I. INTRODUCTION

Evolution of a gravitational field within a finite tube with a timelike boundary was derived using a consistent, Hamiltonian formulation in [1] and then reformulated in [2]. Here we extend this description to the case of a null-like boundary or a wave front, i.e., a three-dimensional submanifold whose internal metric is degenerate. Restricting our result to the special case of wave fronts, namely, to nonexpanding horizons, we obtain a generalization of the first law of thermodynamics for black holes as a simple consequence.

Contrary to the Iyer and Wald approach (see [3]), no assumption about stationarity (existence of a Killing field) is necessary here. Such an assumption finally reduces our formula to the standard first law.

In many presentations of the Hamiltonian field theory (cf. [4]) boundary problems are neglected. Consequently, all the surface integrals arising from the integration by parts are assumed to vanish. Here we use the formulation proposed in [5,6], where the field boundary data are treated on the same footing as the Cauchy data. This is the only way to obtain a mathematically consistent (infinite-dimensional) Hamiltonian description of any field theory if the boundary of the space volume taken into account is nontrivial.

To illustrate our approach, consider as an example the linear theory of an elastic, finite string. Field configuration of the string is described by its displacement function: \( \mathbb{R} \times [a, b] \ni (t, x) \rightarrow \varphi(t, x) \in \mathbb{R} \), fulfilling the wave equation:

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \frac{\partial^2}{\partial x^2} \varphi. \tag{1.1}
\]

Here, velocity “\( c \)” is a combination of the string’s proper density (per unit length) and its elasticity coefficient. Passing to appropriate time and length units, we may always put \( c = 1 \). Equation (1.1) may be derived from the Lagrangian density

\[
L = -\frac{1}{2} \sqrt{|\det g|} g^{\mu \nu} (\partial_\mu \varphi)(\partial_\nu \varphi) = \frac{1}{2}(\dot{\varphi})^2 - \varphi', \tag{1.2}
\]

where \( \mu, \nu = 0, 1 \) and \( (x^0, x^1) = (t, x) \), \( g_{\mu \nu} = \text{diag}(-1, +1) \), “dot” denotes the time derivative, and “prime” denotes the space derivative. A convenient way to encode the entire information about the dynamics of the string is to write it in a form of the following equation:

\[
\delta L(\varphi, \partial_\nu \varphi) = \partial_\mu (p^\mu \delta \varphi) = (\partial_\mu p^\mu) \delta \varphi + p^\mu \delta (\partial_\mu \varphi), \tag{1.3}
\]

equivalent to the Euler-Lagrange equations. Indeed, condition (1.3) is satisfied if and only if the volume part of the variation of \( L \) (normally present on the right-hand side) vanishes. The above equation (which we refer to as the Lagrangian generating formula of the dynamics) has a beautiful symplectic interpretation (see [5] or [6]) as a definition of a Lagrangian submanifold of physically admissible states within the symplectic space of the first jets of sections of the state bundle. Here \( (\varphi, \partial_\nu \varphi, \partial_\mu p^\mu, p^\nu) \) are local canonical coordinates in this symplectic space. Without going into deep, mathematical details, Eq. (1.3) may be simply read as the following condition imposed on the string configuration: the “response parameters” \( (\partial_\mu p^\mu, p^\nu) \) must be equal to partial derivatives of the Lagrangian \( L \) with respect to the corresponding “control parameters” \( (\varphi, \partial_\nu \varphi) \). Hence, we have the following dynamical equations of the theory:

(i) Definition of the canonical momenta:

\[
p^0 = \frac{\partial L}{\partial (\partial_0 \varphi)} = \partial_0 \varphi = \dot{\varphi}, \tag{ii}
\]

(ii) stress density

\[
p^1 = \frac{\partial L}{\partial (\partial_1 \varphi)} = -\partial_1 \varphi = -\dot{\varphi}'.
\]
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(2) Euler-Lagrange equation, equivalent to (1.1):
\[ \partial_{\mu} p^\mu = \frac{\partial L}{\partial \dot{\varphi}} = 0. \]

For purposes of the Hamiltonian description of the theory we introduce the following notation:
\[ \pi := p^0, \quad \pi^\perp := p^1. \]

Integrating the infinitesimal generating formula (1.3) over the entire string \([a, b]\) we obtain the spatially finite generating formula (it is still infinitesimal with respect to the time variable):
\[ \delta \int_a^b L = \int_a^b (\pi \delta \varphi + \pi^\perp \delta \pi) + [\pi^\perp \delta \varphi]^b_a, \]

A Hamiltonian description of the same dynamics is obtained via the Legendre transformation between \(\pi\) and \(\varphi\), putting \(\pi \delta \varphi = \delta (\pi \varphi) - \varphi \delta \pi\):
\[ -\delta \tilde{\mathcal{H}} = \int_a^b (\pi \delta \varphi - \varphi \delta \pi) + [\pi^\perp \delta \varphi]^b_a, \]

with
\[ \mathcal{H} := \int_a^b (\pi \varphi - L) = \frac{1}{2} \int_a^b (\pi^2 + (\varphi')^2). \]

Equation (1.6) acquires an infinitely dimensional, Hamiltonian meaning
\[ \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \varphi}, \quad \dot{\varphi} = \frac{\delta \mathcal{H}}{\delta \pi}, \]

as soon as the boundary term in (1.6) is killed. This may be done in many ways, by imposing appropriate boundary conditions. Physically, this corresponds to a choice of a specific device controlling the behavior of the extremal points of the string. Mathematically, such a choice implies a specific self-adjoint extension of the second order differential operator on the right-hand side of the dynamical Eq. (1.1). This makes the dynamics uniquely defined. As an example consider two such choices: the Dirichlet conditions and the Neumann conditions. In the Dirichlet mode we restrict ourselves to an infinitely dimensional phase space of initial data \((\varphi, \pi)\), defined on \([a, b]\) and fulfilling conditions: \(\varphi(a) = A, \varphi(b) = B\). Within this phase space we have \(\delta \varphi(a) = \delta \varphi(b) = 0\) and Eqs. (1.8) hold.

Instead of controlling the string configuration \(\varphi\) at the boundary, we may control its stress by applying an appropriate force \(F\). This leads to the Neumann control mode: \(\pi^\perp(a) = F_{\text{left}}, \pi^\perp(b) = F_{\text{right}}\). Consequently, \(\delta \pi^\perp\) vanishes at the boundary. Performing the Legendre transformation between \(\varphi\) and \(\pi^\perp\) at the boundary \(\pi^\perp \delta \varphi = \delta (\pi^\perp \varphi) - \varphi \delta \pi^\perp\), we obtain again a legitimate Hamiltonian system, defined in a phase space which is completely different from the previous one and with a new Hamiltonian \(\tilde{\mathcal{H}}\) playing the role of free energy:
\[ -\delta \tilde{\mathcal{H}} = \int_a^b (\pi \delta \varphi - \varphi \delta \pi) - [\varphi \delta \pi^\perp]^b_a, \]

where
\[ \tilde{\mathcal{H}} := \mathcal{H} + [\varphi \pi^\perp]^b_a = \mathcal{H} - [\varphi \pi^\perp]^a_b = \frac{1}{2} \int_a^b (\pi^2 - (\varphi')^2 - 2 \varphi \pi^\perp). \]

Again, the boundary term in (1.9) vanishes due to Neumann conditions and the field dynamics reduces to (1.8).

Consider now the subspace of the static solutions \(\pi = 0 = \varphi\). Because of (1.8), the functional derivative of the \(\mathcal{H}\) vanishes at those points and, whence, the Hamiltonian formula (1.6) describes only the virtual work performed by the configuration controlling device at the boundary:
\[ \delta \tilde{\mathcal{H}} = -[\pi^\perp \delta \varphi]^b_a. \]

But, due to (1.7), \(\tilde{\mathcal{H}}\) is manifestly convex. This implies that every static solution corresponds to the minimal value of the Hamiltonian in the phase space defined by the Dirichlet conditions. Because of Eq. (1.1) and boundary conditions, such a solution is given by \(\pi = 0\) and \(\varphi(x) = A + (x - a) \frac{B - A}{b - a}\). Inserting this value into (1.7) we obtain the following “Penrose-like inequality”:
\[ \left( \frac{B - A}{b - a} \right)^2 \leq \mathcal{H}, \]

analogous to the gravitational Penrose inequality relating the energy carried by the Cauchy data outside of a horizon \(S\) and the energy of a black hole corresponding to the same value of appropriate boundary data on \(S\).

In the Neumann mode, the new Hamiltonian \(\tilde{\mathcal{H}}\) is obviously nonconvex. It is easy to check that the free energy (1.10) is unbounded neither from below nor from above and possesses no stationary points as soon as \(F_{\text{left}} - F_{\text{right}} \neq 0\). There is, therefore, no Penrose-like inequality in this mode.

In [1] the dynamics of the gravitational field within a timelike world tube \(S\) was analyzed in a similar way. For this purpose the so-called “affine variational principle” was used, where the Lagrangian function depends on the Ricci tensor of a spacetime connection \(\Gamma\). In this picture, the metric tensor \(g\) arises only in the Hamiltonian formulation as the momentum canonically conjugate to \(\Gamma\). Later, it was proved in [2] that the Hamiltonian dynamics obtained this way is universal and does not depend upon a specific variational formulation we start with (actually, it can be derived from field equations only, without any use of variational principles, the existence of them being a consequence of the “reciprocity” of Einstein equations—see [5,6]). On the contrary, the Hamiltonian picture is
very sensitive to the method of controlling the boundary data. A list of natural control modes, leading to different “quasilocally Hamiltonians,” is given in [2]. Each of them is related with a specific choice of control variables at the boundary. The “true mass,” which tends to the Arnowitt-Deser-Misner (ADM) mass when shifting the boundary to infinity, is one of them (see also an analysis of the linearized theory [7]).

The aim of the present paper is to generalize the above results to the case when the boundary $S$ is a wave front (a three-dimensional submanifold of spacetime $M$ whose internal three-metric $g_{ab}$ is degenerate). This way we obtain a general Hamiltonian formula for the gravitational field dynamics within (or outside) a wave front, which is very much analogous to Eq. (1.6) for the string theory. As a by-product, assuming that the wave front $S$ is very special, namely, is a nonexpanding horizon, we obtain a generalization of the first law of thermodynamics for black holes [see Eq. (4.1)]. Some of the results have been already published in [8].

II. DYNAMICS OF THE GRAVITATIONAL FIELD WITHIN A NULL HYPERSURFACE

Consider gravitational field dynamics inside a null hypersurface $S$:

Parameter $s = \pm 1$ labels two possible situations: an expanding or a shrinking wave front (if $S$ is a horizon, these correspond to a black hole or a white hole case). To simplify the notation we use coordinates $x^i$, $\mu = 0, 1, 2, 3$, adapted to the above situation: $x^0 = t$ is constant on a chosen family of (spacelike) Cauchy surfaces $\Sigma_t$ whereas $x^3$ is constant on the boundary $S$. This does not mean that $x^3$ is null like everywhere, but only on $S$. We may imagine that $x^3 = r$ coincides, far away from $S$, with the spacelike radial coordinate. Consider the three-dimensional volume $V \subset \Sigma_t$ defined as the set of those points of $\Sigma_t$ which are situated inside $S$. Coordinates $x^3, A = 1, 2$ are “angular” coordinates on the two-surface $\partial V = V \cap S$ whose topology is assumed to be that of a two-sphere. Finally, $x^k, k = 1, 2, 3$ are spatial coordinates on the Cauchy surfaces $\{x^0 = \text{const}\}$ and $x^\mu, \alpha = 0, 1, 2$ are coordinates on $S$. We stress, however, that our results are coordinate independent and will be expressed in terms of relations between geometric objects defined on $V$ and $S$.

In Appendix B we prove the following identity fulfilled by any one-parameter family of solutions of Einstein equations (“variation” operator $\delta$ may be understood as a derivative with respect to this parameter and the dot denotes the time derivative):

$$-\delta \mathcal{H} = \frac{1}{16\pi} \int_V (\dot{P}^{kl} \delta g_{kl} - \dot{g}_{kl} \delta P^{kl}) + \frac{s}{8\pi} \int_{\partial V} (\lambda \delta a - \dot{a} \delta \lambda) + \frac{s}{16\pi} \int_{\partial V} [\Lambda^{lAB} \delta g_{AB} - 2(w_0 \delta \Lambda^0 - \Lambda^B \delta w_B)].$$

(2.1)

where

$$\mathcal{H} = \frac{1}{8\pi} \int_V G^0_0 + \frac{s}{8\pi} \int_{\partial V} \lambda l = \frac{s}{8\pi} \int_{\partial V} \lambda l,$$

(2.2)

and $P^{kl}$ denotes external curvature of the Cauchy surface, written in the ADM form (cf. [9]). Moreover, $\lambda = \sqrt{\text{det} g_{AB}}$ is the two-dimensional volume form on $\partial V$ and $a = -\frac{1}{2} \log |g^{00}| = \log N$, where $N$ is the lapse function. To define the remaining objects we must choose a null (time oriented) field $K$ tangent to $S$. It is not unique, since $fK$ is also a null (time oriented) field for any (positive) function $f$ on $S$. For purposes of the Hamiltonian equation (2.1) we always choose the normalization compatible with the $(3 + 1)$ decomposition used here: $(K, dx^0) = 1$. Hence, $K = \partial_0 - n^A \partial_A$. On the contrary, the vector density $\Lambda^a = \lambda K^a = (\lambda, -\lambda n^4)$ is uniquely defined on $S$ and does not depend upon a particular choice of $K$. Now, we define

$$I_{ab} := -g(\partial_b, \nabla_a K) = -\frac{1}{2} L_{K} g_{ab},$$

(2.3)

$$w_a := -\langle \nabla_a K, dx^0 \rangle,$$

(2.4)

where $g_{ab}$ is the induced (degenerate) metric on $S$. Because of the identity $I_{ab} K^a = 0$, the null mean curvature $I = \Sigma^{AB} I_{AB}$ may be defined (it is often denoted by $\theta$ — see [11,12]), where by $\Sigma^{AB}$ we denote the inverse two-metric.

The volume term in (2.2) vanishes due to constraint equations

$$G^0_0 = 0.$$  

(2.5)

and also vanishes due to constraint equations.

The volume term in (2.2) vanishes due to constraint equations.

In Appendix C we give two independent proofs of the identity (2.1). The first one is analogous to the transition from Eq. (1.3) to Eq. (1.6). For this purpose we use Einstein equations written analogously to (1.3) (cf.

1This is the $(2 + 1)$ decomposition of the extrinsic curvature $Q^a(K)$ defined in [10,11].

In the presence of matter the volume term equals $G^0_0 - 8\pi T_0^0$ and also vanishes due to constraint equations.
geodesics. This means that five components of the boundary data: the two-metric 
\( g_{\mu\nu} \), the entire approach gauge dependent. Integrating (2.5) over a volume \( V \) and using metric constraints for the connection \( \Gamma \), we directly prove (2.1).

However, in Appendix C, an indirect proof is also provided, based on a limiting procedure, when a family \( S_e \) of timelike surfaces tends to a lightlike surface \( S \). It is shown that the nondegenerate formula derived in [1,2] gives (2.1) as a limiting case for \( e \to 0 \).

The last term in (2.1) may be written in the following way:
\[-\Lambda^A \delta w_A = \lambda n^A \delta W_A = n^A \delta W_A - n^A w_A \delta \lambda,\]
where \( W_A := \lambda w_A \) and \( n^A := g^{AB} g_{0B} \). Denoting \( \kappa := n^A w_A - w_0 = -K^a w_a \) we finally obtain the following generating formula:
\[-\delta \mathcal{H} = \frac{1}{16\pi} \int_V \left( p^{kl} \delta g_{kl} - \dot{g}_{kl} \delta p^{kl} \right) \]
\[ + \frac{s}{8\pi} \int_{\partial V} (\lambda \delta a - \dot{a} \delta \lambda) \]
\[ + \frac{s}{16\pi} \int_{\partial V} \left[ \lambda \delta a \delta g_{AB} + 2(\kappa \delta \lambda - n^A \delta W_A) \right].\] (2.6c)

It is easy to prove that the integral lines of \( K \) are null geodesics. This means that \( K^a \nabla_a K \) is always proportional to \( K \). Hence, quantity \( \kappa \) (traditionally called a “surface gravity” on \( S \)) fulfills equation \( K^a \nabla_a K = \kappa K \), which may be used as its alternative definition. We stress that its value does not correspond to any intrinsic property of the surface \( S \), but depends upon a choice of the null field \( K \) on \( S \) (i.e., upon a \( (3 + 1) \) decomposition of spacetime).

However, in a special case of a black hole thermodynamics, there is a privileged choice of \( K \), compatible with the Killing field of the stationary solution and its normalization to unity at infinity. In this case \( \kappa \) is an intrinsic property of the hole and the above formula provides, as will be seen later, the so-called first law of black hole thermodynamics.

We stress that the symplectic structure of gravitational Cauchy data is given here by the two first terms on the right-hand side of (2.6). Neglecting the second (surface) integral, the symplectic form would not be gauge invariant with respect to spacetime diffeomorphisms (see [2]). The sum of these two terms plays, therefore, the role of the integral over the string interval \([a, b]\) in Eq. (1.6). Most authors analyzing these problems take only the first (volume) integral as the symplectic form, which makes the entire approach gauge dependent.

The last integral (2.6c) is responsible for the control of five components of the boundary data: the two-metric \( g_{AB} \) on \( \partial V \) and the “curvature” \( W_A \). Assuming the null-like character of \( S \) we already control the sixth parameter: \( g^{33} \equiv 0 \) [see Eq. (A2)]. This corresponds to the general observation (cf. [2]) that we must always control four gauge parameters of the boundary \( S \), together with boundary data of the 2 “true degrees of freedom” of the gravitational field.

III. DYNAMICS OF THE GRAVITATIONAL FIELD OUTSIDE OF A NULL SURFACE

Consider now dynamics of the gravitational field outside of a wave front \( S^- \). We first add an external, timelike (nondegenerate) boundary \( S^+ \) and the situation is illustrated by the following figure:

\[ \partial V^+ \]
\[ V \]
\[ \partial V^- \]
\[ S^- \]
\[ S^+ \]
\[ S^- \]

where \( \partial V^+ = V \cap S^+ \), and \( \partial V^- = V \cap S^- \). Because \( \partial V^- \) enters with negative orientation, we have \( \int_{\partial V^-} = \int_{\partial V^+} - \int_{\partial V^-} \). Integrating again Einstein equations written in the form (2.5), over \( V \), we use techniques derived in [1,2] to handle surface integrals over the timelike surface \( S^+ \). To handle surface integrals over \( S^- \) we use our formula (2.1). This way we obtain
\[-\delta \mathcal{H} = -\delta \mathcal{H}^+ - \delta \mathcal{H}^-\]
\[ = \frac{1}{16\pi} \int_V \left( p^{kl} \delta g_{kl} - \dot{g}_{kl} \delta p^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^+} (\lambda \delta a - \dot{a} \delta \lambda) \]
\[ - \frac{s}{16\pi} \int_{\partial V} [\lambda \delta a \delta g_{AB} + 2(\kappa \delta \lambda - n^A \delta W_A)] \]
(3.1)

where \( \alpha \) is the “hyperbolic angle” between \( V \) and \( S^+ \), whereas \( Q^{ab} \) is the external curvature of \( S^+ \) written in the ADM form (cf. [2]). The contribution \( \delta \mathcal{H}^+ \) to the total Hamiltonian from the external boundary is written here in the form of a “free energy” proposed in [2]:
\[ \mathcal{H}^+ = -\frac{1}{8\pi} \int_{\partial V^+} Q_0^0 - E_0, \] (3.2)

where the additive gauge \( E_0 \) is chosen in such a way that the entire quantity vanishes if \( \partial V^+ \) is a round sphere in a flat space. The internal contribution \( \delta \mathcal{H}^- \) to the energy is given by Eq. (2.2) with \( \partial V \) replaced by \( \partial V^- \). It was proved in [2] that shifting the external boundary to space infinity \( \partial V^+ \to \infty \), the external energy \( \delta \mathcal{H}^+ \) gives the ADM mass, which we denote by \( M \), whereas the remaining surface integrals over \( \partial V^- \) vanish. Using this procedure we obtain the following generating formula for the field dy-
namics outside of an arbitrary wave front $S^-$ in an asymptotically flat spacetime:

$$-\delta \mathcal{M} - \delta \mathcal{H}^- = \frac{1}{16\pi} \int_V \left( \Phi^{kl} \delta g_{kl} - \delta g_{kl} \Phi^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^-} \left( \lambda \delta a - \dot{a} \delta \lambda \right) + \frac{s}{16\pi} \int_{\partial V^-} \left( 2(k \delta \lambda - n^A \delta W_A) \right).$$

(3.3)

**IV. BLACK HOLE THERMODYNAMICS**

In this Section we apply the above result to the situation, when the wave front $S^-$ is a nonexpanding horizon, i.e., $l = 0$ (see [11]). In this case the “internal energy” $\mathcal{H}^-$ given by (2.2) vanishes. Moreover, Einstein equations imply $\Gamma^{AB} = 0$ and the definition (2.4) of $w_a$ reduces to $\nabla_{\dot{a}} K = -w_a K$ (see [14]). Hence, we obtain the following generating formula for the black hole dynamics:

$$-\delta \mathcal{M} = \frac{1}{16\pi} \int_V \left( \Phi^{kl} \delta g_{kl} - \delta g_{kl} \Phi^{kl} \right) + \frac{s}{8\pi} \int_{\partial V^-} \left( \lambda \delta a - \dot{a} \delta \lambda \right) + \frac{s}{16\pi} \int_{\partial V^-} \left( 2(k \delta \lambda - n^A \delta W_A) \right).$$

(4.1)

where $s = 1$ for a white hole, and $s = -1$ for a black hole.

The so-called “black hole thermodynamics” consists in restricting the above analysis only to stationary situations. By stationarity we understand the existence of a timelike symmetry (Killing) vector field outside of the horizon. If such a field exists, we may always choose a coordinate system such that the Killing field becomes $\partial / \partial \chi^0$ and all the time derivatives (dots) vanish. Hence, Eq. (4.1) reduces to:

$$\delta \mathcal{M} = -\frac{s}{8\pi} \int_{\partial V^-} \left( k \delta \lambda - n^A \delta W_A \right).$$

(4.2)

Moreover, we assume that $\partial / \partial \chi^0$ is tangent to $S$. If this was not the case, we would have had a one-parameter family of horizons. Such phenomenon corresponds to the Kundt’s class of metrics (see, e.g., [15]). The known metrics of this class are not asymptotically flat. However, we do not know whether or not this is a universal property and we exclude such a pathology by the above assumption.

We have shown in [14] that there is a canonical affine fibration $\pi: S \to B$ over a base manifold $B$, whose topology is assumed to be that of a sphere $S^2$. The affine structure of the fibers is implied by the fact that they are null geodesic lines in $M$. Identity $-2l_{ab} = \mathcal{L}_K g_{ab} = 0$ implies that the metric $g$ on $S$ may be projected onto the base manifold $B$, which acquires a Riemannian two-metric tensor $h_{AB}$. The degenerate metric $g_{ab}$ on the manifold $S$ is simply the pull back of $h_{AB}$ from $B$ to $S$: $g = \pi^* h$.

The quantity $w_a$ is not an intrinsic property of the surface itself, but depends upon a choice of the null field $K$ on $S$. Indeed, if $K = \exp (-\gamma) K$ then $\tilde{w}_a = w_a + \partial_a \gamma$. In particular, there are on $S$ vector fields $K$ such that $K^a \nabla_a K = 0$ and, consequently, $\kappa = 0$. They correspond to the affine parametrization of the fibers of $\pi: S \to B$.

In the case of a black hole, there is a privileged field $K$, compatible with the timelike symmetry of the solution, which is normalized to unity at infinity. This way the quantities $\kappa$ and $w_A$ in Eq. (4.2) become uniquely defined.

We have, therefore, two symmetry fields of the metric $g_{ab}$ on $S$: $\partial_0$ and $K$. Because of normalization chosen above, we have $\langle \partial_0 - K, \delta x \rangle = 0$. Hence, the field $\tilde{n} := \partial_0 - K = n^A \partial_A$ is purely spacelike and projects on $B$. Moreover, it is a symmetry field of the Riemannian two-metric $h_{AB}$.

Because the conformal structure of $h_{AB}$ is always isomorphic to the conformal structure of the unit sphere $S^2$, we are free to choose a coordinate system in which $h_{AB} = f h_{AB}$ (and $\tilde{h}_{AB}$ denotes the standard unit two-sphere metric). The field $\tilde{n}$ is, therefore, the symmetry field of this conformal structure. Consequently, $\tilde{n}$ belongs to the six-dimensional space of conformal fields on the two-sphere. Using the remaining gauge freedom, we may choose angular coordinates $(x^A = (\theta, \phi))$ in such a way that $\tilde{n}$ becomes a rotation field on the two-sphere. This means (cf. [16] or Appendix D) that there exists a coordinate system in which the following holds:

$$\tilde{n} = -\Omega^k \epsilon_{k \ell m} \delta^\ell \delta^m.$$

(4.3)

Here $\Omega^k$ are components of a three-dimensional vector called angular velocity of the black hole, and $y^k$ are functions on $S^2$ created by restricting Cartesian coordinates on $\mathbb{R}^3$ to a unit two-sphere. We can also set the $z$-coordinate axis parallel to the angular velocity vector field. After a suitable rotation we have $(\Omega^k) = (0, 0, \Omega)$, $z = y^3 = \cos \theta$, and

$$\tilde{n} = -\Omega^k \frac{\partial}{\partial \varphi}.$$

(4.4)

Inserting this into (4.2) we obtain:

$$-\frac{1}{8\pi} \int_{\partial V^-} n^A \delta W_A = \Omega \delta J,$$

(4.5)

where

$$J = J_z := \frac{1}{8\pi} \int_{\partial V^-} W \varphi$$

(4.6)

is the $z$ component of the black hole angular momentum.

Up to now we have used only the symmetry of the conformal structure carried by $h_{AB}$. The symmetry of the
metric itself implies that the conformal factor \( f \) is constant along the field \( \hat{n} \). This follows from the observation that the trace of the Killing equation implies vanishing of divergence of the field \( \hat{n} \):

\[
0 = \partial_A (\sqrt{\det \hat{h}_{CD}} n^A) = n^A \sqrt{\det \hat{h}_{CD}} \partial_A f, \quad (4.7)
\]

where the fact that \( \hat{n} \) is the symmetry field of the metric \( \hat{h} \) has been used. Equation (4.4) implies that \( \partial_A f = 0 \) and the conformal factor \( f \) must be a function of the variable \( \theta \) only.

It turns out that also its canonical conjugate \( \kappa \) may be gauged in such a way that it is constant along the field \( \hat{n} \) (see Appendix E for a proof).³

This result was obtained locally, or rather quasi-locally—i.e., from the analysis of the field on the horizon itself. However, the global theorems on the existence of stationary solutions possessing a horizon imply the so-called zeroth law of thermodynamics of black holes (see [17]), according to which the surface gravity \( \kappa \) must be constant along the horizon. But \( A := \int_S \kappa \) is the area of the horizon \( S \). Taking this into account and using (4.5), we derive from (4.2) the “first law of black holes thermodynamics”:

\[
-s \delta \mathcal{M} = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J. \quad (4.8)
\]

Contrary to the theory proposed by Wald and Iyer in [3], the first law (4.8) is, in our approach, a simple consequence of the complete Hamiltonian equation (4.1), restricted to the stationary case. As illustrated by an example of the string dynamics, where Eq. (1.11) for virtual work was a consequence of the Hamiltonian equation (1.6), a similar “thermodynamics of boundary data” may be expected in any Hamiltonian field theory (see, e.g., [2] for the corresponding analysis of the Maxwell electrodynamics). Also a “Penrose-like” inequality (analogous to (1.12) in the string theory) is satisfied as soon as the Hamiltonian is convex. We hope that the gravitational Penrose inequality can be proved along these lines. Preliminary results in this direction, based on the analysis of the field Hamiltonian in linearized gravity (see [7]), are promising.

APPENDIX A: NOTATION AND PRELIMINARY FORMULAS

A considerable simplification of the proofs is obtained if we use in the neighborhood of \( S = \{x^3 = \text{const}\} \) a special coordinate system introduced in [14], which reduces the metric to the following special form:

\[
g^{\mu\nu} = \begin{bmatrix}
n^A n_A & n_A & s M + m^A n_A \\
n_A & g_{AB} & m_A \\
s M + m^A n_A & m_A & (\frac{M}{s})^2 + m^A m_A
\end{bmatrix}. \quad (A1)
\]

Consequently, we have

\[
g^{\mu\nu} = \begin{bmatrix}
\frac{n^A n_A}{s} & s M - s m^A M & s \frac{n^A n_A}{s} M \\
\frac{s M - s m^A M}{s} & \frac{s M M}{s} - s m^A s m^A M & \frac{s}{s} M \\
\frac{s M - s m^A M}{s} & \frac{s}{s} M - s m^A s m^A M & 0
\end{bmatrix}. \quad (A2)
\]

where \( M > 0, s := \text{sgn} \frac{\partial}{\partial \theta} \), \( g_{AB} \) is the induced two-metric on surfaces \( \{x^0 = \text{const}, x^3 = \text{const}\} \), and \( \tilde{g}^{AB} \) is its inverse (contravariant) metric. Both \( \tilde{g}^{AB} \) and \( g_{AB} \) are used to raise and lower indices \( A, B = 1, 2 \) of the two-vectors \( n^A \) and \( m^A \).

We denote by \( \lambda \) the two-dimensional volume form on each two-surface \( \{x^0 = \text{const}, x^3 = \text{const}\} \):

\[
\lambda := \sqrt{\det g_{AB}}. \quad (A3)
\]

For any degeneracy field \( K \) of \( g_{ab} \) the following object

\[
v_K := \frac{\lambda}{K(x^0)}
\]

is a scalar density on \( S \). The vector density

\[
\Lambda = v_K K = \lambda (\delta_0 - n^A \partial_A) \quad (A4)
\]

is well defined (i.e., coordinate independent) and, obviously, does not depend upon any choice of the field \( K \). Hence, it is an intrinsic property of \( S \).

The external geometry of \( S \) is described in terms of the following tensor density:

\[
Q^b_b(K) := -s \{v_K \nabla_b K^a - \delta^b_a \nabla_c K^c + \delta^b_a \partial_c \Lambda^c\} \quad (A5)
\]

which is fully analyzed in our previous paper [10].

In our calculations we shall use also quantities which are not geometric objects (are coordinate dependent). All of them drop out in the final result, where only well-defined geometric objects remain. More precisely, we consider the following combination of the connection coefficients:

\[
\tilde{\tilde{Q}}^a_b := \sqrt{|g|} (g^{\alpha\beta} g^{\nu\mu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta}) (\Gamma^a_{\alpha\beta} - \delta^a_{\alpha} \Gamma^a_{\beta\lambda})
\]

(6)

and the two-dimensional inverse metric \( \tilde{g}^{AB} \) rewritten in a “three-dimensional notation,” where we put \( \tilde{g}^{0a} := 0 \). The resulting matrix \( \tilde{g}^{ac} \) does not define any tensor on \( S \) and satisfies the obvious identity:

\[
\tilde{g}^{ac} g_{cb} = \delta^a_b - K^a \delta^b_b.
\]

Hence, the \( (0, 1, 2) \) block of the contravariant metric (A2) may be rewritten as follows:

\[
g^{ab} = \tilde{g}^{ab} - \frac{1}{N^2} K^a K^b - \frac{s}{s} (m^a K^b + m^b K^a), \quad (A7)
\]

where \( m^a := \tilde{g}^{ab} m_b \), so that \( m^0 := 0 \) and

³In the case \( \Omega = 0 \), quantities \( \kappa \) and \( f \) are arbitrary functions on \( S^2 \).
Using the above definition we may write that
\[
\begin{align*}
\bar{Q}^a_{b} &= \lambda \tilde{\rho}^{ac} l_{cb} - \frac{1}{2} \delta^a_b l^c + \Lambda^a w_b - \delta^a_b \Lambda^c w_c + \Lambda^a \chi_b - \delta^a_b \Lambda^c \chi_c. \\
&= \lambda \tilde{\rho}^{ac} l_{cb} + \Lambda^a w_b - \delta^a_b \Lambda^c w_c. \\
&= \lambda \tilde{\rho}^{ac} l_{cb} + \Lambda^a w_b - \delta^a_b \Lambda^c w_c. \\
\end{align*}
\]
(A8)

where \( l_{ab} \) and \( w_a \) are defined by (2.3) and (2.4) correspondingly, \( \chi_c := \frac{1}{2} \partial_c \ln \left( \frac{\tilde{\rho}}{\Lambda} \right) \). From (A5) we also have the following:
\[
\begin{align*}
\bar{Q}^a_{b} &= \lambda \tilde{\rho}^{ac} \partial_c K^b - \lambda \tilde{\rho}^{ac} \partial_c \Lambda^b - \delta^a_b \partial_c \Lambda^c \\
&= - \lambda \tilde{\rho}^{ac} \left( w_c K^b + l \right) + \Lambda \left( \tilde{\rho}^{ac} l_{cb} + \delta^a_b \Lambda^c \right) + \delta^a_b \partial_c \Lambda \\
&= \lambda \tilde{\rho}^{ac} l_{cb} + \Lambda^a w_b - \delta^a_b \Lambda^c w_c. \\
\end{align*}
\]
(A9)

Equation (A9) expresses the (2 + 1) decomposition of the three-dimensional density \( \bar{Q}^a_{b} \). As a consequence of (A8) and (A9) we obtain the following:
\[
\begin{align*}
\bar{Q}^a_{b} &= \bar{Q}^a_{b} + \frac{1}{2} \Lambda \Lambda^c + \Lambda^a \chi_b - \delta^a_b \Lambda^c \chi_c. \\
\end{align*}
\]
(A10)

The detailed proof of these formulas is contained in [10], where we derive also the following equality:
\[
\begin{align*}
\bar{Q}^a_{b} = \lambda \tilde{\rho}^{ac} \partial_c K^b - \lambda \tilde{\rho}^{ac} \partial_c \Lambda^b - \delta^a_b \partial_c \Lambda^c \\
&= - \lambda \tilde{\rho}^{ac} \left( w_c K^b + l \right) + \Lambda \left( \tilde{\rho}^{ac} l_{cb} + \delta^a_b \Lambda^c \right) + \delta^a_b \partial_c \Lambda \\
&= \lambda \tilde{\rho}^{ac} l_{cb} + \Lambda^a w_b - \delta^a_b \Lambda^c w_c. \\
\end{align*}
\]
(A11)

Replacing the partial derivative \( \partial_a \) by the variation operator \( \delta \), we get an analogous formula
\[
\begin{align*}
\bar{Q}^a_{b} \delta g_{a\beta,c} &= \lambda \left( g^{be} g^{cd} l_{cd} - \frac{1}{2} \tilde{\rho}^{bc} \right) \delta g_{be,c} + \left( \Lambda^b g^{cd} + \Lambda^c g^{bd} - \Lambda^d g^{bc} \right) \delta g_{bc} + 2 \bar{Q}_3 \\
&\quad \times \left( \frac{1}{M} \delta M + \frac{\tilde{\rho}}{M} m_B \delta n_B \right), \\
\end{align*}
\]
which may be further simplified to
\[
\begin{align*}
\bar{Q}^a_{b} \delta g_{a\beta} &= \bar{Q}^a_{b} \delta g_{ab} - \frac{1}{2} \delta \tilde{\rho} \delta g_{ab} - \lambda \delta \log M \\
&\quad + \chi_d \left( \Lambda^a g^{bd} + \Lambda^b g^{ad} - \Lambda^d g^{ab} \right) \\
&= \bar{Q}^a_{b} \delta g_{ab} - \lambda \delta \log M \\
\end{align*}
\]
(A12)

Moreover, we need the following identities from [10]:
\[
\bar{Q}^a_{b} \delta g_{a\beta} = - s \lambda \left[ 2 K^d (w_d + \chi_d) \right], \\
\end{align*}
\]
(A13)

and
\[
\Lambda l = - \lambda + \partial_a (n^a \lambda). \\
\end{align*}
\]
(A14)

**APPENDIX B: PROOFS OF THE GENERATING FORMULAS FOR DYNAMICS OF THE GRAVITATIONAL FIELD**

Dynamics of the gravitational field is derived from the principle of the least action \( \delta \mathcal{A} = 0 \), where the action of the gravitational field is defined as the integral of the
\[
\mathcal{A} = \int_V \left[ \frac{2}{\lambda} \bar{Q}^a_{b} \delta g_{a\beta} - \lambda \bar{Q}^a_{b} \delta g_{ab} - \frac{1}{2} \bar{Q}^a_{b} \delta g_{ab} - \lambda \delta \log M \right] dV. \\
\]

Hilbert Lagrangian:
\[
L = \frac{1}{16 \pi} \sqrt{|g|} R. \\
\]
(B1)

The method proposed by one of us in [1,2] leads to Einstein equations written in the following form:
\[
\delta L = \partial_a \left( \pi^{\mu\nu} \delta A_{\mu\nu}^a \right), \\
\]
(B2)

where
\[
\pi^{\mu\nu} := \frac{1}{16 \pi} \sqrt{|g|} g^{\mu\nu}, \\
A_{\mu\nu}^a := \Gamma_{\mu\nu} - \delta^a_{\mu} \Gamma_{\nu}^a. \\
\]
As soon as we choose a (3 + 1) decomposition of the spacetime \( M \), our field theory will be converted into a Hamiltonian system, with the space of Cauchy data on each of the three-dimensional surfaces playing the role of an infinite-dimensional phase space. Let us choose a coordinate system adapted to this (3 + 1) decomposition. This means that the time variable \( t = x^0 \) is constant on three-dimensional surfaces of this foliation. We assume that these surfaces are spacelike. To obtain the Hamiltonian formulation of our theory we shall simply integrate Eq. (B2) over each a Cauchy surface \( \mathcal{C}_t \subset M \) and then perform a Legendre transformation between time derivatives and corresponding momenta.

We consider the case of an asymptotically flat spacetime and assume that also leaves \( \mathcal{C}_t \) of our (3 + 1) decomposition asymptotically flat at infinity. To maintain control over two-dimensional surface integrals at spatial infinity, we first describe dynamics of our “matter + gravity” system within a finite volume \( V \). Integration of (B2) over \( V \) yields
\[
\begin{align*}
\delta \int_V L &= \int_V \partial_a \left( \pi^{\mu\nu} \delta A_{\mu\nu}^a \right) + \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^a, \\
&\quad \int_{\partial V} \pi^{\mu\nu} \delta A_{\mu\nu}^a, \\
\end{align*}
\]
(B3)

where the dot denotes the time derivative [the two-dimensional divergence \( \partial_a \left( \pi^{\mu\nu} \delta A_{\mu\nu}^a \right) \) vanishes when integrated over \( \partial V \)].

In [1] the Legendre transformation between time derivatives and corresponding momenta was performed in the case of a nondegenerate (one-timelike, two-spacelike) surface \( S \). Here we generalize this method to the case of a wave front. The first step in this construction consists in observation that, due to metricity of the connection \( \Gamma \), the following identity holds:
\[
\pi^{\mu\nu} \delta A_{\mu\nu}^a = - \frac{1}{16 \pi} \tilde{\rho} g_{ab} \partial c l^k + \partial_a \left[ \pi^{00} \delta \left( \pi^{0c} \right) \right]. \\
\]
(B4)

where \( P^{kl} \) denotes the external curvature of \( \Sigma \) written in the ADM form. A simple proof of this formula is also contained in [10].

On the other hand, direct calculations of the variation of the quantity \( \bar{Q}^a_{b} \) given by (A6) lead to the following...
reduction of the boundary term $\pi^{\mu \nu} \delta A_{\mu \nu} = \pi^{\mu \nu} \delta A_{\mu \nu}$:

$$\pi^{\mu \nu} \delta A_{\mu \nu} = - \frac{1}{16 \pi} g_{\mu \nu} \delta \tilde{Q}^{\mu \nu}. \quad (B5)$$

Skipping the two-dimensional divergencies which vanish after integration and using (B4) and (B5), we may rewrite the right-hand side of (B3) in the following way:

$$\delta \int_V L = \frac{1}{16 \pi} \int_V (\tilde{p}^{kl} \delta g_{kl} - \tilde{g}_{kl} \delta P^{kl}) - \frac{1}{16 \pi} \int_V \left[ \pi^{00} \delta \left( \frac{n^0}{\pi^{00}} \right) \right]$$

$$- \int V \delta \tilde{g}_{\mu \nu} + \int_V \left[ \frac{\pi^{03}}{\pi^{00}} \delta g^{03} \right]$$

$$- \left( \frac{\pi^{03}}{\pi^{00}} \right) \delta g^{00} + \frac{1}{8 \pi} \delta \int_V \sqrt{\left| \delta g \right|} R^0_{0 \mu}. \quad (B10)$$

Using the form of the metric (A1) and (A2) we express $\pi^{\mu \nu}$ in terms of the metric. Denoting

$$a := \log N, \quad (B11)$$

we obtain

$$\delta \int_V \left[ \frac{\pi^{00} \delta (n^0)}{\pi^{00}} - \frac{\pi^{03}}{\pi^{00}} \delta g^{00} \right]$$

$$= \frac{s}{16 \pi} \int_V (2 \lambda \delta a - 2 \delta a) - \frac{s}{16 \pi} \delta \left[ \lambda \delta \log M - (\log M) \delta a \right]. \quad (B12)$$

Similarly, we rearrange the boundary term:

$$\delta (g_{\mu \nu} \tilde{Q}^{\mu \nu}) = \delta (g_{\mu \nu} \tilde{Q}^{\mu \nu}) - \tilde{Q}^{\mu \nu} \delta g_{\mu \nu}.$$  

For this purpose we use the formulas (A12) and (A13). Therefore, our generating formula takes the following form:
Finally, expression (B14) takes the following form (modulo two-dimensional divergencies):

$$\delta [(\partial_A \Lambda^A) \log M + \lambda \partial_A n^A],$$

and Eq. (B13) reads as

$$\delta \frac{s}{8\pi} \int_{\partial V} [s(\partial_A \Lambda^A) \log M + s\lambda \partial_A n^A + s[2K^d(w_d + \chi_d) + l]$$

$$+ \sqrt{|g|(g^{3\mu} \Gamma_{0\mu}^0 - g^{0\mu} \Gamma_{0\mu}^3)},$$

Now, we simplify the expression

$$\delta \frac{s}{8\pi} \int_{\partial V} [s(\partial_A \Lambda^A) \log M + s\lambda \partial_A n^A + s\lambda [2K^d(w_d + \chi_d) + l]$$

$$+ \sqrt{|g|(g^{3\mu} \Gamma_{0\mu}^0 - g^{0\mu} \Gamma_{0\mu}^3)].}$$

Using

$$w_a = \Gamma_3^a - \frac{s}{M} m^c l_{ca} - \frac{1}{M} M_a,$$

and taking into account that $K^\mu \Gamma_{\mu a}^0 = -w_a$ (2.4) and $K^\mu \Gamma_{ab}^0 = I_{ab}$ (2.3), we can pass to the metric derivatives contained in \( \Gamma_s \). This way (B19) reduces (modulo two-dimensional divergencies) to

$$-2s \int_{\partial V} I \delta \lambda + 2s \delta \int_{\partial V} \lambda (l - n^a w_a).$$

Therefore, we obtain the following boundary formula:

$$\frac{-s}{8\pi} \int_{\partial V} G_0 = \frac{1}{16\pi} \int_{\partial V} [P^{ki} \delta g_{kl} - \tilde{g}_{kl} \delta P^{kl}] + \frac{s}{8\pi}$$

$$\times \int_{\partial V} (\lambda \delta a - \tilde{a} \delta a) + \frac{1}{16\pi}$$

$$\times \int_{\partial V} (Q^{ab}(K) \delta g_{ab} - 2s \delta \lambda a)$$

$$+ \frac{s}{8\pi} \delta \int_{\partial V} \lambda (l - n^a w_a).$$

Using identities $Q^{ab}(K) \delta g_{ab} = 2Q_0^0(K) \delta n^A - Q_{AB}(K) \delta \tilde{g}^{AB}$ and $s\lambda n A^A = Q_0^0(K) \delta n^A$ we get

$$\frac{-s}{8\pi} \int_{\partial V} G_0 = \frac{1}{16\pi} \int_{\partial V} [P^{ki} \delta g_{kl} - \tilde{g}_{kl} \delta P^{kl}] + \frac{s}{8\pi}$$

$$\times \int_{\partial V} (\lambda \delta a - \tilde{a} \delta a) - \frac{1}{16\pi}$$

$$\times \int_{\partial V} (Q^{ab}(K) \delta \tilde{g}^{AB} + 2n^A \delta Q_0^0(K)$$

$$- 2s \lambda \delta a).$$

Finally, expression (B14) takes the following form (modulo two-dimensional divergencies):

$$\delta [(\partial_A \Lambda^A) \log M + \lambda \partial_A n^A],$$

and Eq. (B13) reads as

$$\delta \frac{s}{8\pi} \int_{\partial V} [s(\partial_A \Lambda^A) \log M + s\lambda \partial_A n^A + s\lambda [2K^d(w_d + \chi_d) + l]$$

$$+ \sqrt{|g|(g^{3\mu} \Gamma_{0\mu}^0 - g^{0\mu} \Gamma_{0\mu}^3)].}$$

Using Eq. (A9) we express $Q$ in terms of independent objects $l_{ab}$ and $w_a$, and finally obtain

$$-\delta \mathcal{H} = \frac{1}{16\pi} \int_{\partial V} [P^{ki} \delta g_{kl} - \tilde{g}_{kl} \delta P^{kl}] + \frac{s}{8\pi}$$

$$\times \int_{\partial V} (\lambda \delta a - \tilde{a} \delta a) + \frac{s}{16\pi} \int_{\partial V} [\Lambda^{AB} \delta g_{AB}$$

$$- 2(w_o \delta \Lambda^0 - \Lambda^A \delta w_A)],$$

where

$$\mathcal{H} = \frac{1}{8\pi} \int_{\partial V} G_0 + \frac{s}{8\pi} \int_{\partial V} \lambda l.$$

APPENDIX C: DERIVATION OF GENERATING FORMULA FROM THE NONDEGENERATE CASE

Consider a one-parameter family of hypersurfaces $S_\epsilon := \{ r_\epsilon := r - s \epsilon t = \text{const} \}$ parametrized by a real $\epsilon$, such that for $\epsilon = 0$ we have $S_0 = S$ and for each $\epsilon \neq 0$ the induced three-metric on $S_\epsilon$ is nondegenerate.

Take the external curvature tensor $\tilde{Q}^{ab}$ of $S_\epsilon$ and the three-dimensional contravariant metric $\tilde{g}^{ab}$, inverse to $g_{ab}$. Define

$$Q := \frac{1}{\sqrt{|\tilde{g}|}} Q^{00},$$

$$Q_A := Q^0_A,$$

$$\tilde{Q}^{AB} := Q_{CD} \tilde{g}^{CA} \tilde{g}^{DB}.$$

Observe that these are two-dimensional objects defined on $\partial V$: $Q$ is a scalar density, $Q_A$ is a covector density, and $\tilde{Q}^{AB}$ is a symmetric tensor density. The following identity (a homogeneous generating formula for the field dynamics) was proved in [2] for a timelike boundary and gen-
Generalized in [18] for any nondegenerate (e.g., a spacelike) boundary:

\[ 0 = \int_V (\delta g^{kl} - \delta g_{kl}) + 2 \int_V (\lambda \delta \alpha - \alpha \delta \lambda) + \int_{\partial V} \left( \frac{1}{2} Q^{AB} \delta g_{AB} - 2n^A \delta Q_A + 2n \delta Q \right), \]  

(C4)

where

\[ n := \frac{1}{\sqrt{\text{det} g^{00}}}, \quad n^A := \frac{\tilde{g}^{AB} g_{0B}}{\text{det} g^{00}}. \]

The boundary objects \( \alpha, Q, Q_A, \) and \( Q^{AB} \) diverge under the limit \( S_e \to S \) — or, equivalently, \( g^{33} \to 0 \). We are going to rearrange them in such a way that the resulting objects do behave well under this limit.

For this purpose the three-dimensional metric component \( \tilde{g}^{00} \) defining the lapse function \( n \) can be expressed by the following components of the spacetime metric:

\[ \tilde{g}^{00} = g^{00} - \frac{(g^{03})^2}{g^{33}} = g^{00} g^{33} - \frac{(g^{03})^2}{g^{33}}. \]  

(C5)

The four-dimensional component \( g^{33} \) of the inverse metric can be expressed in terms of the three-dimensional metric component \( g^{33} \) of the inverse \( \tilde{g}_{kl} \) induced on \( V \):

\[ g^{33} = \tilde{g}^{33} + \frac{(g^{03})^2}{g^{00}}. \]  

(C6)

Then

\[ \tilde{g}^{00} = \frac{\tilde{g}^{00} g^{33}}{g^{33}}. \]  

(C7)

Restricting Eq. (A1) to the hypersurface \( \{x^0 = \text{const}\} \) we obtain

\[ g_{kl} = \left[ \frac{g_{AB}}{m_A} \begin{pmatrix} m_A & m_A & m_A \end{pmatrix} \right]. \]

Consequently, its three-dimensional inverse equals

\[ \tilde{g}^{kl} = \left( \frac{N}{M} \right)^2 \begin{pmatrix} \frac{(m_A)^2 + m_A m_A}{m_A} & -m_A & -m_A \\ -m_A & 1 & -m_A \\ -m_A & -m_A & 1 \end{pmatrix}. \]

In this notation \( g^{00} = -1/N^2 \), hence we obtain that \( g^{00} = -1/(M^2 g^{33}) \) and the lapse \( n \) equals

\[ n = M \sqrt{g^{33}}. \]  

(C8)

Using the definition of the hyperbolic angle \( \alpha = \text{arsinh}(q) \), where \( q = g^{00}/\sqrt{g^{00} g^{33}} \), we obtain

\[ \delta \alpha = \frac{q}{\sqrt{1 + q^2}} = \frac{1}{\sqrt{1 + q^2}} \delta \log q. \]  

(C9)

Taking into account that

\[ \log q = \log g^{03} - \frac{1}{2} \log |g^{00}| + \log g^{33}, \]  

(C10)

we obtain

\[ \frac{q}{\sqrt{1 + q^2}} \delta \alpha = \delta \log N - \delta \log (M \sqrt{g^{33}}) = \delta \alpha - \delta \log n, \]  

(C11)

where

\[ \alpha := -\log \sqrt{|g^{00}|} = \log N \]  

(C12)

is a regular part of \( \alpha \). The second term \( \delta \log n \) diverges in the limit, but cancels the divergent part of \( Q \) what would be proved in the sequel.

Moreover, we have

\[ n Q = n^2 Q^{00} = \lambda M \tilde{g}^{AB} = \lambda I, \]  

(C13)

hence the following formula holds:

\[ 2n \delta Q = 2n Q \delta (\log Q) = 2 \lambda I \delta (\log \lambda) \]  

(C14)

Continuing, we have

\[ Q_A = Q_A^0 = \tilde{g}^{0b} Q_{bA} \]

\[ = \lambda M \left( \tilde{g}^{06} \frac{g^{33}}{g^{00}} \right) \left( \Gamma^3_{bA} - g_{bA} \Gamma^{3cd} g^{cd} \right). \]  

(C15)

Using the following formula:

\[ -\frac{1}{2} (\log g^{33})_a = \Gamma^3_{a3} + \frac{1}{g^{33}} \Gamma^{3ab} g^{3b}, \]  

(C16)

we obtain

\[ Q_A^0 = \lambda (\Gamma^3_{3A} - m^B \Gamma^3_{BA}) + \frac{1}{2} \lambda (\log g^{33})_A. \]  

(C17)

Now, we define the quantity:

\[ \omega_a := -\frac{g^{3\mu}}{g^{00}} g^{0\mu}. \]  

(C18)

Taking into account that

\[ \Gamma^3_{3a} - m^B \Gamma^3_{BA} = \omega_a + (\log M)_a, \]  

(C19)

it follows that \( Q_A^0 \) may be written in the following form:

\[ Q_A^0 = \lambda \omega_A + \lambda (\log M)_A + \frac{1}{2} \lambda (\log g^{33})_A. \]  

(C20)

Similarly, \( Q^{AB} \) takes the following form:

\[ \frac{1}{2} Q^{AB} = Q^{CD} \tilde{g}^{CA} \tilde{g}^{DB} = \lambda M \tilde{g}^{CA} \tilde{g}^{DB} \left( \Gamma^3_{CD} - g_{CD} \Gamma^{3ef} \tilde{g}_{ef} \right), \]  

(C21)

hence the contraction \( \frac{1}{2} Q^{AB} \) gives us

\[ \frac{1}{2} Q^{AB} g_{AB} = -\lambda I - 2\lambda \omega_0 + 2n^A \omega_A - 2\lambda (\partial_0 - n^A \partial_A) \]

\[ \times \log M - \lambda (\partial_0 - n^A \partial_A) (\log g^{33}), \]  

(C22)
and the traceless part of $\mathcal{Q}^{AB}$ takes the form:

$$\frac{1}{2} \mathcal{Q}^{AB} - \frac{1}{2} \mathcal{Q}^{CD} g_{CDB} \mathcal{S}^{AB} = \lambda l^{AB} - \frac{1}{2} g^{AB}. \quad (C22)$$

The above results may be gathered as follows:

$$2\lambda \delta \lambda = \frac{\sqrt{1 + q^2}}{q} (2\lambda \delta \alpha - 2\lambda \delta \log n),$$

$$-2\alpha \delta \lambda = \frac{\sqrt{1 + q^2}}{q} (-2\delta_0 \alpha \delta \lambda + 2\delta_0 \log \alpha \delta \lambda),$$

$$2n \delta \mathcal{Q} = 2\delta(l \lambda) - 2\lambda \delta \log \lambda,$$

$$-2n^A \delta \mathcal{Q}_A = -2n^A \delta (\alpha \lambda \omega_A) - 2n^A \delta [\alpha (\log n)_A],$$

$$\frac{1}{2} \mathcal{Q}^{AB} g_{AB} \delta \log \lambda = \left[-[l + 2\lambda_0 - 2n^A \omega_A + 2(\delta_0 - n^A \alpha) \right] \times (\log n)] \delta \lambda,$$

$$\frac{1}{2} \left( \mathcal{Q}^{AB} - \mathcal{Q}^{CD} g_{CDB} \mathcal{S}^{AB} \right) \delta g_{AB} = \lambda (l^{AB} - \frac{1}{2} g^{AB}) \delta g_{AB}. \quad (C23)$$

Finally, we express quantities appearing in the boundary formula (C4) in terms of the above objects. Taking into account identity (A14) and omitting two-dimensional divergencies we obtain

$$0 = \int_V \left( \mathcal{P}^{k_1} g_{k_1} - \mathcal{S}_{k_1} \delta P^{k_1} \right)$$

$$+ 2 \int_{\partial V} \left( \lambda \delta \alpha - \delta \alpha \delta \lambda + 2 \delta \int_{\partial V} \lambda l \right) \quad (C23b)$$

$$+ \int_{\partial V} \left( \lambda l^{AB} \delta g_{AB} - 2 \lambda n^A \delta \omega_A - 2 \lambda \delta_0 \delta \lambda \right) \quad (C23c)$$

$$+ 2 \int_{\partial V} \left( \lambda \delta \log n - \frac{\sqrt{1 + q^2}}{q} \lambda \delta \log n \right) \quad (C23d)$$

$$+ 2 \int_{\partial V} \left( \frac{\sqrt{1 + q^2}}{q} \right) (\log n) \delta \lambda - (\log n) \delta \lambda \right). \quad (C23e)$$

In the null limit $\epsilon \to 0$, we have that $\alpha = a$ and $\omega_a = w_a$. Moreover, $q \to \infty$ and, whence, $\sqrt{1 + q^2} / q \to 1$. We shall prove that the last two boundary terms (C23d) and (C23e) vanish in this limit. Indeed, we have

$$\lim_{\epsilon \to 0} \left( \frac{\sqrt{1 + q^2}}{q} - 1 \right) \delta \log n = \frac{s M}{2 \lambda} \delta \left[ \log \left( g^{33} \right) \right]. \quad (C24)$$

A similar formula with $\delta$ replaced by $\delta \lambda_0$ is also true. But $\delta g^{33} \to 0$ and $\delta \mathcal{S} \to 0$ on $\partial V$. This implies

$$\lim_{\epsilon \to 0} \left( \frac{\sqrt{1 + q^2}}{q} - 1 \right) \delta \log n \bigg|_{\partial V} = 0,$$

$$\lim_{\epsilon \to 0} \left( \frac{\sqrt{1 + q^2}}{q} - 1 \right) \delta_0 (\log \lambda) \bigg|_{\partial V} = 0.$$

Hence, boundary formula (C23) takes the following form:

$$0 = \int_V \left( \mathcal{P}^{k_1} g_{k_1} - \mathcal{S}_{k_1} \delta P^{k_1} \right)$$

$$+ 2 \delta \int_{\partial V} \lambda l + \int_{\partial V} \left( \lambda l^{AB} \delta g_{AB} - 2 \lambda n^A \delta \omega_A - 2 \lambda \delta_0 \delta \lambda \right). \quad (C25)$$

Moving $2\delta \int_{\partial V} \lambda l$ to the left-hand side, we obtain Eq. (2.1).

**APPENDIX D: DERIVATION OF THE FORMULA (4.4)**

Being a difference between two symmetry fields: the four-dimensional Killing field $\partial_0$ and the null field $K$ on $S$, the field $\mathcal{N} = (n^A)$ is a symmetry field of the two-metric $g_{AB}$ and, whence, satisfies the two-dimensional Killing equation:

$$n_{AB} + n_{BA} = 0. \quad (D1)$$

As was already discussed, we may choose a coordinate system such that $g_{AB} = f h_{AB}$, and $h_{AB}$ is a standard two-sphere metric. The field $\mathcal{N}$ is also the symmetry field of conformal structure given by the metric $h_{AB}$, hence it has to fulfill the equation:

$$n_{AB} + n_{BA} = h_{AB} n^C_{||} c = 0. \quad (D2)$$

where $\parallel$ denotes the two-dimensional derivative with respect to metric $h$ on the two-sphere. Therefore, $\mathcal{N}$ belongs to the six-dimensional space of conformal fields on two-sphere and is of the following form:

$$n^A = e^{AB} v^B + \frac{1}{2} v^B h_{AB}, \quad (D3)$$

where $v, \tilde{v}$ are dipole functions, i.e., of the form $v^B = a^B k^i$, $	ilde{v}^B = b^B k^i$, where $k^i$ are coordinates of the unit vector:

$$k^1 = \sin \theta \cos \phi, \quad k^2 = \sin \theta \sin \phi, \quad k^3 = \cos \theta.$$}

Taking all this into account we write the field $\mathcal{N}$ in the following form:

$$n^A \partial_A = \left[ e^{A\theta} (a^B k^i)_{,\theta} + (b^B k^i)_{,\theta} \right] \partial_\theta + \left[ e^{A\theta} (a^B k^i)_{,\theta} + (b^B k^i)_{,\theta} \frac{1}{\sin 2\theta} \right] \partial_\phi.$$
\[ n^\theta = (\cos \theta - 1)(a_1 \sin \varphi + b_1 \cos \varphi) - b_3 \sin \theta, \]
\[ n^\varphi = (\cos \theta - 1)(b_1 \sin \varphi - a_1 \cos \varphi) \frac{1}{\sin \theta} + a_3. \]

We can rotate the coordinate system in such a way that the components \( n^\theta \) and \( n^\varphi \) will take the form:
\[ n^\theta = (\cos \theta - 1)a \sin \varphi - b \sin \theta, \]
\[ n^\varphi = (\cos \theta - 1)(-a \cos \varphi) \frac{1}{\sin \theta} + c, \]
and \( a, b, \) and \( c \) are some new parameters. Because \( S \) is a horizon, we have
\[ (\lambda n^\lambda)_A = 0, \]
and in our coordinate system \( \lambda = f \sin \theta \), hence we have the following equation:
\[ 0 = \{ f \sin \theta[(\cos \theta - 1)a \sin \varphi - b \sin \theta]\}_\theta \]
\[ + \left[ f \sin \theta[(\cos \theta - 1)(-a \cos \varphi) \frac{1}{\sin \theta} + c] \right]_\varphi. \]  
(D4)

Integrating the above equation over \( \varphi \) and omitting the vanishing integral \( \int(\lambda n^\varphi) \varphi d\varphi \) we obtain
\[ \frac{\partial}{\partial \theta} \int_0^{2\pi} f \sin \theta[(\cos \theta - 1)a \sin \varphi - b \sin \theta] d\varphi = 0, \]
\[ \frac{\partial}{\partial \varphi} \int_0^{2\pi} f \sin \varphi d\varphi = 0. \]  
(D5)

hence
\[ \frac{\cos \theta - 1}{\sin \theta} a \int_0^{2\pi} f \sin \varphi d\varphi + b \int_0^{2\pi} f d\varphi = 0. \]  
(D6)

Because the first summand vanishes for \( \theta \to 0 \), we have
\[ 2\pi bf(\theta = 0) = b \int_0^{2\pi} f d\varphi = 0. \]
This straightforwardly implies \( b = 0 \), because the conformal factor \( f \) is positive. Hence, \( \vec{n} \) is of the following form:
\[ n^\theta = (\cos \theta - 1)a \sin \varphi, \]
\[ n^\varphi = (\cos \theta - 1)(-a \cos \varphi) \frac{1}{\sin \theta} + c. \]

Let us write \( \vec{n} \) in stereographic coordinates on the plane \((x, y)\) intersecting our two-sphere along the equator, where coordinates \((x, y)\) read as
\[ x = \cot \frac{\theta}{2} \cos \varphi, \quad y = \cot \frac{\theta}{2} \sin \varphi. \]  
(D7)

Then
\[ \frac{\partial}{\partial \theta} = \frac{1}{2} \frac{\cos \varphi}{\sin^2 \frac{\theta}{2}} \frac{\partial}{\partial x} - \frac{1}{2} \frac{\sin \varphi}{\sin^2 \frac{\theta}{2}} \frac{\partial}{\partial y}, \]
\[ \frac{\partial}{\partial \varphi} = -\cot \frac{\theta}{2} \cos \varphi \frac{\partial}{\partial x} + \cot \frac{\theta}{2} \sin \varphi \frac{\partial}{\partial y}. \]

and \( n^A \partial_A \) is of the following form:
\[ n^A \partial_A = a \frac{\partial}{\partial y} + c \left( -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right). \]  
(D8)

Let us consider two cases: \( c = 0 \) and \( c \neq 0 \):
(i) \( c = 0 \)
In this case Eq. (D4) takes the following form:
\[ [f \sin \theta(\cos \theta - 1)a \sin \varphi]_\theta + \left[ f \sin \theta(\cos \theta - 1) \times \right. \]
\[ \left. (-a \cos \varphi) \frac{1}{\sin \theta} \right]_\varphi = 0. \]  
(D9)

In stereographic coordinates (D7) the above equation reads
\[ (\log f)_y = \frac{2y}{1 + x^2 + y^2}, \]  
(D10)
and implies
\[ f(x, y) = C(x)(1 + x^2 + y^2), \]  
(D11)
where \( C(x) \) is an arbitrary function of \( x \). Keep \( x \) constant and pass to the limit \( y \to \infty \). If \( C(x) \neq 0 \) then \( f \to \infty \), otherwise \( f \equiv 0 \). But the conformal factor \( f \) must be finite and different from zero. Hence, the case \( c = 0 \) is incompatible with the properties of \( f \).
(ii) \( c \neq 0 \)
In that case we may write
\[ n^A \partial_A = c \left( \left( x + \frac{a}{c} \right) \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \]  
(D12)

Putting \( \tilde{x} = x + \frac{a}{c} \), we obtain
\[ n^A \partial_A = c \left( \tilde{x} \frac{\partial}{\partial y} - y \frac{\partial}{\partial \tilde{x}} \right). \]  
(D13)

hence \( n^A \partial_A \) is a field of rotations and can be written as follows:
\[ n^A \partial_A = -\Omega \frac{\partial}{\partial \tilde{\varphi}}. \]  
(D14)

A different proof of this fact is given in [16].

APPENDIX E: SURFACE GRAVITY $\kappa$ IS CONSTANT ALONG $\vec{n}$

We are going to perform a gauge transformation of $\kappa$, such that the resulting quantity $\tilde{\kappa}$ remains constant along $\vec{n}$. For this purpose we leave the spacelike coordinates unchanged ($x^i = x^i$), whereas the time coordinate is translated by a constant which depends upon them. This implies the following transformation on the surface $S$, where $x^3$ is constant:
\[
\tilde{x}^0 = x^0 + \alpha(x^4).
\]  
(E1)

Then the field \( K \) transforms as \( \tilde{K} = cK \), where
\[
c = (1 - n^A \partial_A \alpha)^{-1},
\]  
(E2)

and quantities \( w_a \) transform as follows:
\[
\tilde{w}_a = w_a + \partial_a \ln c
\]  
(E3)

(the transformation law for objects \( w_a \) is given in [14]). After this transformation the surface gravity \( \kappa \) takes the following value:
\[
\tilde{\kappa} = -K^a \tilde{w}_a = -cK^a \left( w_a + \partial_a \ln c \right) = c \kappa - K(c).
\]  
(E4)

Using \( K = \partial_0 - n^A \partial_A \) and (DI4) for \( n^A \) we obtain
\[
\Omega \frac{\partial c}{\partial \varphi} - c \kappa + \kappa = 0.
\]  
(E5)

Consequently,
\[
c = -\int \frac{\tilde{\kappa}}{\Omega} e^{-\Omega^{-1} \int \kappa d\varphi} d\varphi.
\]  
(E6)

Denoting
\[
F(\theta, \varphi) = \int e^{-\Omega^{-1} \int \kappa d\varphi} d\varphi
\]  
(E7)

we obtain
\[
c^{-1} = -\Omega \frac{\partial}{\partial \varphi} \ln F.
\]  
(E8)

We can compare this with the form of \( c^{-1} \) implied by (E2)
\[
c^{-1} = 1 - \frac{1}{\Omega} \frac{\partial}{\partial \varphi} \ln \kappa
\]  
(E9)

which leads to the following equation:
\[
\frac{\partial \alpha}{\partial \varphi} = \frac{1}{\Omega} + \frac{1}{\kappa} \frac{\partial}{\partial \varphi} \ln F.
\]  
(E10)

Solutions of the above equation read as
\[
\alpha = \frac{\varphi}{\Omega} + \frac{1}{\kappa} \ln F.
\]  
(E11)

where
\[
F(\varphi) = C_1 \left( \int_0^\varphi e^{\Omega^{-1} \int_0^s \kappa(s) ds} ds + C_2 \right)
\]

and \( C_1, C_3 \) are integration constants. Applying periodicity conditions:
\[
\alpha(0) = \alpha(2\pi), \quad \frac{d}{d\varphi} \ln F(0) = \frac{d}{d\varphi} \ln F(2\pi),
\]

we obtain
\[
\tilde{\kappa} = \frac{\Omega}{2\pi} \int_0^{2\pi} \frac{F(\varphi)}{F(2\pi)} d\varphi.
\]  
(E12)

where the values of \( F(0) \) and \( F(2\pi) \) remain to be determined. Denote by \( f(\varphi) \) the expression:
\[
f(\varphi) := \int_0^\varphi e^{\Omega^{-1} \int_0^s \kappa(s) ds} ds.
\]  
(E13)

Therefore \( F \) and \( \log F \) are of the following form:
\[
F = C_1 (f + C_2), \quad \log F = \log C_1 + \log(f + C_2).
\]

The above equations imply
\[
\frac{f'(0)}{f(0) + C_1} = \frac{f'(2\pi)}{f(2\pi) + C_1}, \quad \text{and} \quad f(0) = 1,
\]  
(E14)

hence
\[
C_3 = \frac{f(2\pi)}{f'(2\pi) - 1},
\]  
(E15)

and we have
\[
\tilde{\kappa} = \frac{\Omega}{2\pi} \log \frac{C_3}{f'(2\pi) + C_3}.
\]  
(E16)

Finally, we obtain
\[
\tilde{\kappa} = -\frac{\Omega}{2\pi} \log f'(2\pi) = \frac{1}{2\pi} \int_0^{2\pi} \kappa(s) ds \equiv \text{const}.
\]  
(E17)

Hence, we have performed such a gauge transformation from \( \kappa \) to \( \tilde{\kappa} \) that the new quantity \( \tilde{\kappa} \) is constant along parallels of the sphere \( S^2 \).

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