On the Algebra of Gauge Invariants for One–Flavour Chromodynamics

J. Kijowski
Center for Theor. Phys., Polish Academy of Sciences
al. Lotników 32/46, 02 - 668 Warsaw, Poland

G. Rudolph, M. Rudolph
Institut für Theoretische Physik, Univ. Leipzig
Augustusplatz 10/11, 04109 Leipzig, Germany

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Abstract

The structure of the algebra of Grassmann-algebra valued gauge invariant differential forms built from $SU(3)$-gauge potentials as well as quark and antiquark fields is discussed. The relevance for one-flavour chromodynamics is outlined.

1 Introduction

In [17] and [18] we have shown, that it is possible to reformulate the functional integral of quantum electrodynamics as well as of one–flavour chromodynamics completely in terms of local gauge invariant quantities. The result differs essentially from the effective functional integral obtained via the Faddeev-Popov procedure [2]. In the case of QCD we end up with a description in terms of a set $(j^{ab}, c_{\mu K}^L)$ of purely bosonic invariants, where $j^{ab}$ is built from bilinear combinations of quarks and antiquarks (mesons) and $c_{\mu K}^L$ is a set of complex-valued vector bosons built from the gauge potential and the quark fields. A naive counting of the degrees of freedom encoded in these quantities yields the correct result: The field $j^{ab}$ is Hermitean and carries, therefore, 16 degrees of freedom, whereas $c_{\mu K}^L$ is complex-valued and carries, therefore, 32 degrees of freedom. On the other hand, the original configuration $(A_\mu, A_\mu^B, \psi^A)$ carries $32 + 24 = 56$ degrees of freedom. Thus, exactly 8 gauge degrees of freedom have been removed. The main technical difficulty in our construction comes, of course, from the fact that the quark
fields are Grassmann-algebra-valued. In some sense, one has to start with all invariants one can write down. Next one finds identities relating these invariants, which however — due to their Grassmann character — cannot be “solved” with respect to the correct number of effective invariants. But there exists a scheme, which enables us to implement these identities under the functional integral and to integrate out the original quarks and gauge bosons. This is the main idea of the above mentioned papers. As a result we obtain a functional integral in terms of the correct number of effective gauge invariant bosonic quantities. Thus, our procedure consists in a certain reduction to a sector, where we have mesons $j^{ab}$, whose interaction is mediated by vector bosons $c_{\mu KL}$. For further discussion of this point see Section 4.

In [18] we have only considered invariants and relations, which were necessary for reformulating the functional integral of QCD. In this paper we present a full analysis of the algebra of gauge invariants for this model. More precisely, we restrict ourselves to the physically most interesting algebra of invariants consisting of Grassmann algebra valued differential forms with the natural derivation given by the covariant derivative. This is a finitely generated differential algebra. Thus, one can give a minimal set of generators and, moreover, one can write down a complete list of relations in terms of these generators.

The paper is organized as follows: In Section 2 we define the algebra of gauge invariant Grassmann algebra valued differential forms, built from the gauge potential and the (anticommuting) quark fields. In Section 3 we present a minimal set of generators for this algebra and give the complete set of relations for these generators. In Section 4 we shortly discuss their relevance for QCD.

2 The Algebra of Gauge Invariants

A classical field configuration of one-flavour chromodynamics consists of an SU(3)-gauge potential $(A_{\mu A}^B)$ and a four-component colored quark field $(\phi^a_A)$, where $A, B, \ldots = 1, 2, 3$ are color indices, $a, b, \ldots = 1, 2, 1, 2$ denote bispinor indices and $\mu, \nu, \ldots = 0, 1, 2, 3$ space-time indices. In differential geometric terms, $(A_{\mu A}^B)$ represents a connection form in a principal bundle $P$ with structure group $SU(3)$ over space-time $M$ and $(\psi^a_A)$ is a section in the associated bundle $E = P \times_{SL(2, \mathbb{C}) \times SU(3)} F$, where $F = \mathbb{C}^4 \otimes \mathbb{C}^3$ denotes the typical fibre. On $\mathbb{C}^4$ acts the spinor group $SL(2, \mathbb{C})$ and $\mathbb{C}^3$ carries the fundamental representation of $SU(3)$. Due to Berezin, see [12] and [13], the $\psi$-fields should become Grassmann-algebra valued under the functional integral. The underlying geometric construction goes – roughly speaking – as follows:

The space of anticommuting (Berezin) variables is the Grassmann-algebra

\[ G_M := \bigwedge \mathcal{E}_M \, . \] (2.1)
built of the infinite-dimensional direct sum of vector spaces

$$\mathcal{E}_M = \bigoplus_{x \in M} F_x.$$  \hfill (2.2)

where all $F_x$ are isomorphic to the vector space $F$.

Since we have an overcountable set of points $x \in M$, different inequivalent direct sums and its different tensor (exterior) products could be taken. The choice of an appropriate topology is a subtle problem, which we are not going to discuss here. Here we limit ourselves to the algebraic direct sum and the algebraic exterior product. Consequently, we choose the strongest topology compatible with the linear structure of $\mathcal{E}_M$ and $\mathcal{G}_M$. That means, we assume that $\mathcal{G}_M$ is generated by elements of the following form

$$g = g_{x_1} \wedge g_{x_2} \wedge \ldots \wedge g_{x_k},$$  \hfill (2.3)

where $g_{x_i} \in F_{x_i}$ (the points $x_i$ are not necessarily different).

A “classical fermionic field” $Z$ is a local, linear and continuous mapping $Z : \Gamma^\infty(E) \to \mathcal{G}_M$ (or a $\mathcal{G}_M$-valued distribution), where by $\Gamma^\infty(E)$ we denote the space of $C^\infty$ sections of the bundle $E$. By locality we mean the following property: If $\text{supp} \, \phi = K \subset M$, with $\phi \in \Gamma^\infty(E)$, then

$$< \phi, Z > \in \mathcal{G}_K = \bigwedge \mathcal{E}_K \subset \mathcal{G}_M.$$  \hfill (2.4)

Again, different topologies on the “test function space” $\Gamma^\infty(E)$ may be taken. Here we choose the Schwartz–topology of almost uniform convergence (together with all derivatives). The set of classical fermionic fields is naturally equipped with a linear structure. Moreover, we have the following natural operations of multiplication and (both ordinary and covariant) differentiation:

$$< \phi, Z_1 \cdot Z_2 >= < \phi, Z_1 > \wedge < \phi, Z_2 >, \hfill (2.5)$$

$$< \phi, \partial_\mu Z >= - < \partial_\mu \phi, Z >, \hfill (2.6)$$

$$< \phi, D_\mu Z >= - < D_\mu \phi, Z >, \hfill (2.7)$$

where

$$D_\mu \phi^A_B = \partial_\mu \phi^A_A + i A_\mu^A B \phi^B_B.$$  \hfill (2.8)

In our paper we are going to use only the above structure. For a deeper understanding of (Berezin) path integrals one also has to choose a topology in the space of such mappings, i.e. in $\Gamma^\infty(E)^* \otimes \mathcal{G}_M$.

A basic example of a classical fermionic field is the “classical quark field” $\psi^a_A(x)$. Given a trivialization of $E$, a section $\phi$ is uniquely represented by its coefficient functions $\phi^a_A$. Hence, the quantity $\psi^a_A(x)$ is given by:

$$< \phi, \psi^a_A(x) >= \phi^a_A(x).$$  \hfill (2.9)
Thus, the \( \{ \psi^a_A(x) \} \) can be taken as generators of \( \mathcal{G}_M \). Since the group of local gauge transformations acts on the space \( \Gamma^\infty(E) \), the \( \psi^a_A(x) \) inherit this action. Therefore, we have

\[
D_\mu \psi^a_A = \partial_\mu \psi^a_A + i A^B_\mu \psi^a_B.
\]

Replacing \( \mathcal{G}_M \) by \( \mathcal{G}_M \otimes \Lambda \), where \( \Lambda = \oplus_i \Lambda^i \) denotes the algebra of exterior forms on \( M \), we may organize the above derivatives into a new object:

\[
D \psi^a_A = D_\mu \psi^a_A \, dx^\mu,
\]

which is a \( \mathcal{G}_M \otimes \Lambda \)-valued distribution or a “\( \mathcal{G}_M \)-valued differential form on \( M \) with distribution coefficients”. For such objects we have the following natural multiplication (denoted by the same symbol):

\[
(\psi \otimes \alpha) \bullet (\psi' \otimes \alpha') = \psi \bullet \psi' \otimes \alpha \wedge \alpha'.
\]

Moreover, we lift the covariant derivative to these objects:

\[
D(\psi \otimes \alpha) = D\psi \bullet (1 \otimes \alpha) + (-1)^{\alpha|} \psi \otimes d\alpha,
\]

where \( |\alpha| \) denotes the degree of \( \alpha \). With this multiplication there is a natural \( \mathbb{Z}_2 \)-grading in the space of these objects:

\[
(\psi \otimes \alpha) \bullet (\psi' \otimes \alpha') = (-1)^{|\psi||\psi'|+|\alpha||\alpha'|} (\psi' \otimes \alpha') \bullet (\psi \otimes \alpha).
\]

It is easy to check, that \( D \) defined by (2.13) is indeed the derivation with respect to multiplication defined by (2.14).

Now we are prepared to discuss gauge invariants. Using a classical result by H. Weyl [19], there are only two tensors in color space, which can be used to construct invariants, the completely antisymmetric symbol \( \epsilon^{ABC} \) and the Hermitean metric \( g^{AB} \). In [19] one can find a complete list of identities for invariant tensors built from \( \epsilon \) and \( g \). They are obtained by the following relations (and further relations by taking contractions):

\[
\epsilon^{*ABC} \epsilon^{DEF} = g^{AD} g^{BE} g^{CF} + g^{AF} g^{BD} g^{CE} + g^{AE} g^{BF} g^{CD} - g^{AF} g^{BE} g^{CD} - g^{AE} g^{BD} g^{CF} - g^{AD} g^{BF} g^{CE},
\]

\[
\epsilon^{ABC} g^{EF} = \epsilon^{ABF} g^{EC} + \epsilon^{BCF} g^{EA} + \epsilon^{CAF} g^{EB},
\]

\[
\epsilon^{*ABC} g^{EF} = \epsilon^{*ABE} g^{CF} + \epsilon^{*BCE} g^{BF} + \epsilon^{*CAE} g^{AF}.
\]

**Definition 1** We put

\[
\mathcal{J}^{ab}_{(i,j)} := g^{AB} \left( D^i \psi^a_A \right) \bullet \left( D^j \psi^b_B \right),
\]

where \( i, j = 0, \ldots, 4 \) and \( D^i \psi \equiv \psi \) for \( i = 0 \).
On 4-dimensional Minkowski space we have $0 \leq i + j \leq 4$. Obviously, all gauge invariant differential forms built from quark and antiquark fields are combinations of these quantities. In local coordinates we have

\[ \mathcal{J}_{(i,j)}^{ab} = g^{AB} (D_{\mu_1} \ldots D_{\mu_i} \bar{\psi}_A^a) (D_{\mu_{i+1}} \ldots D_{\mu_{i+j}} \bar{\psi}_B^b) \ dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{i+j}}. \]  

(2.19)

**Definition 2** We put

\[ \mathcal{X}^{abc}_{(i,j,k)} := \epsilon^{ABC} (D^i \bar{\psi}_A^a) \bullet (D^j \bar{\psi}_B^b) \bullet (D^k \bar{\psi}_C^c), \]

with $i, j, k = 0, \ldots, 4$ and $0 \leq i + j + k \leq 4$.

All gauge invariant forms built from three quarks and $\epsilon^{ABC}$ are combinations of these quantities. In local coordinates we have

\[ \mathcal{X}^{abc}_{(i,j,k)} = \epsilon^{ABC} (D_{\mu_1} \ldots D_{\mu_i} \bar{\psi}_A^a) (D_{\mu_{i+1}} \ldots D_{\mu_{i+j}} \bar{\psi}_B^b) (D_{\mu_{i+j+1}} \ldots D_{\mu_{i+j+k}} \bar{\psi}_C^c) \]

\[ \times dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{i+j+k}}. \]

(2.21)

Moreover, for simplicity let us denote

\[ \mathcal{J}^{ab} := \mathcal{J}_{(0,0)}^{ab}, \]

(2.22)

\[ \mathcal{X}^{abc} := \mathcal{X}_{(0,0,0)}^{abc}. \]

(2.23)

Obviously, the $(i+j)$–forms $\mathcal{J}^{ab}_{(i,j)}$ obey the following identity under complex conjugation:

\[ (\mathcal{J}^{ab}_{(i,j)})^* = (-1)^{ij} \mathcal{J}^{ba}_{(j,i)}. \]

(2.24)

Thus, for instance, the 0–form $\mathcal{J}^{ab}$ is a Hermitean field of even (commuting) type (rank 2 in the Grassmann-algebra).

The 0–forms $\mathcal{X}^{abc}$ and $\mathcal{X}^{abc}_{\alpha\beta\gamma}$ are of rank 3 in the Grassmann-algebra, and thus of anticommuting type. Because of the antisymmetry of the $\epsilon$ tensor the $(i + j + k)$–forms built from three quarks or three antiquarks obey the following symmetry relations:

\[ \mathcal{X}^{abc}_{(i,j,k)} = (-1)^{ij} \mathcal{X}^{bac}_{(j,i,k)} = (-1)^{jk} \mathcal{X}^{acb}_{(i,k,j)} = (-1)^{ik} \mathcal{X}^{bca}_{(j,k,i)} \]

\[ = (-1)^{ik+jk} \mathcal{X}^{cab}_{(k,i,j)} = (-1)^{ij+ik+jk} \mathcal{X}^{cba}_{(k,j,i)}. \]

(2.25)

**Definition 3** The $\ast$-algebra $\mathcal{A}$ of gauge invariant Grassmann-algebra valued differential forms is the algebra generated by the set $\{ \mathcal{J}^{ab}_{(i,j)}, \mathcal{X}^{abc}_{(i,j,k)} \}$.

Since the action of SU(3) on invariants is trivial, the restriction of $D$ to $\mathcal{A}$ coincides with the ordinary external differential $d$ in the following sense:

\[ D(a \otimes \alpha) = da \bullet (1 \otimes \alpha) + (-1)^{|\alpha|} a \otimes da, \]

(2.26)

where $a \in \mathcal{A}$. Thus $\mathcal{A}$ carries the structure of a $\mathbb{Z}_2$-graded differential $\ast$-algebra.
3 The Structure of the Differential Algebra $\mathcal{A}$

Proposition 1 The generators of the $\mathcal{A}$ obey the following identities

\[ \mathcal{X}^{abc*}(i,j,k) \cdot \mathcal{X}^{def}(i,m,n) = (-1)^{j+i+k+m+n+l+m} \mathcal{J}^{ae}(i,l) \cdot \mathcal{J}^{be}(j,m) + (-1)^{k(n+l)+n(j+l+m)} \mathcal{J}^{af}(i,n) \cdot \mathcal{J}^{bd}(j,l) \cdot \mathcal{J}^{ce}(k,m) \]  
\[ + (-1)^{(n+m)(k+l)+n(j+m)} \mathcal{J}^{af}(i,n) \cdot \mathcal{J}^{be}(j,m) \cdot \mathcal{J}^{cd}(k,l) \]  
\[ + (-1)^{(k+m)+m(j+k)} \mathcal{J}^{ae}(i,m) \cdot \mathcal{J}^{bd}(j,l) \cdot \mathcal{J}^{cf}(k,n) + (-1)^{l(j+k)+n(k+m)} \mathcal{J}^{ad}(i,l) \cdot \mathcal{J}^{bf}(j,n) \cdot \mathcal{J}^{ce}(k,m) \]  
\[ + (-1)^{(j+m)+m(n+j)} \mathcal{J}^{ae}(i,m) \cdot \mathcal{J}^{bd}(j,l) \cdot \mathcal{J}^{cf}(k,n) + (-1)^{l(j+k)+n(k+m)} \mathcal{J}^{ad}(i,l) \cdot \mathcal{J}^{bf}(j,n) \cdot \mathcal{J}^{ce}(k,m) \]  
and

\[ \mathcal{X}^{abc}(i,j,k) \cdot \mathcal{J}^{de}(m,n) \]  
\[ = (-1)^{1+i(k+n)+n(m+j)+m+j} \mathcal{X}^{abe}(i,j,n) \cdot \mathcal{J}^{de}(m,k) + (-1)^{1+i(k+1)+n(m+j)+m+j} \mathcal{X}^{ace}(i,j,n) \cdot \mathcal{J}^{dh}(m,j) \]  
\[ + (-1)^{1+i(k+j)+n(m+i+j)+im} \mathcal{X}^{abe}(i,j,n) \cdot \mathcal{J}^{da}(m,i) \]  
\[ + (-1)^{1+i(k+j)+n(i+m)+jm} \mathcal{X}^{ace}(i,j,n) \cdot \mathcal{J}^{da}(m,i) \cdot \mathcal{J}^{bh}(m,j) \]  
\[ + (-1)^{1+i(k+1)+n(j+i)+ij} \mathcal{X}^{abe}(i,j,n) \cdot \mathcal{J}^{da}(m,i) \cdot \mathcal{J}^{bh}(m,j) \cdot \mathcal{J}^{ge}(m,k) \]  
\[ + (-1)^{1+i(k+1)+n(m+j)+m+j} \mathcal{X}^{ace}(i,j,n) \cdot \mathcal{J}^{da}(m,i) \cdot \mathcal{J}^{bh}(m,j) \cdot \mathcal{J}^{ge}(m,k) \]  
\[ \cdot \mathcal{J}^{ge}(m,k). \]

Proof. The proof of the relations given in the proposition is straightforward using identities (2.15) and (2.16). Moreover, note that $\mathcal{X}^{abc*}(i,j,k)$ is given by

\[ \mathcal{X}^{abc*} = -\epsilon^{ABC} (D^i \psi_{A}^{a*}) \cdot (D^j \psi_{B}^{b*}) \cdot (D^k \psi_{C}^{c*}). \]  

First let us show (3.1):

\[ \mathcal{X}^{abc*}(i,j,k) \cdot \mathcal{X}^{def}(i,m,n) \]  
\[ = -\epsilon^{ABC} \epsilon^{DEF} (D^i \psi_{A}^{a*}) \cdot (D^j \psi_{B}^{b*}) \cdot (D^k \psi_{C}^{c*}) \cdot (D^l \psi_{D}^{d*}) \cdot (D^m \psi_{E}^{e*}) \cdot (D^n \psi_{F}^{f*}) \]  
\[ = -\left( g^{AD} g^{BE} g^{CF} + g^{AF} g^{BD} g^{CE} + g^{AE} g^{BF} g^{CD} \right) \]  
\[ \times (D^i \psi_{A}^{a*}) \cdot (D^j \psi_{B}^{b*}) \cdot (D^k \psi_{C}^{c*}) \cdot (D^l \psi_{D}^{d*}) \cdot (D^m \psi_{E}^{e*}) \cdot (D^n \psi_{F}^{f*}) \]  
\[ = (-1)^{j+i+k+m+n+l+m} \mathcal{J}^{ae}(i,l) \cdot \mathcal{J}^{be}(j,m) + (-1)^{(n+l)+n(j+l+m)} \mathcal{J}^{af}(i,n) \cdot \mathcal{J}^{bd}(j,l) \cdot \mathcal{J}^{ce}(k,m) \]  
\[ + (-1)^{(n+m)(k+l)+n(j+m)} \mathcal{J}^{af}(i,n) \cdot \mathcal{J}^{be}(j,m) \cdot \mathcal{J}^{cd}(k,l) \]  
\[ + (-1)^{(k+m)+m(j+k)} \mathcal{J}^{ae}(i,m) \cdot \mathcal{J}^{bd}(j,l) \cdot \mathcal{J}^{cf}(k,n) + (-1)^{l(j+k)+n(k+m)} \mathcal{J}^{ad}(i,l) \cdot \mathcal{J}^{bf}(j,n) \cdot \mathcal{J}^{ce}(k,m). \]

Next we prove relation (3.2):
\[\epsilon^{ABC} g^{DE} \left( D^i \psi^a_A \right) \cdot \left( D^j \psi^b_B \right) \cdot \left( D^k \psi^c_C \right) \cdot \left( D^m \psi^d_D \right) \cdot \left( D^n \psi^e_E \right) \]
\[= \left( \epsilon^{ABE} g^{DC} + \epsilon^{CAE} g^{DB} + \epsilon^{BCE} g^{DA} \right) \times \left( D^i \psi^a_A \right) \cdot \left( D^j \psi^b_B \right) \cdot \left( D^k \psi^c_C \right) \cdot \left( D^m \psi^d_D \right) \cdot \left( D^n \psi^e_E \right) \]
\[= (-1)^{1+nk+nm+km} \mathcal{X}^{abc}_{(i,j,k)} \mathcal{J}^{dc}_{(m,k)} + (-1)^{1+k(i+j)+n(m+j)+mj} \mathcal{X}^{cde}_{(i,j,k)} \mathcal{J}^{db}_{(m,j)} + (-1)^{1+i(j+k)+n(i+m)+jm} \mathcal{X}^{abe}_{(i,j,k)} \mathcal{J}^{da}_{(m,i)}.\]

As already mentioned, \( \mathcal{A} \) has the structure of a differential algebra. Therefore we can reduce the generating set \( \{ \mathcal{J}^{ab}, \mathcal{X}^{abc} \} \) essentially:

**Proposition 2** The \( \ast \)-algebra \( \mathcal{A} \) is as a differential algebra generated by the following set of invariants: \( \{ \mathcal{J}^{ab}_{(0,n)}, \mathcal{X}^{abc}_{(0,0,1)}, \mathcal{X}^{abc}_{(0,1,1)}, \mathcal{X}^{abc}_{(0,2,2)}, \mathcal{X}^{abc}_{(1,1,2)} \} \), where \( 0 \leq n \leq 4 \).

**Proof.** First note that, since \( \epsilon \) and \( g \) are the only tensors in color space which can be used to construct gauge invariants, an arbitrary gauge invariant \( n \)-form, \( n \leq 4 \), built from quark and antiquark fields and their covariant derivatives can be expressed as a linear combination of products of gauge invariants (2.18), (2.20) and their complex conjugates. Thus we have to show, that the invariant forms given in Definition 1 and 2 are generated by the generating set given in the proposition. This will be done by explicit construction.

We start with invariants given by Definition 1. The 0–form \( \mathcal{J}^{ab}_{(0,n)} \) is a generator. There are two 1–forms: \( \mathcal{J}^{ab}_{(0,1)} \) is a generator and the remaining invariant 1–forms can be obtained by acting with the exterior derivative on \( \mathcal{J}^{ab}_{(0,1)} \):

\[d \mathcal{J}^{ab}_{(0,1)} = \mathcal{J}^{ab}_{(1,0)} + \mathcal{J}^{ab}_{(0,1)}.\]  (3.4)

This leads to

\[\mathcal{J}^{ab}_{(1,0)} = d \mathcal{J}^{ab}_{(0,1)} - \mathcal{J}^{ab}_{(0,1)}.\]  (3.5)

There are three types of 2–forms. \( \mathcal{J}^{ab}_{(0,2)} \) is a generator and the remaining invariant 2–forms can be obtained by acting with the exterior derivative on 1–forms:

\[d \mathcal{J}^{ab}_{(0,1)} = \mathcal{J}^{ab}_{(1,1)} + \mathcal{J}^{ab}_{(0,2)}\]  (3.6)

\[d \mathcal{J}^{ab}_{(1,0)} = \mathcal{J}^{ab}_{(2,0)} - \mathcal{J}^{ab}_{(1,1)}.\]  (3.7)

Equation (3.6) leads to

\[\mathcal{J}^{ab}_{(1,1)} = d \mathcal{J}^{ab}_{(0,1)} - \mathcal{J}^{ab}_{(0,2)},\]  (3.8)

and from (3.7) together with (3.5) we obtain

\[\mathcal{J}^{ab}_{(2,0)} = - \mathcal{J}^{ab}_{(0,2)}.\]  (3.9)
Analogously we get for the 3–forms: \( J_{(0,3)}^{ab} \) is a generator,

\[
\begin{align*}
    d J_{(0,2)}^{ab} &= J_{(1,2)}^{ab} + J_{(0,3)}^{ab}, \\
    d J_{(1,1)}^{ab} &= J_{(2,1)}^{ab} - J_{(1,2)}^{ab}, \\
    d J_{(2,0)}^{ab} &= J_{(3,0)}^{ab} + J_{(2,1)}^{ab}.
\end{align*}
\]  (3.10)

(3.10) gives

\[
J_{(1,2)}^{ab} = d J_{(0,2)}^{ab} - J_{(0,3)}^{ab}.
\]  (3.13)

From (3.13) we get

\[
J_{(1,2)}^{ab} = d J_{(0,2)}^{ab} - J_{(0,3)}^{ab}.
\]  (3.14)

Inserting (3.8) in (3.11) leads to

\[
J_{(2,1)}^{ab} = -J_{(0,3)}^{ab}.
\]  (3.15)

(3.17) together with (3.14) leads to

\[
J_{(2,2)}^{ab} = -J_{(0,4)}^{ab}.
\]  (3.21)

(3.16) gives

\[
J_{(1,3)}^{ab} = d J_{(0,3)}^{ab} - J_{(0,4)}^{ab}.
\]  (3.20)

Finally, the 4–forms \( J_{(0,4)}^{ab} \) is a generator. The remaining invariant 4–forms one gets

by acting with the exterior derivative on all 3–forms:

\[
\begin{align*}
    d J_{(0,3)}^{ab} &= J_{(1,3)}^{ab} + J_{(0,4)}^{ab}, \\
    d J_{(1,2)}^{ab} &= J_{(2,2)}^{ab} - J_{(1,3)}^{ab}, \\
    d J_{(2,1)}^{ab} &= J_{(3,1)}^{ab} + J_{(2,2)}^{ab}, \\
    d J_{(3,0)}^{ab} &= J_{(4,0)}^{ab} - J_{(3,1)}^{ab}.
\end{align*}
\]  (3.16)

(3.16) gives

\[
J_{(1,3)}^{ab} = d J_{(0,3)}^{ab} - J_{(0,4)}^{ab}.
\]  (3.20)

(3.17) together with (3.14) leads to

\[
J_{(2,2)}^{ab} = -J_{(0,4)}^{ab}.
\]  (3.21)

(3.18) together with (3.14) and (3.21) gives

\[
J_{(3,1)}^{ab} = -d J_{(0,3)}^{ab} + J_{(0,4)}^{ab}.
\]  (3.22)

and finally inserting (3.15), (3.20) and (3.22) in (3.19) one obtains

\[
J_{(4,0)}^{ab} = J_{(0,4)}^{ab}.
\]  (3.23)

In a similar way we can generate the invariant forms built from trilinear combinations
of quarks and their derivatives given by (2.20). There is only one, 0–form \( X^{abc} \), which is a
generator. Because of the symmetry (2.25) the invariant 1–forms are generated by \( X^{abc}_{(0,0,1)} \).
Furthermore this invariants obey the following identity, which we obtain by acting with the external derivative on the 0–form $\mathcal{X}^{abc}$:

$$
d\mathcal{X}^{abc} = \mathcal{X}^{abc}_{(1,0,0)} + \mathcal{X}^{abc}_{(0,1,0)} + \mathcal{X}^{abc}_{(0,0,1)},
$$

where in the last step the symmetry properties (2.25) were used.

For the 2–forms we get the following: $\mathcal{X}^{abc}_{(0,1,1)}$ is a generator. Acting with $d$ on $\mathcal{X}^{abc}_{(0,0,1)}$ we get

$$
d\mathcal{X}^{abc}_{(0,0,1)} = \mathcal{X}^{abc}_{(1,0,1)} + \mathcal{X}^{abc}_{(0,1,1)} + \mathcal{X}^{abc}_{(0,0,2)}.
$$

This leads to

$$
\mathcal{X}^{abc}_{(0,0,2)} = d\mathcal{X}^{abc}_{(0,0,1)} - \mathcal{X}^{abc}_{(0,1,1)} - \mathcal{X}^{abc}_{(0,1,1)}.
$$

All remaining 2–forms can be expressed in a similar way using the symmetry properties (2.25).

Analogously, there are three types of invariant 3–forms. $\mathcal{X}^{abc}_{(0,1,2)}$ is a generator. Acting with $d$ on all types of 2–forms one gets:

$$
d\mathcal{X}^{abc}_{(0,0,2)} = \mathcal{X}^{abc}_{(1,0,2)} + \mathcal{X}^{abc}_{(0,1,2)} + \mathcal{X}^{abc}_{(0,0,3)},
$$

$$
d\mathcal{X}^{abc}_{(0,1,1)} = \mathcal{X}^{abc}_{(1,1,1)} + \mathcal{X}^{abc}_{(0,1,2)} + \mathcal{X}^{abc}_{(0,0,3)}.
$$

Inserting (3.26) in (3.27) gives

$$
\mathcal{X}^{abc}_{(0,0,3)} = -(d\mathcal{X}^{abc}_{(0,1,1)} + d\mathcal{X}^{abc}_{(0,1,1)}) - (\mathcal{X}^{abc}_{(0,1,2)} + \mathcal{X}^{abc}_{(0,1,2)}),
$$

whereas (3.28) leads to

$$
\mathcal{X}^{abc}_{(1,1,1)} = d\mathcal{X}^{abc}_{(0,1,1)} + \mathcal{X}^{abc}_{(0,1,2)} - \mathcal{X}^{abc}_{(0,1,2)}.
$$

Finally, there are four types of invariant 4–forms. Acting with the exterior derivative on all 3–forms gives the following set of equations:

$$
d\mathcal{X}^{abc}_{(0,0,3)} = \mathcal{X}^{abc}_{(1,0,3)} + \mathcal{X}^{abc}_{(0,1,3)} + \mathcal{X}^{abc}_{(0,0,4)},
$$

$$
d\mathcal{X}^{abc}_{(0,1,2)} = \mathcal{X}^{abc}_{(1,1,2)} + \mathcal{X}^{abc}_{(0,2,2)} - \mathcal{X}^{abc}_{(0,1,3)},
$$

$$
d\mathcal{X}^{abc}_{(1,1,1)} = \mathcal{X}^{abc}_{(2,1,1)} - \mathcal{X}^{abc}_{(1,2,1)} + \mathcal{X}^{abc}_{(1,1,2)}
$$

$$
= -\mathcal{X}^{abc}_{(1,1,2)} - \mathcal{X}^{abc}_{(1,1,2)} - \mathcal{X}^{abc}_{(1,1,2)}.
$$

(3.32) leads to

$$
\mathcal{X}^{abc}_{(0,0,4)} = -\mathcal{X}^{abc}_{(0,2,2)} - \mathcal{X}^{abc}_{(0,2,2)},
$$

whereas this equation together with (3.31) gives

$$
\mathcal{X}^{abc}_{(0,1,3)} = -d\mathcal{X}^{abc}_{(0,1,2)} + \mathcal{X}^{abc}_{(1,1,2)} + \mathcal{X}^{abc}_{(0,2,2)}.
$$
Thus we expressed all 4–forms in terms of the generators $X_{(1,1,2)}^{abc}, X_{(0,2,2)}^{abc}$ and exterior derivatives of generating 3–forms. This completes the proof of Proposition 2.

We have to remark, that the set of generators of the differential $\ast$–algebra $A$ given in Proposition 2 is not minimal. First of all the set $\{\mathcal{J}_{(0,r)}^{ab}, X_{(0,0,1)}^{abc}, X_{(0,1,1)}^{abc}, X_{(0,1,2)}^{abc}, X_{(0,2,2)}^{abc}, X_{(1,1,2)}^{abc}\}$ can be reduced due to the symmetry properties (2.25). Moreover, in the proof of Proposition 2 we obtained the relations (3.24) and (3.33), which further reduce the number of independent generating 1– and 4–forms. A simple counting leads to the following corollary:

**Corollary 1** The minimal set of generators of the differential algebra $A$ has a total of 56 independent generating 0–forms, 56 generating 1–forms, 96 generating 2–forms, 144 independent generating 3–forms and 136 independent generating 4–forms.

**Proof.** First we count the number of independent 0–forms. There are 16 invariants $\mathcal{J}_{(0,0)}^{ab}$ and 64 invariants $X_{(0,0,1)}^{abc}$. Because of the symmetry properties (2.25) the number of the latter 0–forms are reduced by 44. The conjugate complex quantities $X_{(0,0,1)}^{abc}$ obey the same symmetry relations, giving a total of $(16 + 2 \cdot (64 - 44)) = 56$ independent generating 0–forms.

There are 16 invariant 1–forms $\mathcal{J}_{(0,1)}^{ab}$. Because of the symmetry of $X_{(0,0,1)}^{abc}$ in $(a, b)$ we can reduce this set of 64 invariants by 24. Moreover, equation (3.24) shows, that the elements of this reduced set are not independent. Indeed, we have the following relations:

for $a = b = c$

$$X_{(0,0,1)}^{aaa} = \frac{1}{3} dX_{(0,0,1)}^{aaa},$$

(3.36)

for $a = b \neq c, a < c$

$$X_{(0,0,1)}^{aac} = dX_{(0,0,1)}^{aac} - 2X_{(0,0,1)}^{aca},$$

(3.37)

for $a \neq b = c, a < c$

$$X_{(0,0,1)}^{cca} = dX_{(0,0,1)}^{acc} - 2X_{(0,0,1)}^{acc},$$

(3.38)

and finally for $a \neq b \neq c, a < b < c$

$$X_{(0,0,1)}^{abc} = dX_{(0,0,1)}^{abc} - X_{(0,0,1)}^{bca} - X_{(0,0,1)}^{cab},$$

(3.39)

(3.36) reduces the number of invariant 1–forms $X_{(0,0,1)}^{abc}$ by 4, (3.37) and (3.38 each by 6, and (3.39) by 4. Thus we are left with a number of 20 independent generators $X_{(0,0,1)}^{abc}$, giving a total of $(16 + 2 \cdot 20) = 56$ independent generating 1–forms.

There are 16 generating 2–forms $\mathcal{J}_{(0,2)}^{ab}$. Using the symmetry in $(b, c)$ we get 40 independent 2–forms of type $X_{(0,1,1)}^{abc}$. There are no further identities, thus giving a total of $(16 + 2 \cdot 40) = 96$ generating 2–forms.
We have 16 3–forms $J_{(0,3)}^{ab}$ and 64 independent invariants $X_{(0,1,2)}^{abc}$, because there is no symmetry relation which further reduces this set. Thus there are $(16 + 2 \cdot 64) = 144$ independent generating 3–forms.

Finally, there are 16 independent 4–forms of type $J_{(0,4)}^{ab}$, 64 forms of type $X_{(1,1,2)}^{abc}$, and 64 forms of type $X_{(0,2,2)}^{abc}$. The set $X_{(0,2,2)}^{abc}$ is reduced due to the symmetry in $(b, c)$ by 24. The set $X_{(1,1,2)}^{abc}$ is reduced using the identity (3.33) given in the proof of Proposition 2 and the symmetry equations (2.25). This is done analogously to the reduction of the 1–forms $X_{(0,0,1)}^{abc}$ above. Thus we are left with a total of $(16 + 2 \cdot (40 + 20)) = 136$ independent generating 4–forms, which finally proves the corollary. □

4 Discussion

In the previous section we have constructed the complete algebra $\mathcal{A}$ of gauge invariant differential forms in one–flavour chromodynamics. There are two types of generators: one type built from quark and antiquark fields as well as their covariant derivatives and a second type containing trilinear combinations of quarks respectively antiquarks and their covariant derivatives. From the physical point of view, one would like to interpret the bilinear invariant scalars as mesons and the trilinear invariant scalars as baryons. It is a challenge to try to reformulate the whole quantum theory, say on the level of the functional integral, as a theory of interacting hadrons understood in the above sense. Then, at least three questions arise immediately:

1. Is it possible to implement the complicated algebraic relations between invariants on the quantum level?
2. Is there a natural choice of the correct number of degrees of freedom parameterizing the effective theory?
3. Is there a natural gauge invariant field, which mediates the interaction of hadrons?

Concerning the first question, we have developed a general scheme, see [18], which can be applied to any theory containing fermions. Concerning the remaining two questions, we have only partial results. In [17] we have demonstrated that it is possible to reformulate the functional integral of one–flavour chromodynamics in terms of effective bosonic degrees of freedom ($j_{ab}^{\mu}$ and $c_{\mu K}^{\nu}$). Here $j_{ab}^{\mu}$ is built from bilinear combinations of quarks and antiquarks (mesons) and $c_{\mu K}^{\nu}$ is a set of complex-valued vector bosons built from the gauge potential and the quark fields describing the interaction of mesons. Here we outline the basic ideas leading to this result.

Let us denote the Hermitean metric in bispinor space by $\beta_{ab}$ and denote

$$\beta_{abcd} := \frac{1}{2} (\gamma^{\mu})^{a}_{\alpha} (\gamma^{\mu})^{\beta}_{\gamma},$$

where $(\gamma^{\mu})^{a}_{\alpha}$ are the Dirac matrices. For a discussion of the spin tensor calculus relevant for this model we refer to [18]. The first basic idea is, to find a (nonvanishing) invariant
field of maximal rank in the Grassmann algebra. For this purpose, we define:
\[ X^2 := \beta_{bcef} \beta_{ad} X^{abe} \cdots X^{def}. \] (4.1)

**Lemma 1** The quantity \((X^2)^4\) is a nonvanishing element of maximal rank in the Grassmann-algebra.

The proof of this Lemma is technical and can be found in Appendix B of [17].

**Lemma 2** The field \(X^2\) obeys the following algebraic identity
\[ X^2 := 4 \beta_{bcef} \beta_{ad} \left\{ \mathcal{J}^{ad} \mathcal{J}^{be} \mathcal{J}^{cf} + 2 \mathcal{J}^{ae} \mathcal{J}^{bf} \mathcal{J}^{cd} \right\} \equiv 4 \mathcal{J}^{ad} \mathcal{J}^{be} \mathcal{J}^{cf} \left\{ \beta_{bcef} \beta_{ad} + 2 \beta_{bcde} \beta_{af} \right\}. \] (4.2)

**Proof.** To prove (4.2) we simply multiply equation (3.1) by \(\beta_{bcef} \beta_{ad}\) and use the symmetry properties of the \(\beta\) tensor. \[ \square \]

It is shown in [18] that the nonvanishing invariant of maximal rank \((X^2)^4\) can be used to rewrite the gauge field action in terms of invariants \(\mathcal{J}^{ab}\) and
\[ C^{ab} := \mathcal{J}^{ab}_{(0,1)} - \mathcal{J}^{ba}_{(1,0)} = 2 \mathcal{J}^{ab}_{(0,1)} - d \mathcal{J}^{ab}. \] (4.3)

The 1-form \(C^{ab}\) is an anti-Hermitean covector–field:
\[ \overline{C}_{ab} = - C^{ba}, \] (4.4)
which in local coordinates takes the form
\[ C^{ab} = C^{ab}_{\mu} dx^\mu. \] (4.5)

Similarly, the matter field action can be rewritten using exactly these quantities. Thus, applying the above mentioned procedure of implementing identities under the functional integral, we are able to reformulate this integral in terms of \(\mathcal{J}^{ab}\) and \(C^{ab}_{\mu}\), or more precisely, in terms of their c-number mates \(j^{ab}\) and \(c^{ab}_{\mu}\) – see [18].

Counting the degrees of freedom carried by \(j^{ab}\) and \(c^{ab}_{\mu}\) (16 + 64), one finds that this set of invariants should be further reduced. This can be done, indeed, using the following relations:

**Lemma 3** The following identities hold
\[ X^2 \mathcal{J}^{ab}_{(0,1)} = -4 \mathcal{J}^{ab}_{(0,1)} \beta_{cf} \left( \mathcal{J}^2 \mathcal{J}^{af} + 2 \beta_{ghde} \mathcal{J}^{gd} \mathcal{J}^{hf} \mathcal{J}^{ae} \right) \]
\[ -8 \mathcal{J}^{gb}_{(0,1)} \beta_{cf} \beta_{ghde} \left( \mathcal{J}^{he} \mathcal{J}^{cf} \mathcal{J}^{ad} + \mathcal{J}^{cd} \mathcal{J}^{hf} \mathcal{J}^{ae} + \mathcal{J}^{hd} \mathcal{J}^{ce} \mathcal{J}^{af} \right), \] (4.6)

where
\[ \mathcal{J}^2 := \mathcal{J}^{ab} \beta_{abcd} \mathcal{J}^{cd}. \] (4.7)
Proof. Multiplying equation
\[
\left( \lambda^{abc*} J^{de}_{(0,\mu)} \right) dx^\mu = \left( - \lambda^{abcd*} J^{ce}_{(0,\mu)} - \lambda^{cad*} J^{be}_{(0,\mu)} - \lambda^{bed*} J^{ae}_{(0,\mu)} \right) dx^\mu
\] (4.8)
by \( \lambda^{gh*} \beta_{fc} \beta_{ghde} \) and using identity (3.1) leads by a straightforward calculation to (4.6).
\[ \square \]

Substituting
\[
J^{ab}_{(0,1)} = \frac{1}{2} \left\{ C^{ab} + d J^{ab} \right\},
\]
which follows from (4.5), into (4.6) leads to
\[
\lambda^2 \left( C^{ab} + d J^{ab} \right) = -4 \left( C^{cb} + d J^{cb} \right) \beta_{cf} \left( J^{2} J^{af} + 2 \beta_{ghde} J^{gd} J^{hf} J^{ae} \right) - 8 \left( C^{gb} + d J^{gb} \right) \beta_{cf} \beta_{ghde} \left( J^{hc} J^{cf} J^{ad} + J^{cd} J^{hf} J^{ae} + J^{hd} J^{ce} J^{af} \right).
\] (4.9)

These relations can be used to eliminate half of the 64 \( c^{ab} \) fields under the functional integral leading to a description in terms of the set \( j^{ab}, c^{K}_{\mu L} \), which carries the correct number (16 + 32) of degrees of freedom. Exactly 8 gauge degrees of freedom have been eliminated.

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References


