LAGRANGIAN AND HAMILTONIAN FORMULATION
OF SPHERICAL SHELL DYNAMICS

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Lagrangian and Hamiltonian descriptions of the dynamics of a self-gravitating matter shell in General Relativity are discussed in general. The case of a spherical shell composed of an elastic fluid is then considered, its Lagrangian function is derived from first principles and the Hamiltonian is calculated. Known results for dust shells are recovered as particular cases.

Keywords: Variational principles; relativistic shell dynamics; boundary corrections; Hilbert action.

1. Introduction
In his seminal paper [1], Werner Israel considered the dynamics of a self-gravitating thin matter shell as the simplest toy-model describing gravitational collapse. Field equations for realistic collapse are very difficult to handle, whereas many aspects of this phenomenon may be investigated within the model consisting of a two-dimensional matter shell and the surrounding gravitational field (cf. [2] and [3]).
The dynamics of such a system reduces to a proper tailoring of the two different space–times, describing the two sides of the shell.

We assume, therefore, that the space–time $\mathcal{M}$ consists of two parts, tailored together along a hypersurface $\Sigma$, which describes a moving matter shell. “Tailoring” means that the induced metric $g_{ab}$ on $\Sigma$ is continuous, whereas its derivatives (i.e., also the four-dimensional connection coefficients $\Gamma^\lambda_{\mu\nu}$) may be discontinuous on $\Sigma$. Consequently, Einstein curvature tensor density contains derivatives of those discontinuities defined in the sense of distributions (cf. [4] and [8]). These distributional derivatives give the singular part of the curvature, equal to $G^\mu_{\nu} = G^\mu_{\nu}\delta_\Sigma$, where $\delta_\Sigma$ denotes the Dirac delta distribution concentrated on $\Sigma$. The following relation may be easily proved:

$$G^a_b = [Q^a_b], \quad (1.1)$$

where the square brackets denote the jump of the extrinsic curvature $Q^a_b$ of $\Sigma$, written in the ADM form, whereas the components of $G^\mu_{\nu}$ transversal with respect to $\Sigma$ vanish identically. The singular part of Einstein equations reads: $G^a_b = 8\pi T^a_b$, where by $T^a_b$ we denote the singular (concentrated on $\Sigma$) energy–momentum tensor density of the matter filling shell. The remaining dynamical equations of the theory are the vacuum Einstein equations outside of $\Sigma$ and the mechanical equations of motion of the matter, implied by its constitutive equation.

2. Variational Principle

In order to obtain the dynamics of the above “matter + gravity” system, consider first the Hilbert action $\mathcal{A}$ composed of two parts: the gravitational part being the integral over the four-dimensional domain $D$ of the gravitational Lagrangian and the matter part concentrated on the hypersurface and carrying the information about the matter content of the shell:

$$\mathcal{A} = \int_D L_{\text{grav}} + \int_{D \cap \Sigma} L_{\text{mat}}. \quad (2.1)$$

The gravitational Lagrangian splits into two parts: a regular part outside of the shell and a singular part on the shell

$$L_{\text{grav}} = \frac{1}{16\pi} \sqrt{\det g} \ R = L_{\text{grav}}^{\text{sing}} + L_{\text{grav}}^{\text{reg}}, \quad (2.2)$$

with $R = R_{\text{reg}} + R_{\text{sing}}$. We shall assume the simplest — hydrodynamical model for matter. Consequently, (cf. [4]) the matter Lagrangian is equal to the rest frame energy density of the matter. The dust case corresponds, e.g., to the constant function $m(\nu) = m_0$, where the per mole rest-frame energy $m_0$ of the matter does not depend upon its specific volume $\nu$, because it is equal to the sum of the rest-frame masses of the non-interacting dust particles. For a generic fluid, the total energy
contains also the interaction energy depending on the local density of the fluid or, equivalently, upon $\nu$. We assume, therefore, that the rest frame energy of the fluid is a function $m = m(\nu)$, which plays the role of the state equation and implies all the mechanical properties of the fluid composing the shell.

In the present paper we consider the simplest, spherically symmetric case. Assuming the trivial topology of each Cauchy surfaces $\{ t = \text{const.} \}$ and the $S^2 \times \mathbb{R}^1$ topology of the world tube $\Sigma$, we conclude that the interior must be the flat Minkowski space

$$ds_+^2 = -dt^2 + d\rho^2 + \rho^2 d\Omega^2$$

for $\rho \leq \phi(\tau)$, with the shell located at $\rho = \phi(\tau)$, whereas the exterior carries the standard Schwarzschild geometry

$$ds_-^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

for $r \geq \psi(t)$, with the shell located at $r = \psi(t)$. Tailoring implies that both metric tensors give the same induced geometry on $\Sigma$. It is easy to check that this condition implies $\tau = f(t)$, where the function $f$ fulfills the following first order differential equation:

$$f^2 = 1 - \frac{2M}{\psi} \frac{1 - \frac{2M}{\psi} + \dot{\psi}^2}{1 - \frac{2M}{\psi}} ,$$

whereas continuity of the spherical part of the metric requires $\phi(f(t)) = \psi(t)$.

The fluid is homogeneous and, therefore, its specific volume $\nu$ equals to the total (two-dimensional) volume $4\pi \psi^2$ of the shell, divided by the total amount of the fluid. To simplify further our notation we choose units such that the total amount of the fluid contained in the shell equals $8\pi$. This leads to the following formula:

$$\nu := \frac{1}{2} \psi^2 .$$

A considerable simplification is obtained if we introduce the quantity $\mu$ defined as the hyperbolic angle between the vector $n$ (ortho-)normal to the surfaces $\{ t = \text{const.} \}$ on the Schwarzschild side of $\Sigma$ and the vector $m$ (ortho-)normal to surfaces $\{ \tau = \text{const.} \}$ on the Minkowski side: $\mu := \arccosh(n \cdot m)$ (see also [5]). It can be shown that $\mu$ may be calculated as a function of $\psi$ and $\dot{\psi}$ by solving the following equation:

$$\frac{\sinh \mu}{\cosh \mu - \sqrt{1 - \frac{2M}{\psi}}} = \frac{\dot{\psi}}{1 - \frac{2M}{\psi}} .$$
We denote, therefore, by \( \mu(\psi, \dot{\psi}) \) the function defined implicitly by the above constraint.

Since both parts of \( \mathcal{M} \) are already Ricci-flat, we have \( R_{\text{reg}} = 0 \) and the action \( (2.1) \) reduces to the singular part concentrated on \( \Sigma \). Taking the spatially unbounded domain \( D = \{ t_1 \leq t \leq t_2 \} \), the total action may be easily computed. We obtain:

\[
A = \int_{t_1}^{t_2} L \, dt + F(t_2) - F(t_1),
\]

where \( F(t) = -\frac{1}{2} \psi^2 \mu \). Of course, the boundary term \( F(t_2) - F(t_1) \) can be neglected.

In the above variational principle the status of the (arbitrarily chosen) value \( M \) of the Schwarzschild mass is highly unclear. Physically, being equal to the ADM mass, it describes the total mass (i.e., the total energy) of the interacting “matter + gravity” system and, therefore, we would expect it to be equal to the Hamiltonian of the system, which is not the case here.

We conclude that fixing \textit{a priori} the value of \( M \) leads rather to an analog of the Maupertuis–Lagrange variational principle in classical mechanics, where the total energy of the system is given in advance. However, this is not a genuine Maupertuis–Lagrange approach since the relation between \( (\psi, \dot{\psi}) \) and the energy \( M \) is still missing here and cannot be derived from the above formula. Many authors have noticed these difficulties and the correct variational formula for the matter shell was never derived from first principles.

We propose the following, a simple remedy for all these problems, which leads to the correct variational and Hamiltonian formulation of the model. Our method is based on the analysis of the boundary terms (usually neglected) arising in the variational principle. It was noticed (see [6] and the references herein) that the variation of the gravitational Lagrangian \( (2.2) \) contains, besides the standard volume part responsible for the field equations, also the boundary part, containing the variation of the extrinsic curvature of the boundary \( \partial D \) of the domain \( D \) in question. More precisely, by also taking into account the matter Lagrangian, we have:

\[
\delta A = \int_D \frac{\delta \mathcal{L}}{\delta g} \delta g_{\mu\nu} + \int_{D \cap \Sigma} \frac{\delta \mathcal{L}}{\delta \varphi} \delta \varphi - \frac{1}{16\pi} \int_{\partial D} g_{ab} \delta Q^{ab},
\]

where \( \varphi \) denotes matter fields on the shell, \( Q^{ab} = \sqrt{|\text{det} \hat{g}|} (\hat{L} \hat{g}^{ab} - L^{ab}) \), \( L_{ab} \) is the extrinsic curvature of \( \partial D \) and \( \hat{g}^{ab} \) is its three-dimensional inverse metric. Choose
now the spatially bounded domain $D_R = \{ t_1 \leq t \leq t_2; r \leq R \}$ and denote $t = x^0$ and $r = x^1$, where $x^A$ are angular coordinates, $A,B = \{ 2,3 \}$. Because of the spherical symmetry of the field configuration we have $g_{0A} = 0$, $Q^{0A} = 0$ and, therefore,

\[ g_{ab} \delta Q^{ab} = g_{00} \delta Q^{00} + g_{AB} \delta Q^{AB} = g_{00} \delta Q^{00} - Q^{AB} \delta g_{AB} + \delta (g_{AB} Q^{AB}). \]

Now, we put this into the last term on the left hand side of (2.9) and consider the new action, improved by this boundary term:

\[ A_{\text{tot}} = \int_D L_{\text{grav}} + \int_{D \cap \Sigma} L_{\text{mat}} + \int_{\partial D} L_{\text{boundary}}, \]

where

\[ \int_{\partial D} L_{\text{boundary}} = \frac{1}{16\pi} \int_{\partial D} g_{AB} Q^{AB}. \]

Of course, the new action implies the same field equations, because the volume part of (2.9) remains the same. However, the quantity $g_{ab} \delta Q^{ab}$ in the boundary term is now replaced by $g_{00} \delta Q^{00} - Q^{AB} \delta g_{AB}$. Hence, “variation with fixed boundary” now means something different. Now, we must fixed on the boundary the following quantities: $Q^{00}$ (as before) and the two-geometry $g_{AB}$ (instead of $Q^{AB}$ as before). This gives us the freedom to choose a much less restricted family of field configurations. Instead of the external Schwarzschild geometry (2.4) let us consider now the “Schwarzschild-like” geometry parametrized by an arbitrary function of time $M = M(t)$:

\[ ds^2 = -\left(1 - \frac{2M(t)}{r}\right) dt^2 + \left(1 - \frac{2M(t)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \]

for $r \geq \psi(t)$. This family of configurations is useless for the previous variational principle because $Q^{AB}$ depend on the function $M(t)$ and, therefore, does not correspond to the “variation with fixed boundary values” of $Q^{AB}$. On the other hand, the angular part $r^2 d\Omega^2$ of the above metric is fixed. This means that $g_{AB}$ is fixed. Also the value $Q^{00}$ on $\partial D_R$, calculated for the above metric, is fixed and equal to the corresponding value for the flat metric $M = 0$. We conclude, that the boundary term in $\delta A_{\text{tot}}$ vanishes and we really obtain this way a genuine variation principle for two arbitrary functions $\psi(t)$ and $M(t)$.

It may be easily checked that, when calculated on the above configuration, the value of $A$ is given by almost the same formula (2.7) (with $M$ being now an arbitrary function of time). More precisely, the formula (2.7) must be corrected by
an additional term $G(t_2) - G(t_1)$, where

$$G(t) = \dot{M}\left\{\frac{R^2}{2(1 - \frac{2M}{R})^2} - \frac{\psi^2}{2(1 - \frac{2M}{\psi})^2} + 2M(R - \psi) + 12M^2\ln\left(\frac{2M - R}{2M - \psi}\right) - 2M^2\left(\frac{4}{(1 - \frac{2M}{R})^2} + \frac{1}{(1 - \frac{2M}{\psi})^2} - \frac{4}{(1 - \frac{2M}{\psi})} - \frac{1}{(1 - \frac{2M}{R})^2}\right)\right\}. \quad (2.13)$$

Moreover, the boundary contribution to the action equals:

$$\int_{\partial D_R} L_{\text{boundary}} = \frac{1}{2} \int_{t_1}^{t_2} (R - M(t)) dt. \quad (2.14)$$

This implies:

$$A_{\text{tot}} = \int_{t_1}^{t_2} L_{\text{tot}} dt + F(t_2) + G(t_2) - F(t_1) - G(t_1) + \frac{R}{2}(t_2 - t_1), \quad (2.15)$$

where

$$L_{\text{tot}} = m(\nu) \sqrt{\left(1 - \frac{2M}{\psi}\right) - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}} + \frac{2M \cosh \mu}{\cosh \mu - \sqrt{1 - \frac{2M}{\psi}}} - 2\psi + \psi\dot{\psi}\mu + M. \quad (2.16)$$

The relation (2.6) between the hyperbolic angle $\mu(\psi, \dot{\psi})$ and $(\psi, \dot{\psi})$ remains valid for $M = M(t)$ as time varies.

We see that the Lagrangian depends on the (arbitrarily chosen) radius $R$ only via the boundary terms $G_1$ and $G_1$, which may be simply neglected. Hence, the remaining (“bulk”) part $L_{\text{tot}}$ of the Lagrangian remains valid when we pass to the limit $R \to \infty$.

A remarkable feature of these calculations is that all the terms containing the time derivative $\dot{M}$ in $L_{\text{tot}}$ cancel in the final result. Hence, the Lagrangian function $L_{\text{tot}}$ does not depend on $\dot{M}$. Variation with respect to the function $M(t)$ may thus be simply performed, which leads to an algebraic (instead of the differential) equation:

$$\frac{\delta L_{\text{tot}}}{\delta M} = \frac{\partial L_{\text{tot}}}{\partial M} = 0,$$

and implies the following Euler–Lagrange equation:

$$M(\psi, \mu) = \frac{1}{2} \psi \left\{1 - \left(\cosh \mu - \sqrt{\frac{m(\nu)\dot{\psi}^2}{\psi^2} + \sinh^2 \mu}\right)^2\right\}. \quad (2.17)$$

Unlike in the previous, naive approach, the total mass $M$ of the system, seen by an observer at infinity, is not fixed a priori but now is defined as a function of the variables $\psi$ and $\dot{\psi}$ (we remember that $\mu$ is also a function of $\psi$ and $\dot{\psi}$ defined
implicitly by (2.6)). Substituting (2.17) into (2.16) we express the total Lagrangian only in terms of $\psi$ and $\dot{\psi}$. The following formula is easy to prove:

$$L_{\text{tot}}(\psi, \dot{\psi}) = \psi \dot{\psi} \mu(\psi, \dot{\psi}) - M(\psi, \mu(\psi, \dot{\psi})).$$

(2.18)

It turns out that this Lagrangian gives correct Israel equations of motion (with respect to the Schwarzschild time $t$).

3. Hamiltonian Formulation

To obtain the Hamiltonian version of the above model, we first calculate the momentum canonically conjugated to the variable $\psi$. The following result can be proved:

$$p_\psi := \frac{dL_{\text{tot}}}{d\dot{\psi}} = \psi \mu.$$  

(3.1)

Performing the usual Legendre transformation, it finally follows that the Hamiltonian function of the system is equal to its ADM mass at infinity:

$$\mathcal{H}(\psi, p_\psi) := p_\psi \dot{\psi} - L_{\text{tot}} = M.$$  

(3.2)

Its conservation $M(t) = \text{const.}$, not postulated \textit{a priori}, is now obtained as a consequence of the equations of motion.

The symplectic structure of the theory may be calculated as follows:

$$\omega = dp_\psi \wedge d\psi = d(\psi \mu) \wedge d\psi = d\mu \wedge d\left(\frac{1}{2} \psi^2 \right) = d\mu \wedge dv.$$  

(3.3)

We now see that the hyperbolic angle $\mu$ can be interpreted as the momentum canonically conjugate to the proper volume $\nu = \frac{1}{2} \psi^2$. The Hamiltonian written in terms of these canonical variables reads:

$$\mathcal{H}(\mu, \nu) = \sqrt{\frac{\nu}{2}} \left\{ 1 - \left( \cosh \mu - \sqrt{\frac{m(\nu)^2}{2\nu} + \sinh^2 \mu} \right)^2 \right\}.$$  

(3.4)

For the dust matter ($m(\nu) = m_0$) the canonical structure (3.3) and the Hamiltonian (3.4) have been already derived in [5] by a different method. Namely, the Hamiltonian structure for a generic “shell + gravity” system (not necessarily spherically symmetric) was obtained first (cf. also [8]), and then this structure was reduced to the spherical symmetry. (The dependence of the Hamiltonian structure on a specific choice of the time parametrization was later discussed in [7].) A direct derivation of the correct Lagrangian (2.18) for the spherical shell, starting from the standard Hilbert variational principle and reduced to the spherical case, however, was never performed before. Moreover, the present result is much more general because it is true for any constitutive equation $m = m(\nu)$ of the fluid and not only for dust.
References


