},
\]
\[
h := \bar{\varphi} K \phi K, \quad b_\mu := \text{Im} \left\{ \bar{h} \left( \bar{\varphi} K (D_\mu \phi K) + \phi K (D_\mu \bar{\varphi} K) \right) \right\}. \quad (5)
\]

**Lemma 1** We have the following identities:
\[
l_\mu l_\mu = 0, \quad r_\mu r_\mu = 0, \quad 2l_\mu r_\mu = |h|^2. \quad (6)
\]

**Theorem 1** Every class \(\{A_\mu, \psi^a\}\) of generic gauge equivalent configurations is in one-to-one correspondence with a set \(\{l_\mu, r_\mu, \chi, b_\mu\}\), where \(\chi\) is the phase of the complex field \(h\).

**Proposition 1** Field dynamics of the classical Dirac–Maxwell–system in terms of the above invariants is given by the Lagrangian
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2m \sqrt{2(l \cdot r)} \text{Re} \chi - \frac{i}{4} (l_\mu - r_\mu) \chi \partial_\mu \chi - 2e (l_\mu + r_\mu) B_\mu
\]
\[
+ \frac{1}{l \cdot r} \epsilon^{\alpha\beta\mu\nu} l_\alpha r_\beta \nabla_\mu (l_\gamma + r_\gamma), \quad (7)
\]
with \(B_\mu = \frac{1}{2e|h|^2} b_\mu\) and \(F_{\mu\nu} = \partial_\mu B_\nu + \frac{1}{2e(l \cdot r)^2} \epsilon^{\alpha\beta\gamma\delta} l_\alpha r_\beta \left( \nabla_\mu r_\gamma \mid \nabla_\nu l_\delta \right)\).

For the notion “generic” we refer to [1]. The proofs of Theorem 1 and Proposition 1 can be also found in [1].

### 3 The Functional Integral of QED in Terms of Gauge Invariants

A field configuration of QED consists of \(\{A_\mu, \psi^a\}\) with the notation of the previous section. The Lagrangian is given by (2). In contrast to the (classical) Dirac–Maxwell–system, the components of \((\psi^a)\) are now anticommuting quantities, which (pointwise) build up a Grassmann-algebra generated by 8 elements. In the ordinary approach, the reduction of the naive functional integral
\[
\mathcal{F} = \int \prod dA d\psi d\bar{\psi} e^{i \int d^4 x \mathcal{L}[A, \psi, \bar{\psi}]} \quad (8)
\]
to an integral with the correct number of degrees of freedom is done via the Faddeev-
Popov gauge fixing procedure. Here we propose a completely different approach.
For this purpose, we define the following Grassmann-algebra valued invariants:

\[
L^{\mu} := \frac{1}{2} \bar{\phi} K^{\mu} \sigma_{Kl}, \quad R^{\mu} := \frac{1}{2} \bar{\phi} K^{\mu} \sigma_{Kl},
H := \bar{\phi} K \phi, \quad B^{\mu} := \text{Im} \left\{ \bar{H} \left( \bar{\phi} K_{\mu} \phi + \phi K_{\mu} \bar{\phi} K \right) \right\}.
\]

**Lemma 2** The following algebraic identities hold:

\[
2L^{\mu} R_{\mu} = -|H|^2 = -H \bar{H},
\]

\[
(|H|^2) F_{\mu \nu} = \frac{1}{2e} \left( |H|^2 (\partial_{[\mu} B_{\nu]} - (\partial_{[\mu} |H|^2) B_{\nu]} \right)
+ \frac{2}{e} \epsilon^{\alpha \beta \gamma \delta} R_{\alpha} L_{\beta} (\partial_{[\mu} R_{\gamma]})(\partial_{\nu]} L_{\delta]},
\]

\[
|H|^2 L^{\mu}_{\text{mat}} = -2m |H|^2 \text{Re} (H) - (L^{\mu} + R^{\mu}) B_{\mu}
+ (L^{\mu} - R^{\mu}) \text{Im} (\bar{H} \partial_{\mu} H) - 2e^{\alpha \beta \gamma \delta} L_{\alpha} R_{\beta} \partial_{\mu} (R_{\gamma} + L_{\gamma}).
\]

The proof of this Lemma and of the following Proposition can be found in [2]. For the invariants \(L^{\mu}, R^{\mu}, B^{\mu}, H, \bar{H}\), we introduce corresponding c-number mates \(l^{\mu}, r^{\mu}, b^{\mu}, h\), which by definition are gauge invariant.

**Proposition 2** The functional integral \(\mathcal{F}\) in terms of invariants is given by

\[
\mathcal{F} = \int \prod dv dl dr dh d\bar{h} K(l, r, h) e^{i \int d^4x \mathcal{L}[v, l, r, h]},
\]

with the integral kernel

\[
K = \frac{1}{2i} \left\{ \frac{1}{16} \frac{\partial^2}{\partial r^\mu \partial r^\mu} + \frac{1}{2} \frac{\partial^2}{\partial h^2 \partial \bar{h}^2} - \frac{1}{2} \frac{\partial^2}{\partial r^\mu \partial \bar{h}^2} \right\} \delta(h) \delta(l) \delta(r),
\]

and the effective Lagrangian

\[
\mathcal{L}[v, l, r, h] = -\frac{1}{4} \left( \partial_{[\mu} v_{\nu]} \right)^2 - 2m \text{Re} \{h\} - 2e(v^{\mu} + r^{\mu}) v_{\mu} - 2(l^{\mu} - r^{\mu}) \frac{\text{Im} (\bar{h} \partial_{\mu} h)}{2|h|^2}.
\]

### 4 The Massless Schwinger model

A field configuration of the Schwinger model is given by a \(U(1)\)-gauge potential \(A_{\mu}\), \(\mu = 1, 2\), and a spinor field \(\psi^K\), \(K = 1, 2\), represented by

\[
\psi^K = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}
\]

with anticommuting components. The functional integral in Euclidean space is defined by

\[
\mathcal{F} = \int \prod dA d\psi d\bar{\psi} e^{-\int d^4x \mathcal{L}[A, \psi, \bar{\psi}]},
\]
with the Lagrangian
\[ L = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \text{Im} \left\{ \bar{\psi} \gamma^\mu (D_\mu \psi) \right\}. \]  
(18)

Here
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]  
(19)
\[ D_\mu \psi^K = \partial_\mu \psi^K - i A_\mu \psi^K. \]  
(20)

For a representation of the Euclidean Dirac matrices see for instance [5].

In complete analogy to QED we define the following set of gauge invariant Grassmann algebra valued quantities:
\[ L := \overline{\varphi} \varphi, \quad R := \overline{\phi} \phi, \quad H := \overline{\phi} \varphi, \quad B_\mu := \text{Im} \left\{ \overline{\psi} \phi (D_\mu \varphi) \right\}, \]  
(21)
\[ J_\mu := \overline{\psi} \gamma_\mu \psi, \quad J^5_\mu := \overline{\psi} \gamma_\mu \gamma^5 \psi. \]  
(22)

Introducing for \((L, R, H, B_\mu, J_\mu, J^5_\mu)\) the corresponding \(c\)-number mates \((l, r, h, b_\mu, j_\mu, j^5_\mu)\), we can formulate the following

**Proposition 3** The functional integral of the Euclidean massless Schwinger model in terms of gauge invariants is given by
\[ \mathcal{F} = \int \prod \left\{ dl \, dj^5_\mu \, dv_\mu \right\} \frac{\partial^2}{\partial r \partial l} + \frac{\partial^2}{\partial j^5_\mu \partial j^5_\mu} \right) \delta(r) \delta(l) \delta(j^5_\mu) \right\} e^{-\int d^2 x \mathcal{L}[l, v_\mu, j^5_\mu]} \]  
(22)

with the Lagrangian
\[ \mathcal{L}[l, v_\mu, j^5_\mu] = \frac{1}{4e^2} \left( \partial_\mu v_\nu \right)^2 + i v_\mu \epsilon^{\mu\nu} j^5_\nu + \frac{i}{2} j^5_\mu \frac{\partial^\mu l}{l}, \]  
(23)
where \(v_\mu := \frac{b_\mu}{ln^2}.\)

Treating \(v_\mu\) as an external field and the \(\delta\)-functions in the integral kernel as in the Faddeev-Popov-procedure (see for instance [6]), we are able to perform a one-loop expansion around classical solutions. Using this technique, which in detail will be discussed elsewhere, we can – for instance – calculate the chiral anomaly \(< \partial_\mu j^5_\mu >\). For the generating functional we get the following expression
\[ Z[\xi] = \mathcal{N} \int \prod \left\{ dl \, dj^5_\mu \right\} e^{-\int d^2 x \mathcal{L}_{\text{eff}}}, \]  
(24)
where \(\mathcal{N}\) denotes the normalization constant and
\[ \mathcal{L}_{\text{eff}} = i v_\mu \epsilon^{\mu\nu} j^5_\nu + \frac{i}{2} j^5_\mu \frac{\partial^\mu l}{l} - \xi_\mu j^5_\mu + \frac{1}{2 \alpha_j} j^5_\mu j^5_\mu + \frac{1}{2 \alpha_l} l^2 - \ln \left[ \frac{j^5_\mu j^5_\mu}{a^2_j} - \frac{2}{a_j} \right]. \]  
(25)
Here, $\xi_\mu j_5^\mu$ is the source term for the axial current and $\alpha_j, \alpha_l$ are parameters coming from the representation of the $\delta$-functions. Performing the above mentioned 1-loop-expansion around a special solution of the classical field equations, we get for the chiral anomaly

$$\langle \partial_\mu j_5^\mu \rangle = \frac{-i\alpha_j}{2} \epsilon_{\mu\nu} \partial_\mu v_\nu.$$  \hspace{1cm} (26)

Thus, choosing $\alpha_j = \frac{2}{\pi}$ we get the correct value of the anomaly, see [5] for other methods.

5 Further Applications and Perspectives

Our constructions can be extended to non-Abelian gauge field theories, for a similar formulation of one-flavour chromodynamics we refer to [4]. In this paper we have obtained a description of QCD in terms of purely bosonic invariants. From the physical point of view, this formulation could serve as an effective model of mesons interacting via gauge invariantly dressed gluons. This may be considered as a step towards understanding QCD as an effective theory of interacting hadrons.

Another interesting line of research within our programme consists in doing similar constructions on the level of field operators (Hamiltonian approach) for models formulated on a lattice. In [6] this has been done for QED. In this paper we have given a complete classification of irreducible representations of the algebra of observables. These representations are labelled by eigenvalues of the total charge. Thus, a decomposition of the physical Hilbert space into charge superselection sectors has been obtained.

References


