A Consistent Canonical Approach to Gravitational Energy

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Abstract

Hamiltonian evolution of the gravitational field within a spatially compact world tube is considered. It is shown that the standard A. D. M.-symplectic structure in the space of Cauchy data must be supplemented by an extra boundary term. Possible “initial value + boundary value” problems, compatible with this structure, are proposed and corresponding quasilocal gravitational Hamiltonians are analyzed.

Dedicated to Professor Giorgio Ferrarese in honour of his 70-th birthday

1 Introduction

Gravitational energy looks somewhat mysterious: it cannot be localized, and many paradoxes arise when we want to define it in a quasilocal way. It seems, however, that at least total energy may be correctly measured in an asymptotically flat spacetime. For that purpose one uses different “superpotentials” (see [5] or [4] and references therein). Yet, there is a frustrating discrepancy between a recent abundance of various superpotentials, and the lack of their canonical (Hamiltonian) interpretation. As we know, a straightforward application of the field-theoretical canonical formalism (as we learn
it from classical textbooks, like e. g. [2]), leads to a somewhat paradoxical result: gravitational energy vanishes identically modulo boundary terms. This happens because the formalism is only “volume sensitive”, but not “boundary sensitive” and the boundary phenomena are simply neglected. But in gravity theory, because of its diffeomorphism-invariance, neglecting boundary terms means neglecting everything. Some authors improve this version of Canonical Gravity by imposing extra requirements on the energy functional in the asymptotically flat case (see e. g. [1]). This way, gravitational Hamiltonian is defined as “zero + boundary corrections”. These corrections, however, are not obtained by any universal procedure, whose validity may be checked independently in another theory (e. g. electrodynamics), but *via ad hoc* improvements, which make no sense outside of the gravity theory.

Recently, some authors use another philosophy: they propose to add boundary corrections to the Lagrange function of the theory (see [3] or [6]). Of course, adding boundary terms to the Lagrangian is irrelevant as far as the field equations are concerned. I will try to convince the reader, that the Lagrangian manipulations are also irrelevant for canonical (Hamiltonian) purposes.

In this paper I propose a certain approach to the energy problem, which is universal: it works not only in standard, special relativistic theories (including theories with constraints, like gauge theories) but also in general relativistic theories of gravitation. However, I would like to separate carefully facts from the ideology. Results presented in Section 4 are facts (even if virtually unknown). Then, I show how different definitions of gravitational energy may be obtained from these facts as soon as we decide how to control gravitational boundary data. Any such decision belongs to the ideology. Among all of them, there is one which is especially promising. My arguments for this choice are not, at the moment, very strong, but are based on some mathematical conjectures, strongly supported by the analysis of the linearized gravity theory. If these conjectures are false, my choice of the “boundary control mode” in canonical gravity and, consequently, the definition of the gravitational energy, must be replaced by something else. But there is no way to avoid the facts which I discuss in Section 4 or to improve them by any – even the most sophisticated – boundary correction to the gravitational action. They must be taken into account as a starting point of any serious approach to the gravitational energy.
2 Role of boundary phenomena in Hamiltonian field theory: an example

As a starting point of our approach we take a version of canonical field theory which is not only “volume sensitive” but also “boundary sensitive” (see [11], and [4]) and where boundary terms are fully legitimate. We illustrate this formalism on the following, simple example:

Suppose that somebody has proved a physical significance of the theory of a vector field $\phi^\mu$, derived from the Lagrangian:

$$L = \frac{1}{2} \left\{ g^{\mu\nu} \phi^\mu \phi^\nu + \frac{1}{m^2} (\partial_\mu \phi^\mu)^2 \right\}, \quad (1)$$

where $g^{\mu\nu}$ denotes the Minkowski flat metric with signature $(-, +, +, +)$. According to Bogoliubov and Shirkov, we define the corresponding field momenta:

$$\Pi_\mu := \frac{\partial L}{\partial (\dot{\phi}^\mu)} \quad , \quad (2)$$

(“dot” denotes always time derivative: $\dot{\phi}^\mu = \partial_0 \phi^\mu$) and define the field Hamiltonian by the formula:

$$H = \Pi_\mu \dot{\phi}^\mu - L \quad . \quad (3)$$

Energy defined this way is not necessarily positive and has further bad properties. To analyze its physical significance, let us describe the theory in a complete canonical way, according e. g. to [11] or [4]. For this purpose we define the complete canonical momentum

$$P^{\nu}_\mu := \frac{\partial L}{\partial (\partial_\nu \phi^\mu)} \quad ; \quad (4)$$

(the momentum $\Pi$ describes only its temporary components: $\Pi_\mu = P^0_\mu$) and write field equations generated by (1) in the following way:

$$\delta L(\phi^\mu, \partial_\nu \phi^\mu) = \left\{ \partial_\nu \left( P^{\nu}_\mu \delta \phi^\mu \right) \right\} = (\partial_\nu P^{\nu}_\mu) \delta \phi^\mu + P^{\nu}_\mu \delta (\partial_\nu \phi^\mu) \quad . \quad (5)$$

(Here, applying the variation operator $\delta$, we always mean that we have a one-parameter family of fields $\phi^\mu(x^\lambda; \epsilon)$. Variation consists in taking derivative with respect to $\epsilon$, at $\epsilon = 0$.) The above equation contains both the
definition (4) of the conjugate momenta and the Euler-Lagrange equations of the theory:

\[ \partial_\nu P_{\mu} = \frac{\partial L}{\partial \dot{\varphi}^\mu}. \]  

(6)

At this point the reader probably noticed that this theory is just the Klein-Gordon theory, written in a fancy way. Indeed, denoting

\[ \varphi := -\frac{1}{m^2} \partial_\mu \phi^\mu, \]

(7)

we obtain from (4) the following constraint for the momenta:

\[ P_{\mu} = -\delta_{\nu}^\mu \varphi. \]

(8)

(By \( \delta_{\mu}^\nu \) we denote the Kronecker “delta”, while \( \delta \) denotes always the field variation.) Hence, Euler-Lagrange equation (6) is equivalent to

\[ -\partial_\mu \varphi = g_{\mu\nu} \phi^{\nu} = \phi_\mu, \]

(9)

and equation (7), which was “merely a notation”, becomes the Klein-Gordon equation:

\[ \Box \varphi = m^2 \varphi. \]

In terms of these quantities, formula (3) for the “energy” \( H \) may be rewritten as follows:

\[ H = \frac{1}{2} \left\{ \dot{\varphi}^2 + \nabla \varphi \cdot \nabla \varphi + m^2 \varphi^2 \right\}. \]

(10)

Putting \( \varphi = \Delta \varphi - m^2 \varphi \) from the Klein-Gordon equation, we have

\[ H = -\nabla (\varphi \nabla \varphi) + \frac{1}{2} \left\{ (\dot{\varphi})^2 + (\nabla \varphi)^2 + m^2 \varphi^2 \right\}, \]

(11)

which differs by a boundary term from the “correct” energy of the Klein-Gordon field, known to be a manifestly positive and convex quantity:

\[ \tilde{H} = \frac{1}{2} \left\{ (\dot{\varphi})^2 + (\nabla \varphi)^2 + m^2 \varphi^2 \right\}. \]

(12)

In spite of this discovery, let us continue our analysis of the theory in terms of the variational principle (1) and the corresponding canonical quantities \( (\Pi_\mu, \phi^\mu) \), because we want to find a universal approach to the energy problem, which does not rely on such “lucky guesses”. For this purpose, we integrate the fundamental generating formula (5) over a three-dimensional region with boundary \( \Sigma \subset \{ t = \text{const.} \} \). Using Stokes theorem to convert
terms containing spacelike derivatives $\partial_k$, $k = 1, 2, 3$ into a surface integral over the boundary $\partial \Sigma$, we obtain:

$$
\delta \int L = \int \partial_0 \left( P^0_\mu \delta \phi^\mu \right) dV + \int_{\partial \Sigma} P^\perp_\mu \delta \phi^\mu d\sigma
= \int \left( \Pi_\mu \delta \phi^\mu + \Pi_\mu \delta \phi^\mu \right) dV + \int_{\partial \Sigma} P^\perp_\mu \delta \phi^\mu d\sigma .
$$

(13)

At a certain point constraints (8) must be implemented, either now or, equivalently, after performing the Legendre transformation. The transformation consists in using identity

$$
\Pi_\mu \delta \phi^\mu \delta \phi^\mu = \delta \left( \Pi_\mu \phi^\mu \right) - \dot{\phi}^\mu \delta \Pi_\mu ,
$$

and leads to the Hamiltonian generating formula:

$$
-\delta \mathcal{H} = \int \left( \dot{\Pi}_\mu \delta \phi^\mu - \dot{\phi}^\mu \delta \Pi_\mu \right) dV + \int_{\partial \Sigma} \left( P^\perp_\mu \delta \phi^\mu \right) d\sigma ,
$$

(14)

where by $\mathcal{H}$ we denote the total Hamiltonian

$$
\mathcal{H} := \int H dV = \int \left( \Pi_\mu \dot{\phi}^\mu - L \right) dV .
$$

Using constraints (8) (i. e. $\Pi_k = 0$, $\Pi_0 = -\varphi$) we rewrite (14) as follows:

$$
-\delta \mathcal{H} = \int \left( -\dot{\varphi} \delta \phi^0 + \dot{\phi}^0 \delta \varphi \right) dV - \int_{\partial \Sigma} \left( \varphi \delta \phi^\perp \right) d\sigma ,
$$

(15)

or, denoting $\phi^0 =: \pi$,

$$
-\delta \mathcal{H} = \int \left( \dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi \right) dV - \int_{\partial \Sigma} \left( \varphi \delta \phi^\perp \right) d\sigma .
$$

(16)

Neglecting boundary terms, both formulae (14) and (16) describe infinite dimensional Hamiltonian systems. The phase space is parameterized by 8 functions ($\Pi_\mu, \phi^\mu$) in the first and by 2 functions ($\pi, \varphi$) in the second case. Both formulae are analogous to formula

$$
-\delta H(p_i, q^i) = \sum_i \dot{p}_i \delta q^i - q^i \delta p_i \quad ( = \omega(\mathcal{X}, \mathcal{Y}) ) .
$$

(17)
equivalent to Hamiltonian equations of motion in mechanics:

\[
\begin{align*}
\dot{p}_i &= -\frac{\partial H}{\partial q^i}, \\
\dot{q}^i &= \frac{\partial H}{\partial p_i}.
\end{align*}
\]

(18)

The right-hand-side of (17) denotes the value of the symplectic 2-form

\[
\omega = dp_i \wedge dq^i,
\]

(19)

taken on a pair \((X,Y)\) of vectors, where

\[
X = \dot{p} \frac{\partial}{\partial p} + \dot{q} \frac{\partial}{\partial q},
\]

(20)

is the Hamiltonian vector field tangent to the trajectory and

\[
Y = \delta p \frac{\partial}{\partial p} + \delta q \frac{\partial}{\partial q},
\]

(21)

is a generic variation, tangent to the phase space (i.e. we have \(\delta H := Y(H)\)).

The above, Hamiltonian interpretation of (16) (or, equivalently, (14)), together with the definition of the corresponding symplectic structures

\[
\omega = \int_\Sigma d\Pi(x) \wedge d\phi^\mu(x) = \int_\Sigma d\pi(x) \wedge d\phi(x),
\]

(22)

(where the second one obtained via reduction from the first one) is possible only when the boundary term is annihilated. For this purpose we must impose the field boundary data on \(\partial \Sigma\). More precisely, we take as a phase space of the system the space \(\mathcal{P}\) of such Cauchy data \((\pi, \varphi)\), that the function \(\varphi\) fulfills Neuman condition: \(\partial^\perp \varphi = p^\perp |_{\partial \Sigma} = f\), with \(f\) given. Within this space we have \(\delta p^\perp |_{\partial \Sigma} = 0\) and, therefore, the boundary term vanishes, which is, indeed, equivalent to the infinite dimensional version of (17):

\[
-\delta \mathcal{H} = \int_\Sigma (\dot{\pi} \delta \varphi - \dot{\varphi} \delta \pi) \, dV.
\]

(23)

The quantity \(\mathcal{H}\), although different from the “true energy” (12), is a fully legitimate field Hamiltonian. Moreover, we understand now its relation with the other energy. Performing boundary Legendre transformation:

\[
\varphi \delta \phi^\perp = \delta (\phi^\perp \varphi) - \phi^\perp \delta \varphi
\]
in (16) and putting the total variation $\delta (\phi^\perp \varphi)$ on the left-hand-side, we obtain:

$$-\delta \tilde{H} = \int_{\Sigma} (\dot{\varphi} \delta \pi - \dot{\pi} \delta \varphi) \, dV + \int_{\partial \Sigma} (\phi^\perp \delta \varphi) \, d\sigma , \quad (24)$$

where $\tilde{H}$, given by density (12), plays role of the Hamiltonian. To be able to interpret it as an infinite dimensional Hamiltonian system we must again annihilate the boundary term. For this purpose we take as phase space of the system the space $\tilde{P}$ of such Cauchy data $(\pi, \varphi)$, that the function $\varphi$ fulfills Dirichlet condition: $\varphi|_{\partial \Sigma} = h$. Within this space we have $\delta \varphi|_{\partial \Sigma} = 0$ and, therefore, the boundary term vanishes, which is, indeed, equivalent to the infinite dimensional version of (17):

$$-\delta \tilde{H} = \int_{\Sigma} (\dot{\varphi} \delta \pi - \dot{\pi} \delta \varphi) \, dV . \quad (25)$$

At a first glance, the right-hand-sides of (23) and (25) are identical, but this is not true. They describe field evolution in two different physical arrangements, described mathematically by two different functional spaces: $P$ in the first and $\tilde{P}$ in the second case. Controlling boundary conditions means controlling the way the field in the interior of $\Sigma$ interacts with the rest of the World. Of course, there are infinitely many Hamiltonian systems which one may assign to a given field theory: one could control various combinations of Dirichlet and Neuman data over different pieces of $\partial \Sigma$. The question arises: Is there any criterion to chose one of them as a “fundamental control mode”, to consider its Hamiltonian as a “true field energy” and to call the corresponding boundary conditions as “adiabatic insulation” of $\Sigma$?

The answer “Dirichlet always better than Neumann”, suggested by the fact that we probably prefer $\tilde{H}$ as the field energy, is not a clever response, because what is “Neuman” for $\varphi$ is “Dirichlet” for $\phi$ and vice-versa. Formula (24), after one more Legendre transformation, may be converted into the standard Klein-Gordon Lagrangian formula:

$$\delta \tilde{L}(\varphi, \partial_{\nu} \varphi) = \{ \partial_{\mu} (p^\mu \delta \varphi) = \} \quad (\partial_{\mu} p^\mu) \delta \varphi + p^\mu \delta (\partial_{\mu} \varphi) , \quad (26)$$

with

$$\tilde{L}(\varphi, \partial_{\nu} \varphi) = -\frac{1}{2} \{ g^{\mu \nu} (\partial_{\mu} \varphi)(\partial_{\nu} \varphi) + m^2 \varphi^2 \} , \quad (27)$$
which is an equivalently correct Lagrangian of the theory. We see, therefore, that there is no correlation between the variational formulation we start with and the possible definition of the filed energy.

In our opinion, the only criterion which distinguishes \( \tilde{H} \) among all other candidates for the energy, is its positivity or, more precisely, the fact that it is bounded from below and convex. These are the fundamental physical properties which ensure stability of the physical system in question. Positivity is, therefore, the very reason to call \( \tilde{H} \) the field energy. Other functionals, like e. g. \( H \), play role of a “free energy” and contain also a part of energy of the device used to control physically the boundary data (a “thermostat”).

We stress, however, that any of the two Hamiltonians (\( H \) or \( \tilde{H} \)) may be obtained from any of the two Lagrangians. We conclude that no Lagrangian manipulation is neither useful nor necessary to recognize the field energy: we need for this purpose only a deep analysis of the field dynamics, and especially of the boundary value problem in the field evolution.

3 Volume part of Canonical Relativity

The above conclusion is especially true in General Relativity theory, where we may use different variational formulations. In the “purely metric” formulation, variation is taken with respect to the metric tensor \( g_{\mu\nu} \). The corresponding Hilbert Lagrangian \( L = \frac{1}{16\pi} \sqrt{|g|} R \) is of the second differential order. We may also use the non-invariant, first order, Einstein Lagrangian, obtained by subtracting a complete divergence from the Hilbert Lagrangian. Palatini proposed another (the so called “metric-affine”) formulation, where variation is taken with respect to both the metric and the connection \( \Gamma^\lambda_{\mu\nu} \), treated \textit{a priori} as independent quantities. Finally, I have proposed (see [10]) a “purely-affine” formulation, where the metric does not enter into the Lagrangian function and variation is taken with respect to the connection only. In this approach, metric tensor arises as a momentum canonically conjugate to the connection. Typically, the affine Lagrangian is of the form 
\[
L(\Gamma, \partial\Gamma) = c \cdot \sqrt{|det R_{(\mu\nu)}|}.
\]

Above variational formulations of General Relativity (and also other formulations, which may be obtained from them \textit{via} boundary manipulations) lead to the same volume part of Canonical Gravity. Its structure may be described as follows. Given a space-like hypersurface \( \Sigma \subset M \) (possibly \textit{with} boundary), embedded in a general relativistic space-time \( M \), denote by \( n \) the
Consider the extrinsic curvature of $\Sigma$:

$$K_{mn} = (\nabla_m \partial_n | n) = -(\partial_n | \nabla_m n) ,$$  

and its trace $K = K_{mn} \tilde{g}^{mn}$, where by $\tilde{g}^{mn}$ we denote the three dimensional inverse to the restriction $g_{kl}$ to $\Sigma$ of the metric $g_{\mu\nu}$ ($k, l = 1, 2, 3; \mu, \nu = 0, 1, 2, 3$). Take the Arnowitt-Deser-Misner momentum defined as follows:

$$P^{kl} = \sqrt{\det \tilde{g}} (K \tilde{g}^{kl} - K^{kl}) .$$  

The volume part $\omega_\Sigma$ of the symplectic form, defined in the space of Cauchy data $(P^{kl}, g_{kl})$ on $\Sigma$, the same for all the four variational formulations, equals

$$\omega_\Sigma = \frac{1}{16\pi} \int_\Sigma dP^{kl}(x) \wedge dg_{kl}(x) .$$  

This description of the phase space for gravity is correct only “in principle” because of the following problems:

- A boundary correction $\omega_\partial \Sigma$ to the symplectic form is necessary. We are going to derive it in the sequel and to show that the correct total symplectic structure is given by $\omega_\Sigma + \omega_\partial \Sigma$.
- There are Gauss-Codazzi constraints imposed on data $(P^{kl}, g_{kl})$.
- There is an extra gauge invariance, dual to constraints. This implies that the “true” phase space of gravity is described by the quotient space of classes of data modulo gauge transformations.

In spite of these problems, formula (30) contains, as will be seen later, the only “variational” ingredient which is necessary for the construction of the satisfactory canonical gravity theory.

Hence, we are going to define gravitational energy contained within a generic, two-dimensional, compact boundary $S = \partial \Sigma$, as a quasi-local quantity (for the “free gravity” version of these results see [7]; for a generalized
version, when the interacting “gravity + matter fields” systems were analyzed, see [8]). Then, total energy may be obtained \textit{via} a limiting procedure, when the surface $S$ goes to infinity (spatial infinity for the A. D. M.-mass and null infinity for the Trautman-Bondi-mass). In this paper we present an improved version of this result, where the vector field $X$ generating dynamics is not necessarily time-like (as was assumed in the old versions of the theory). This way, not only the \textit{quasi}-local energy and static momentum may be defined, but also the \textit{momentum} and the \textit{angular momentum}.

4 Generating formula for Hamiltonian gravity: the facts

In this Section we present the “homogeneous generating formula” for Einstein equations, which has to be used as a starting point for the construction of Canonical Relativity. We stress that the result presented here does not depend upon any “ideology”, which one might choose to formulate General Relativity theory or to derive its equations from any kind of a “least action principle”. The formula is equivalent to Einstein equations.

Suppose that the following three objects have been chosen in a general relativistic spacetime $M$: 1) a two dimensional, spacelike surface $S \subset M$, 2) a three dimensional spacelike hypersurface $\Sigma \subset M$, such that $\partial \Sigma = S$, 3) a vector field $X$ defined in a neighbourhood of $\Sigma$:

Suppose, moreover, that a one-parameter family of solutions of Einstein equations $g_{\mu\nu} = g_{\mu\nu}(x; \sigma)$, defined in a neighbourhood of $\Sigma$, has been chosen. Dragging these solutions along the vector field $X$ we may construct a two-parameter family of solutions $g_{\mu\nu} = g_{\mu\nu}(x; \tau, \sigma)$, where $\tau$ is the parameter of the group of diffeomorphisms generated by $X$. Let $(P^{kl}(\tau, \sigma), g_{kl}(\tau, \sigma))$ denote the corresponding Cauchy data on $\Sigma$. Take the following two vectors
in the space Cauchy data:

\[
\mathcal{X} = \begin{cases} 
\dot{g} = \frac{\partial g}{\partial \tau} = \mathcal{L}_X g , \\
\dot{P} = \frac{\partial P}{\partial \tau} = \mathcal{L}_X P , 
\end{cases} \tag{31}
\]

\[
\mathcal{Y} = \begin{cases} 
\delta g = \frac{\partial g}{\partial \sigma} , \\
\delta P = \frac{\partial P}{\partial \sigma} , 
\end{cases} \tag{32}
\]

and calculate their symplectic product according (30):

\[
\omega_\Sigma(\mathcal{X}, \mathcal{Y}) = \frac{1}{16\pi} \int_\Sigma \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} . \tag{33}
\]

Below, we give the result which is valid not only for pure gravity, but also for a broad class of matter fields \(\phi\) interacting with gravity. We stress that, once we know the symplectic form \(\omega_\Sigma\), no variational principle is necessary to calculate (33), and only the field equations are needed. In fact, the proof given in [7] used the “purely affine” variational principle whereas in [8], the same theorem was derived from the Hilbert “purely metric” Lagrangian. It is, however, an interesting and highly instructive exercise (which we leave to the reader) to derive the formula explicitly from Einstein equations and matter field equations.

**Theorem 1:** If \((g_{\mu\nu}(x; \tau, \sigma), \phi(x; \tau, \sigma))\) is a two parameter family of solutions of the interacting system: “Einstein equations + matter field equations”, if \(X = \partial/\partial \tau\) (i. e. if \(\mathcal{L}_X g = \dot{g}\)) and if \((P^{kl}(\tau, \sigma), g_{kl}(\tau, \sigma), \pi(\tau, \sigma), \phi(\tau, \sigma))\) are corresponding Cauchy data on \(\Sigma\), then the following identity holds

\[
0 = \frac{1}{16\pi} \int_\Sigma \dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} + \int_\Sigma \dot{\pi} \delta \phi - \dot{\phi} \delta \pi + \frac{1}{8\pi} \int_S \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda
\]

\[
+ \frac{1}{16\pi} \int_S \{2n \delta(\lambda k) + 2n^A \delta(\lambda \ell_A) + Q^{AB} \delta g_{AB}\} + \int_S p^+ \delta \varphi , \tag{34}
\]

where the following notation has been used. If \((x^A)\), \(A = 1, 2\); are coordinates on \(S\), then

\[
\lambda = \sqrt{\det g_{AB}} \, d^2x \tag{35}
\]
is the volume form on $S$. The field $X$ has been decomposed into the part tangent to $S$, which we denote by $X^\parallel = n^A \partial_A$, and the part $X^\perp$, orthogonal to $S$. We have, therefore, $X = X^\perp + X^\parallel$ and denote: $n := \pm \sqrt{|(X^\perp X^\perp)|}$ where “+” is taken if $X$ is timelike and “−” if $X$ is spacelike. In the two dimensional plane orthogonal to $S$ (which may be identified with the two dimensional Minkowski space) we use the following three normalized vectors: $\mathbf{N} := \frac{1}{n} X^\perp$, $\mathbf{M}$ – orthogonal to $\mathbf{N}$ and $\mathbf{m}$ – tangent to $\Sigma$, directed outwards (we remind the reader that $\mathbf{n}$ was the unit vector orthogonal to $\Sigma$).

By “$\alpha$” we denote the “hyperbolic angle” between $\mathbf{N}$ and $\mathbf{n}$, defined as follows:

$$
\alpha = \begin{cases} 
\text{arsinh}(\mathbf{N} \mid \mathbf{m}) & \text{for } X^\perp \text{ time-like,} \\
\text{sgn}(\mathbf{N} \mid \mathbf{m})\text{arcosh}(\mathbf{N} \mid \mathbf{m}) & \text{for } X^\perp \text{ space-like.}
\end{cases} 
$$

(36)

We also use the extrinsic geometry of $S$: the torsion covector

$$
\ell_A := (\nabla_A \mathbf{N} \mid \mathbf{M}) = \frac{1}{n}(\nabla_A X^\perp \mid \mathbf{M}) ,
$$

(37)

and the symmetric curvature tensor $k$ in the direction of $\mathbf{M}$

$$
k_{AB} = k_{AB}(\mathbf{M}) := (\nabla_A \partial_B)\mathbf{M} .
$$

(38)

This means, that for any pair $(Y, Z)$ of vector fields tangent to $S$ we have: $k(Y, Z) = (\nabla_Y Z \mid \mathbf{M})$. Finally, we consider also the “acceleration” scalar

$$
s = (\nabla_X X \mid \mathbf{M}) = \mathcal{L}_X g(X, \mathbf{M}) - \frac{1}{2} M(X \mid X) .
$$

(39)
Using the two dimensional inverse $\tilde{g}^{AB}$ to the metric $g_{AB}$ on $S$ we define the trace $k = \tilde{g}^{AB}k_{AB}$ and the following tensor density:

$$Q^{AB} = \lambda \left\{ \left( \frac{8}{n} - 2 n^c \ell_c - n^c n^d k_{cd} \right) \tilde{g}^{AB} + n \left( k^{AB} - k^{AB} \right) \right\}. \tag{40}$$

This completes the list of geometric objects used in (34). The “hamiltonian part” (first three terms) of this formula implies the following symplectic structure in the space of Cauchy data for the total “gravity + matter” system:

$$\omega = \frac{1}{16\pi} \int_{\Sigma} dP^{kl} \wedge dg_{kl} + \frac{1}{8\pi} \int_S d\lambda \wedge d\alpha + \int_{\Sigma} d\pi \wedge d\varphi. \tag{41}$$

It contains not only the gravitational volume part (30) and the matter field part (22), but is supplemented by the gravitational surface part $\omega_{\partial S}$ (the second term on the right hand side). This supplement is necessary for gauge invariance of this symplectic structure.

**Definition:** Given $S$ and $X$, by gauge transformations we mean those space-time diffeomorphisms which do not move points of $S$ and their trajectories under the group generated by $X$.

**Theorem 2:** Symplectic structure (41) is invariant with respect to the above gauge transformations.

The zero on the left hand side of (34) does not mean that the Hamiltonian of the “gravity + matter” system vanishes. Indeed, this formula is analogous to the so called “homogeneous formulation” of mechanics of point particles. Consider spacetime coordinates $(q^\mu)$, $\mu = 0, 1, 2, 3$; of a particle and the corresponding four momenta $p_\mu$. Hamiltonian mechanics (relativistic or non-relativistic) may be formulated in terms of the following two equations:

$$0 = \dot{p}_\mu \delta q^\mu - \dot{q}^\mu \delta p_\mu, \quad \tag{42}$$

$$0 = p_0 + H(p_k, q^k, q^0). \quad \tag{43}$$

In this formulation, the parameter $t$ along a trajectory is a pure gauge quantity and has no physical meaning. The theory is invariant with respect to re-parameterizations of the trajectories. This is a consequence of the fact that the “control parameters” $(p_\mu, q^\mu)$ in generating formula (42) are not free, but subject to constraint (43). We derive the standard (3+1)-formulation of mechanics via gauge fixing:

$$t \equiv q^0 \quad \Rightarrow \quad q^0 = 1 \quad \text{and} \quad \delta q^0 = 0. \quad \tag{44}$$
This implies:

\[ 0 = \dot{p}_0 \delta q^0 - q^0 \delta p_0 + \dot{p}_k \delta q^k - q^k \delta p_k = \delta H + \dot{p}_k \delta q^k - q^k \delta p_k , \tag{45} \]

or, simply

\[ -\delta H(p_k, q^k, t) = \dot{p}_k \delta q^k - q^k \delta p_k . \tag{46} \]

Similarly, boundary control parameters in equation (34) are subject to constraints – see [8]. As a consequence, these parameters do not imply the value of the time lapse at the boundary. To derive Hamiltonian dynamics from (34), we must choose boundary conditions in such a way, that they uniquely fix the time coordinate at the boundary. Here, different choices are possible. They must correspond to well posed boundary value problems for Einstein equations. We believe that the correct choice would be the one which leads to a positive – and, preferably, convex – energy functional. An analysis of the positivity theorem for the global gravitational energy given in [9] is rather encouraging. At the moment, however, we do not know whether or not such a “good choice” is possible and we do not see any uniqueness in the choice of boundary conditions. We have, however, some conjectures which we present in the next Section (see also [8]).

Concluding this Section, we would like to stress that formula (34) is valid not only in the simplest case of a scalar field interacting with gravity, but also for a wide class of matter fields, including electromagnetism and gauge fields.

5 Examples of gravitational Hamiltonians

As an example of a possible choice of boundary conditions we may take the one obtained via the following Legendre transformation at the boundary:

\[ n \delta(\lambda k) = \delta(n\lambda k) - \lambda k \delta(n) , \tag{47} \]

\[ n^A \delta(\lambda \ell_A) = \delta(n^A \ell_A) - \lambda \ell_A \delta(n^A) . \tag{48} \]

This enables us to rewrite (34) in the following way:

\[
-\delta H = \frac{1}{16\pi} \int_\Sigma \dot{\pi}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl} + \int_\Sigma \dot{\pi} \delta \varphi - \varphi \delta \pi + \frac{1}{8\pi} \int_S \dot{\lambda} \delta \alpha - \dot{\alpha} \delta \lambda
+ \frac{1}{16\pi} \int_S -2\lambda k \delta(n) - 2\lambda \ell_A \delta n^A + Q^{AB} \delta g_{AB} + \int_S p^k \delta \varphi . \tag{49} \]
with the Hamiltonian given by:

\[ H = \frac{1}{16\pi} \int_S (2\lambda n^A \ell_A + 2\lambda nk) + E_0 . \]  

(50)

This choice consists, therefore, in controlling internal geometry of the three dimensional world tube obtained from \( S \) be dragging it along \( X \). In particular, putting \( n = 1 \) and \( n^A = 0 \) and controlling in formula (49) also the two dimensional metric \( g_{AB} \) on the boundary (together with Dirichlet data \( \varphi|_S \) for the matter field), we may define the total energy of the “gravity + matter” system:

\[ E = \frac{1}{8\pi} \int_S \lambda k + E_0 . \]  

(51)

where the additive constant \( E_0 \) (always free in the Hamiltonian formalism) may be fixed in such a way that \( E = 0 \) for the empty Minkowski spacetime. Although the above formula resembles the Brown-York proposal of defining the gravitational energy (see [3]), we stress that the latter contains the curvature \( \kappa \) of \( S \), considered as a submanifold embedded in \( \Sigma \) (i.e. \( \kappa \) is taken with respect to the vector \( m \) and not \( M \), as in (51) and, whence, is not gauge invariant). Putting \( n = 0 \) in (50), and taking vector \( n^A \) equal to the generator of a translations or rotations, we may define in a similar way the total momentum and the angular momentum of the system.

In our opinion, the above choice of boundary conditions is not the best one. Analyzing the linearization of the gravitational energy in an asymptotic region, i.e. when gravitational field on \( S \) is very weak, I came to conclusion that the correct energy control mode must be somehow related with the one obtained \textit{via} the following Legendre transformation:

\[ 2n\delta(\lambda k) = \delta(\lambda nk) - \lambda \kappa^2 \delta \left( \frac{\kappa}{2} \right) + nk\delta\lambda . \]  

(52)

The last term vanishes when \( g_{AB} \) is controlled, since

\[ \delta \lambda = \frac{1}{2} \lambda \overset{\sim}{g}^{AB} \delta g_{AB} = 0 . \]  

(53)

Hence, we may define a Hamiltonian related to the control the value of \( b := \frac{k}{n} \). Fixing a “standard” value of \( b \) and keeping \( n^A = 0 \) we thus obtain for energy the following expression:

\[ E = \frac{1}{16\pi} \int_S n\lambda k + E_0 = \frac{1}{16\pi} \int_S \lambda \frac{k^2}{b} + E_0 . \]  

(54)
As the “standard” value of the parameter $b$ we take the extrinsic curvature $k$ of the local, isometric embedding of $S$ into the three dimensional Euclidean space $E^3$. If, e. g., $g_{AB}$ is a sphere of radius $r$, then $b = -\frac{r}{2}$. The constant $E_0$ must be chosen in such a way that $E$ vanishes for the Minkowski space. Hence, $E_0 = \frac{r}{2}$. This formula works especially well for the Schwarzschild solution, where it gives the correct value of mass on any Schwarzschild sphere (see [8] for a more detailed discussion).

6 Rigid shells in General Relativity

In Special Relativity Theory, we do not expect any specific properties of a Hamiltonian assigned to a generic triplet $(S, X, \Sigma)$. Nice properties, which should be fulfilled by a “good energy functional”, are expected only when $\Sigma$ is flat and $X$ is orthogonal to $\Sigma$. In the general relativistic framework, the hypersurface $\Sigma$ is no longer relevant, since it is a gauge quantity, but its flatness may be translated into the following property of $S$.

**Definition:** A two dimensional, spacelike submanifold $S \subset M$, homeomorphic with the sphere $S^2$, is called a rigid shell if there exist a non-vanishing vector field $N$ orthogonal to $S$, such that the external curvature of $S$ in direction of $N$ vanishes: $k_{AB}(N) \equiv 0$. The manifold is called weakly rigid if there is a non-vanishing vector field $N$ orthogonal to $S$, such that the traceless part of $k(N)$ vanishes:

$$k_{AB}(N) - \frac{1}{2}g_{AB} g^{CD}k_{CD}(N) \equiv 0.$$ 

In Minkowski space, every weakly rigid shell is also strongly rigid and may be embedded in a flat Euclidean hyperplane $\Sigma$. Below we illustrate the fact, that folding $\Sigma$ in such a way that its internal geometry does not change, we may obtain a shell $S'$, whose internal geometry is isometric with internal
geometry of a rigid shell $S$, but which is no longer rigid.

A rigid (or weakly rigid) shell defines automatically a reference frame: vector $\mathbf{N}$ from the Definition gives the time direction, whereas its orthonormal vector $\mathbf{M}$ span (together with vectors tangent to $S$) the local space directions. There is a conjecture that, in every asymptotically flat spacetime, there are “sufficiently many” weakly rigid shells, having a given internal geometry $g_{AB}$. If this conjecture is true, rigid spheres might be used to construct “good reference frames” in asymptotically flat regions of spacetime. Such frames would be unique up to an asymptotic Poincaré transformation. This way supertranslation ambiguities would be eliminated. Another conjecture says that a weakly rigid shell, which is a solution of the hamiltonian system (49) with control parameters constant in time and with vanishing matter field $\varphi|_S$, must also be strongly rigid.

Analyzing properties of the quasilocal energy, defined in the previous Section, we came to yet another conjecture, that the positivity theorem might be satisfied for rigid shells, even if it is not universally valid. At the moment, we have no proof of this conjecture, but the work is already in progress.

References


