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SYMPLECTIC GEOMETRY AND SECOND QUANTIZATION

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A construction of the space $\mathscr{S}$ of quantum states is given for a field theory. The space $\mathscr{S}$ is a projective limit of spaces S_P where P is a finite system of measuring instruments and S_P describes only those degrees of freedom of the field which are measured by P. The above construction can also be used in the case of curved space-time.

1. Introduction

The usual aproach to the quantum theory of interacting fields is based on perturbation techniques. As a starting point we take a non-interacting (linear) theory for which the second quantization procedure can be defined. The definition of this procedure is based on the possibility of diagonalizing the Hamiltonian by means of Fourier transformation.

Such an approach is not "structurally stable". Even slight curvature of space-time makes the Fourier diagonalization of the Hamiltonian impossible. But we want to treat quantum field theory as a fundamental theory of local interactions between micro-objects. We cannot thus tolerate the situation where the possibility of describing those interactions depends on the question whether space is curved or not far beyond our Galaxy. There must be room in quantum field theory for gravitation (at least as an external classical force).

Axiomatic approach to quantum field theory (Wightman's or Haag's formulation) also depends on the global structure of space-time. The spectral condition can be formulated only in terms of the Poincaré group (there have been some attempts to replace the Poincaré group by another symmetry group, cf. [2], but the global structure of space-time still plays a fundamental role).

In the present paper we are going to formulate another approach which is purely "local" and mathematically well defined. It is based on the symplectic ("canonical") structure of the classical field theory (cf. [3], [5], [9]) and closely related to classical concepts of "canonical quantization". From a mathematical point of view our theory is a wide generalization of the lattice approximation (cf. [10]).

The present paper contains only a general formulation of kinematics and some problems of dynamics. Further structures will be presented in our next paper.

Our approach can also be applied to gauge theories, where the notion of quantum field is particularly incomprehensible in the standard formulation. Gauge theories will also be discussed in the next paper.

2. Finite systems of observables

In this section we consider classical field theories. The quantum structure will be discussed further.

Let be given a classical field theory (e.g., in the sense of [5]). For a given spacelike surface Σ, the set of all Cauchy data on Σ is an infinite-dimensional symplectic space. We suppose also that it is a linear space (all physical fields like scalar, tensor, spinor fields obey this assumption).

In order to fix our notation we shall speak in the sequel about the scalar field. In this case the state of the field is given by a scalar function φ and a scalar density π, both defined on Σ. Scalar densities on 3-dimensional manifolds can be treated as differential 3-forms (in dynamical considerations the density π will play the role of the derivative $d\varphi/d\mathbf{n}$ where $\mathbf{n}$ is the vector normal to Σ).

We shall use the following notation: F will denote the space of all functions on Σ which are locally square-integrable. By F_0 we shall denote the space of all such functions which have also compact support. By D and D_0 we shall denote the corresponding spaces of scalar densities which have locally square integrable coefficients.

If there is a metric tensor g given on Σ we may, of course, identify F and D with the help of the standard density

$$\mu(x) = \sqrt{g(x)} \cdot dx^1 \wedge dx^2 \wedge dx^3 . \tag{1}$$

We prefer to keep the distinction between F and D, since it shows that the field kinematics can be formulated without metrics. The metric tensor enters only when the dynamics is defined.

By $\mathscr{P}$ (resp. $\mathscr{P}_0$) we denote the space of all locally square integrable Cauchy-data (resp. with compact support):

$$\mathscr{P} = D \times F, \quad \mathscr{P}_0 = D_0 \times F_0 . \tag{2}$$

The symplectic structure in $\mathscr{P}_0$ is given by the 2-form ω which assigns to each pair of vectors in $\mathscr{P}_0$

$$X_1 = (\overset{1}{\delta}\pi, \overset{1}{\delta}\varphi), \quad X_2 = (\overset{2}{\delta}\pi, \overset{2}{\delta}\varphi) \tag{3}$$

a number

$$\langle X_1 \wedge X_2 ; \omega \rangle := \int_\Sigma \overset{1}{\delta}\pi(x) \cdot \overset{2}{\delta}\varphi(x) - \overset{2}{\delta}\pi(x) \cdot \overset{1}{\delta}\varphi(x) . \tag{4}$$

Now we shall consider an instrument measuring the value of the field φ on the space Σ. The sensitivity of our instrument is different at different points of Σ. The simplest example should be something like a Wilson cloud-chamber whose sensitivity depends on

spatial position and vanishes outside the chamber. In the general case the sensitivity of an instrument can be described by the density $d \in D_0$. The numerical value of the measurement is given by

$$\int_\Sigma \varphi(x) \cdot d(x), \tag{5}$$

where φ denotes the state of our field.

In a similar way we may assign to each instrument measuring the value of the field π its "spatial sensitivity-distribution" f. Now $f \in F_0$ and the numerical value of the measurement equals

$$\int_\Sigma \pi(x) \cdot f(x), \tag{6}$$

where π denotes the state of our field.

Now let $(d_1, \ldots, d_n)$ be a set of instruments measuring the field φ and $(f_1, \ldots, f_m)$ the set of instruments measuring the field π. Combining them in different ways we may obtain any instrument whose sensitivity-distribution belongs to subspaces P_D and P_F spanned by $(d_1, \ldots, d_n)$ and $(f_1, \ldots, f_m)$, respectively.

DEFINITION. By a *finite symmetric system of local classical observables* we mean a finite-dimensional subspace

$$P = P_D \times P_F \subset \mathscr{P}_0$$

such that the symplectic form ω remains non-degenerate when restricted to P.

It follows from Darboux's theorem for skew-symmetric non-degenerate forms that $\dim P_D = \dim P_F$, i.e. $n = m$.

Given a finite system P, the space $\mathscr{P}$ of all states of the field splits immediately into classes. Every class is composed of such states which cannot be distinguished one from another by means of our measuring system. We may call those classes "states of the field modulo measuring system P".

Mathematically, the space of states modulo P is a quotient space

$$\mathscr{P}_P = \mathscr{P}/(P_F^\perp \times P_D^\perp), \tag{7}$$

where $P_F^\perp$ and $P_D^\perp$ are composed of all states of the field which give zero results when measured by any instrument belonging to P:

$$P_F^\perp = \left\{ \pi \in D : \int_\Sigma \pi(x) f(x) = 0 \text{ for every } f \in P_F \right\}, \tag{8}$$

$$P_D^\perp = \left\{ \varphi \in F : \int_\Sigma \varphi(x) d(x) = 0 \text{ for every } d \in P_D \right\}. \tag{9}$$

Formula (7) can also be written in the following way:

$$\mathscr{P}_P = (D/P_F^\perp) \times (F/P_D^\perp). \tag{10}$$

The following facts can be deduced from the non-degeneracy axiom for P:

$$P_D \cap P_F^{\perp} = \{0\}, \quad P_F \cap P_D^{\perp} = \{0\}. \tag{11}$$

It means that there is a natural decomposition

$$D = P_D \oplus P_F^{\perp}, \quad F = P_F \oplus P_D^{\perp}. \tag{12}$$

Quotient spaces may thus be represented in the canonical way by P_D and P_F:

$$D/P_F^{\perp} \cong P_D, \quad F/P_D^{\perp} \cong P_F. \tag{13}$$

Using equations (10) and (13), we see that there is a natural isomorphism between the space of all states modulo P, and P itself:

$$\mathscr{P}_P \cong P_D \times P_F = P \subset \mathscr{P}. \tag{14}$$

Formulas (7) and (14) play an essential role in further considerations. The space $\mathscr{P}_P$ can be treated either as a quotient space (7) or as a subspace (14). This will correspond to the construction of the quantum field theory via the projective limit (of quantum states) or the inductive limit (of quantum observables). It follows from our definition that $\mathscr{P}_P$ is a symplectic space with the symplectic form ω_P which is simply the restriction of ω to $P \cong \mathscr{P}_P$. The natural projections $\mathscr{P}_P \to P_D$ and $\mathscr{P}_P \to P_F$ are polarizations in the sense of [1], [4], [7]. We can treat P_F as a "position space" and P_D as "momentum space". Coordinate charts: $(q^1, \ldots, q^n)$ in P_F and $(p_1, \ldots, p_n)$ in P_D may be chosen in such a manner that

$$\omega_P = \sum_{i=1}^{n} dp_i \wedge dq^i. \tag{15}$$

This means that the p's are momenta conjugate to the q's.

The space P (or $\mathscr{P}_P$) describes only a finite number of degrees of freedom of our classical field. The family of all such spaces is an ordered set. The ordering relation is simply inclusion. The more degrees of freedom taken into account, the larger the space P must be. In order to give an exact description of the field we shall pass to the limit over all P, treating them like indices.

3. A special case: lattice approximation

Consider a partition of the space Σ:

$$\Sigma = \bigcup_{i=0}^{n} K_i$$

such that all K_i are finite (relatively-compact) for $i > 0$. Suppose that $K_i \wedge K_j$ has zero measure for $i \neq j$ (see Fig. 1). We suppose moreover that the standard measure (1) has been defined. Take for P_F the space of functions which are constant on every set K_i. Since they have compact support, they have to vanish on K_0. For P_D we take the space of densities which are constant on every K_i. More precisely: $d \in P_D$ if $d = \tilde{d} \cdot \mu$ and $\tilde{d}$

is a function which is constant on every K_i. As a basis in P we may choose

$$f_i(x) = 1_{K_i}(x) = \begin{cases} 1, & x \in K_i, \\ 0, & x \notin K_i \end{cases} \tag{16}$$

$$d_i(x) = 1_{K_i}(x) \cdot \mu, \tag{17}$$

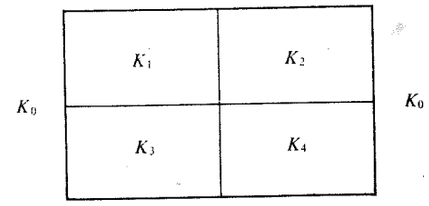

Fig. 1

where $i > 0$. Every element of $\mathscr{P}_P$ can be written

$$\sum_{i=1}^{n} \pi_{K_i} \cdot d_i + \sum_{i=1}^{n} \varphi_{K_i} \cdot f_i.$$

The coefficients $(\pi_{K_i}, \varphi_{K_i})$ form a coordinate chart in $\mathscr{P}_P$. One can immediately check that this chart is "almost canonical"; namely

$$\omega_P = \sum_{i=1}^{n} |K_i| \, d\pi_{K_i} \wedge d\varphi_{K_i}. \tag{18}$$

Here, $|K_i|$ is a measure of the set K_i:

$$|K_i| = \int_{K_i} \mu. \tag{19}$$

Formula (18) shows that Poisson brackets between π_{K_i} and φ_{K_i} equal

$$\{\pi_{K_i}, \varphi_{K_j}\} = \delta_{ij} \cdot \frac{1}{|K_i|}. \tag{20}$$

Of course,

$$\{\pi_{K_i}, \pi_{K_j}\} = 0 = \{\varphi_{K_i}, \varphi_{K_j}\}. \tag{21}$$

The coordinate φ_{K_i} can be called the "mean value of the field φ over the set K_i" and π_{K_i} the "mean value of the field π over the set K_i".

Suppose now that we have two observable systems P^1 and P^2 corresponding to two different partitions of Σ:

$$\Sigma = \bigcup_{i=0}^{n_1} K_i^1 = \bigcup_{j=0}^{n_2} K_j^2.$$

It is easy to check that $P^2 \supset P^1$ if and only if the second partition is finer, i.e. every K_j^2 is contained in some K_i^1.

The above special kind of spaces P have been used in constructive quantum field theory as the so-called "lattice approximation".

4. Quantum description of finite number of degrees of freedom

Given a finite symmetric system of classical observables P, we may define a quantum mechanics describing those degrees of freedom which can be measured by P. We do not need for our purposes any sophisticated definition of a quantum state because of the linear structure of P. The space of states is a projective Hilbert space which can be represented by the space of all square-integrable half-densities defined over "position space" P_F. We may also pass to the "momentum representation" and represent quantum states as half-densities over "momentum space" P_D. The simple geometric definition of this Fourier transformation, which does not depend on the choice of coordinates, may be found, e.g., in [6]. Using the canonical coordinate chart (15), one may write this definition in a very conventional form. If $\psi(q^1, \ldots, q^n)$ is a wave function in the position representation then the corresponding wave function in the momentum representation is equal to

$$\bar{\psi}(p_1, \ldots, p_n) = \int \exp\left\{-\frac{i}{\hbar}\sum_{j=1}^{n} p_j q^j\right\} \cdot \psi(q^1, \ldots, q^n)\, d^n q. \tag{22}$$

It is, however, wiser to write the quantum state as a half-density:

$$\psi(q^1, \ldots, q^n) \cdot \sqrt{|dq^1 \wedge \ldots \wedge dq^n|}$$

and

$$\tilde{\psi}(p_1, \ldots, p_n) \cdot \sqrt{|dp_1 \wedge \ldots \wedge dp_n|}$$

in order to keep in mind its transformation rule when passing to another coordinate chart.

In the case of the coordinates π_{K_i} and φ_{K_i} used in Section 3, we cannot use formula (22) since the condition (15) is not fulfilled. However, we may modify slightly the coordinates combining, e.g., the coefficients $|K_i|$ in (18) with π_{K_i}, and define new momenta:

$$P_i = |K_i| \cdot \pi_{K_i}. \tag{23}$$

Now (22) may be applied but we have to change variables:

$$\tilde{\psi}(p_1, \ldots, p_n) \cdot \sqrt{|dp_1 \wedge \ldots \wedge dp_n|} = \tilde{\psi}(|K_1| \cdot \pi_K, \ldots, |K_n| \cdot \pi_{K_n}) \cdot \sqrt{\prod_{i=1}^{n} |K_i|} \cdot \sqrt{|d\pi_{K_1} \wedge \ldots \wedge d\pi_{K_n}|}. \tag{24}$$

We may define the function

$$\tilde{\tilde{\psi}}(\pi_{K_1}, \ldots, \pi_{K_n}) := \sqrt{\prod_{i=1}^{n} |K_i|} \cdot \tilde{\psi}(|K_1| \cdot \pi_{K_1}, \ldots, |K_n| \cdot \pi_{K_n}) \tag{25}$$

which now plays the role of the wave function with respect to coordinates π_{K_i}. Formula (22) now reads:

$$\tilde{\tilde{\psi}}(\pi_K, \ldots, \pi_{K_n}) = \sqrt{\prod_{i=1}^{n} |K_i|} \int \exp\left\{-\frac{i}{\hbar}\sum_{j=1}^{n} |K_j| \cdot \pi_{K_j} \cdot \varphi_{K_j}\right\} \cdot \psi(\varphi_{K_1}, \ldots, \varphi_{K_n})\, d^n \varphi_K. \tag{26}$$

The Hilbert space corresponding to P will be denoted by H_P. We shall also use mixed states represented by density matrices in H_P (i.e. positive nuclear operators with trace equal to unity). The space of all mixed states will be denoted by S_P.

The last space which we need is the C^*-algebra A_P composed of all bounded operators in the space H_P. Elements of A_P are quantum observables. The algebra A_P (or rather its real part) is a quantum analog of our classical system of observables P.

5. Projection operators

Suppose we have two symmetric systems of classical observables $P^1 \subset P^2$. The system P^2 describes more degrees of freedom of our field than P^1 does. The description of a quantum state with respect to P^2 is more detailed than with respect to P^1. A procedure would thus be defined which assigns to each quantum state in S_{P^2} a state in S_{P^1}:

$$\alpha_{P^1 P^2}\colon S_{P^2} \to S_{P^1}. \tag{27}$$

The projection $\alpha_{P^1 P^2}$ could be called "the procedure of forgetting about those degrees of freedom which are not contained in P^1". The "forgetting operator" cannot be defined on the level of pure states because the image of a pure state is not in general pure. This can easily be understood from the following example: Take a quantum state of the system composed of two (different) particles in standard quantum mechanics. Now we can "forget" about the degrees of freedom connected with the first particle. But the resulting state of the second particle is pure if and only if there were no statistical correlations between the first and second particle (i.e. the wave function of the system was the product of the wave functions of each particle).

In order to define the projecting operator $\alpha_{P^1 P^2}$ we start with the discussion of the simplest case. Let P^1 describe only one degree of freedom, namely the mean value of the field over the set $K \subset \Sigma$. Let P^2 describe mean values of the field over K_1 and K_2 and let $K_1 \cap K_2 = \emptyset$ and $K_1 \cup K_2 = K$.

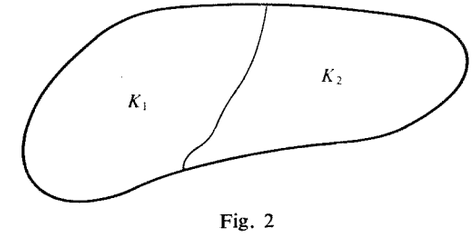

Fig. 2

Let $s \in S_{P^2}$ be a pure state represented by the wave function $\psi(\varphi_{K_1}, \varphi_{K_2})$ (we use for a moment the position representation). Now we change variables:

$$\varphi_K = \frac{|K_1|}{|K|}\varphi_K + \frac{|K_2|}{|K|}\varphi_{K_2}, \quad \xi = \varphi_{K_2} - \varphi_{K_1}. \tag{28}$$

Inversely,

$$\varphi_{K_1} = \varphi_K - \frac{|K_2|}{|K|}\xi, \quad \varphi_{K_2} = \varphi_K + \frac{|K_1|}{|K|}\xi. \tag{29}$$

We substitute (29) into ψ. This enables us to define for a fixed value of ξ a function

$$\psi_\xi(\varphi_K) := \psi\left(\varphi_K - \frac{|K_2|}{|K|}\xi, \varphi_K + \frac{|K_1|}{|K|}\xi\right) \tag{30}$$

which may be considered as a quantum state with one degree of freedom, i.e. a quantum state in S_P. Now the state $\alpha_{P^1P^2}s$ is defined as the mixture of all states ψ_ξ:

$$\alpha_{P^1P^2}s = \int |\psi_\xi\rangle\langle\psi_\xi| d\xi. \tag{31}$$

More precisely: $\alpha_{P^1P^2}s$ is a nuclear operator acting on wave functions depending on the variable φ_K. It may thus be represented by an integral kernel, namely:

$$(\alpha_{P^1P^2}s)(\varphi_K, \bar{\varphi}_K) = \int \psi_\xi(\varphi_K)\psi_\xi^*(\bar{\varphi}_K) d\xi. \tag{32}$$

Having defined $\alpha_{P^1P^2}$ on pure states, we may easily extend this definition by linearity onto the whole S_P, since every state is a combination of pure states.

We could also define $\alpha_{P^1P^2}$ in a similar way using the momentum representation, i.e. wave functions depending on π_{K_1}, π_{K_2} and π_K. It may easily be checked that both definitions give the same result.

Now we define $\alpha_{P^1P^2}$ in the general case. Suppose $P^1 \subset P^2$, i.e. $P_D^1 \subset P_D^2$, $P_F^1 \subset P_F^2$, and, conversely,

$$P_D^1 \supset P_D^{2\perp}, \quad P_F^{1\perp} \supset P_F^{2\perp}.$$

Define subspaces

$$\bar{F} = P_F^2 \cap P_D^{1\perp} \subset F, \quad \bar{D} = P_D^2 \cap P_F^{1\perp} \subset D.$$

Hence it follows from (12) that

$$P_F^2 = P_F^1 \oplus \bar{F}, \quad P_D^2 = P_D^1 \oplus \bar{D}. \tag{33}$$

Take any coordinates $(q^1, \ldots, q^n)$ in P_F^2 such that $(q^1, \ldots, q^m)$ are coordinates in P_F^1 $(m \leqslant n)$ and $(q^{m+1}, \ldots, q^n)$ are coordinates in $\bar{F}$. Let $s \in S_{P^2}$ be any pure state and let

$$\psi(q^1, \ldots, q^n) \cdot \sqrt{|dq^1 \wedge \ldots \wedge dq^n|}$$

be its position representation. Now the state $\alpha_{P^1P^2}s$ may be represented by the integral kernel

$$(\alpha_{P^1P^2}s)(q^1, \ldots, q^m; \bar{q}^1, \ldots, \bar{q}^m)$$
$$= \left\{\int \psi(q^1, \ldots, q^m, q^{m+1}, \ldots, q^n)\psi^*(\bar{q}^1, \ldots, \bar{q}^m, \bar{q}^{m+1}, \ldots, \bar{q}^n) dq^{m+1} \ldots dq^n\right\} \times$$
$$\times \sqrt{|dq^1 \wedge \ldots \wedge dq^m|} \cdot \sqrt{|dq^1 \wedge \ldots \wedge dq^m|}. \tag{34}$$

The above definition may be extended by linearity onto the whole space S_P.

PROPOSITION. *The above definition has a good geometrical meaning (i.e. it is independent of the choice of coordinates q^i).*

A similar definition of $\alpha_{P^1P^2}$ could be given in the momentum picture. The reader may easily check that both definitions of $\alpha_{P^1P^2}$ coincide.

In fact, our definition may be generalized to any linear polarization in P^2 and P^1 (in the sense of [7]), not only positions and momenta (cf. also [6]).

6. Description of an infinite number of degrees of freedom

It is quite easy to check that our "forgetting operators" satisfy the chain rule:

$$\alpha_{P^1P^2}\alpha_{P^2P^3} = \alpha_{P^1P^3} \tag{35}$$

for $P^3 \supset P^2 \supset P^1$. This means that the family S_P together with operators α forms a projective system. On the other hand, we have a dual system of C^*-algebras of quantum observables A_P. The duality between S_P and A_P is given by the mean value of the observable on the state:

$$\langle a\rangle_s := \mathrm{Tr}(s \cdot a). \tag{36}$$

For given $s \in S_P$ the above formula defines a positive linear normalized functional on A_P, i.e. a state in the sense of C^*-algebras. We use the above duality to define the transposed operators

$$\begin{array}{ccc} S_{P^2} & \xrightarrow{\alpha_{P^1P^2}} & S_{P^1} \\ \text{duality}\uparrow\downarrow & & \text{duality}\uparrow\downarrow \\ A_{P^2} & \xleftarrow{\alpha^*_{P^1P^2}} & A_{P^1} \end{array}$$

For given $a \in A_{P^1}$ we define the element $(\alpha^*_{P^1P^2}a) \in A_{P^2}$ by the formula

$$\langle \alpha^*_{P^1P^2}a\rangle_s = \langle a\rangle_{\alpha_{P^1P^2}s} \tag{37}$$

which has to be fulfilled for any $s \in S_{P^2}$. The chain rule for the operators α^* is obviously satisfied. The following theorem can be proved.

THEOREM 1. *The operator $\alpha^*_{P^1P^2}$ is a C^*-isomorphism of A_{P^1} onto its image in A_{P^2}. We may thus identify A_{P^1} with its image and treat it as a subalgebra in A_{P^2}:*

$$A_{P^1} \subset A_{P^2} \tag{38}$$

for $P^1 \subset P^2$.

The family of algebras A_P thus forms an inductive system.

We now take projective and inductive limits of the systems we have defined:

$$\mathscr{S} = \varprojlim_{P} S_P, \tag{39}$$

$$\mathscr{A} = \varinjlim_{P} A_P. \tag{40}$$

Using (38) we may, of course, replace the inductive limit by the completion of the sum:

$$\mathscr{A} = \overline{\bigcup_{P} A_P} \tag{41}$$

(the elements which can be joined by α^* are identified).

Points in the space $\mathscr{S}$ are sequences $\mathbf{s} = (s_P)$, $s_P \in S_P$, which satisfy the coherence condition

$$S_{P^1} = \alpha_{P^1 P^2} S_{P^2} \tag{42}$$

for $P^1 \subset P^2$.

The algebra $\mathscr{A}$ is a full observable algebra of the quantum field. It already describes infinitely many degrees of freedom.

Elements of $\mathscr{S}$ are states on the algebra $\mathscr{A}$ in the mathematical sense (i.e. linear positive normalized functionals). The value of $\mathbf{s} = (s_P)$ on $a \in A_{P^0} \subset \mathscr{A}$ is equal to

$$\langle a \rangle_{\mathbf{s}} = \langle a \rangle_{s_P} \tag{43}$$

where P is any index which follows P^0, i.e. $P \supset P^0$. The above definition may be extended by linearity and continuity onto the whole algebra $\mathscr{A}$.

But the space $\mathscr{S}$ is too large for purposes of quantum field theory. It may be useful in thermodynamics (e.g., as description of the thermodynamical limit or of states with infinitely many particles). Quantum field theory, however, describes only a small sector of $\mathscr{S}$ which corresponds to the simplest physical situations. This sector is composed only of the vacuum-state and all states which may be generated from vacuum by field operators.

In order to define an adequate space of quantum states in our approach we have to choose an element $\Omega = (\Omega_P) \in \mathscr{S}$ which will play the role of the vacuum state.

In the next section we shall discuss the problem of how to find this element. This is mathematically the most difficult problem in our theory. Suppose, however, that Ω has already been found. Finding Ω enables us to define the Hilbert space for our theory through the standard procedure (cf. [8]). We define, namely, the scalar product in $\mathscr{A}$

$$(a|b) := \langle a \cdot b^* \rangle_\Omega \tag{44}$$

which is in general degenerate. Take its null space:

$$N := \{a \in \mathscr{A} \colon (a|a) = 0\}.$$

Then the Hilbert space H is a quotient

$$H = \overline{\mathscr{A}/N}. \tag{45}$$

The vacuum state is now a pure state corresponding to that element of H which is an equivalence class containing the unit operator $I \in \mathscr{A}$. Mixed states of the field are, as usual, positive nuclear operators with unit trace acting in H.

The algebra of observables $\mathscr{A}$ has a natural representation as an operator algebra in the Hilbert space H. For different Ω's we may get different (inequivalent) representations of $\mathscr{A}$. We have thus reduced the problem of "representation of canonical commutation relations" to the problem of the choice of the vacuum state Ω. But it is already a dynamical problem which lies beyond the kinematics.

The above representation of quantum states is, however, not very convenient for further calculations. Another representation can be used which gives much deeper insight into the physical meaning of our theory. In order to define it, take the space $\mathscr{S}^0_\Omega \subset \mathscr{S}$ composed of all states which are equal to Ω "almost everywhere", i.e.

$$\mathscr{S}^0_\Omega = \{(s_P) \in \mathscr{S}\colon \text{there exists } a \in A_{P^0} \text{ such that for } P \supset P^0\ s_P = a^*_P \Omega_P a_P, \text{ where } a_P = \alpha^*_{P^0 P} a\}. \tag{46}$$

Now take its completion

$$\mathscr{S}_\Omega = \overline{\mathscr{S}^0_\Omega}. \tag{47}$$

The following theorem may then be proved:

THEOREM 2. *There exists a natural isomorphism between $\mathscr{S}_\Omega$ and the space of all mixed states in H. Pure states correspond to extremal points of the convex set $\mathscr{S}_\Omega$.*

We see that states of the field form a subspace in $\mathscr{S}$. This is just the "vacuum sector" about which we have spoken.

7. Energy and vacuum

The vacuum is a state corresponding to the lowest energy level. But how to define "energy" in our theory?

Consider first the simplest case: a flat space Σ embedded in flat space-time. The classical Hamiltonian of the field is given by

$$\mathscr{H} = \int_\Sigma \{\tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}(\nabla\varphi(x))^2 + G(\varphi(x))\}\, d^3x, \tag{48}$$

where, e.g., $G(\varphi) = \frac{1}{2}m^2\varphi^2$ (linear case) or $G(\varphi) = \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4$ (the simplest nonlinear case). The simplest way of defining the quantum Hamiltonian consists in replacing values of the field in formula (48) by mean values over sets K_i and taking the sum instead of the integral:

$$\mathscr{H}_P = \tfrac{1}{2}\sum_i (\pi_{K_i})^2 + \tfrac{1}{2}\sum_{i,j}{}' \left(\frac{\varphi_{K_i} - \varphi_{K_j}}{l_{ij}}\right)^2 + \sum_i G(\varphi_{K_i}). \tag{49}$$

Here, P is used as in Section 3. The primed sum Σ' means that we sum only over the set of nearest neighbours and l_{ij} denotes the distance between mass-centers of K_i and K_j. Now we define the approximate quantum Hamiltonian $\hat{\mathscr{H}}_P$, replacing φ_{K_i} and π_{K_i} by

operators. In the position representation, e.g., we put

$$\pi_{K_j} = \frac{1}{|K_j|} \cdot \frac{\hbar}{i} \frac{\partial}{\partial \varphi_{K_j}}$$

by virtue of formula (26) or (23). The operator $\hat{\mathscr{H}}_P$ is thus equivalent to the Hamiltonian of a many-particle system in the ordinary quantum mechanics. We recognize in equation (49) the standard terms: (1) kinetic energy, (2) interaction via harmonic-oscillator forces, (3) external forces whose potential equals G. The existence of the unique lowest-energy state $\tilde{\Omega}_P$ for the Hamiltonian $\hat{\mathscr{H}}_P$ depends on the shape of the potential G. If $\tilde{\Omega}_P$ does exist (like in $\lambda\varphi^4$-theory), then we ask about the existence of the following limit:

$$\Omega_P = \lim_{P^1} \alpha_{PP^1} \tilde{\Omega}_{P^1}. \tag{50}$$

If the answer is "yes" again, then it can easily be proved that the sequence $\Omega = (\Omega_P)$ is coherent with respect to projections $\alpha_{P^1P^2}$, i.e. $\Omega \in \mathscr{S}$.

It may happen, however, that $\hat{\mathscr{H}}_P$ has several lowest-energy states. For example, this is the case where $G(\varphi)$ has the shape shown in Fig. 3.

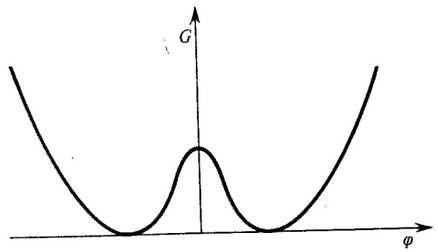

Fig. 3

Such a situation may give rise to the field theory containing many vacuum states and the phenomenon of spontaneous symmetry breaking. It may also happen that the limit (50) does not exist. Such a theory cannot be quantized (at least in our sense).

In the non-flat case the situation becomes a little bit more complicated. In that case there does not exist a unique Hamiltonian. But for every non-vanishing timelike vector field ξ in space-time we may define the classical Hamiltonian $\mathscr{H}_\xi$ as a generator of the 1-parameter group of transformations defined by ξ. Now we may use $\mathscr{H}_\xi$ in the same way as we previously used $\mathscr{H}$. It remains to check whether the vacuum state is independent of the choice of ξ.

In the case of general P, the essential problem is how to find the approximate quantum Hamiltonian $\hat{\mathscr{H}}_P$. We may always take the value of the classical Hamiltonian $\mathscr{H}$ on the subspace P (i.e. $\mathscr{H}_P = \mathscr{H}|P$) and quantize it as in ordinary quantum mechanics. Such a procedure leads, however, to certain divergences and needs some "renormalization".

Physical intuition suggests that $\mathscr{H}_P$ (or $\hat{\mathscr{H}}_P$) should be obtained by some projection from $\mathscr{P}$ to $\mathscr{P}_P$ rather than by restriction to $P \subset \mathscr{P}$.

There is, however, the possibility of another approach. The dynamics may already be defined at the level of the space $\mathscr{S}_P$ (before the choice of Ω) by means of a procedure similar to Feynmann integrals. Now $\hat{\mathscr{H}}_P$ should be defined to be a generator of this dynamics. This approach will be discussed in the next paper.

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