Can twistors be useful in physics?

Iwo Bialynicki-Birula
Center for Theoretical Physics
and Warsaw University
Warsaw, Poland

Summary

A formulation of relativistic wave mechanics is proposed
in terms of twistors
It enables one to easily generate the solutions of the Maxwell, Weyl,
and other equations for massless particles
The key role in this formulation is played by
a 4+4 dimensional phase-space
Twistors arise as a convenient parametrization of this phase-space.

Main idea:

Phase-space is completely separated from space-time

Main advantage:

Fully symmetric treatment of space and time

Main result:

Twistor kinematics generates solutions of wave equations

I shall discuss only a theory of massless particles but a generalization to the massive case is also given

Wigner function and its time evolution

$$W(\vec{r},\vec{p},t) = \int d^3\eta\ \Psi(\vec{r}+\frac{\vec{\eta}}{2},t)\Psi^*(\vec{r}-\frac{\vec{\eta}}{2},t)e^{-i\vec{p}\cdot\vec{\eta}/\hbar}$$

The time evolution of the Wigner function for a linear system is the same as that of a classical distribution function

$$W(\vec{r},\vec{p},t) = W(\vec{r}(t),\vec{p}(t))$$

$(\vec{r}(t),\vec{p}(t))$ is the classical trajectory in phase space

In particular, for a free particle $\vec{r}(t) = \vec{r} + \vec{p}\,t/m,\ \vec{p}(t) = \vec{p}$

In this case, the time evolved Wigner function is obtained from its initial form by a simple substitution

$$W(\vec{r},\vec{p},t) = W(\vec{r}+\vec{p}\,t/m,\vec{p})$$

New phase-space

In the nonrelativistic theory the vector $\vec{r}$ plays a dual role:
it lives simultaneously in the phase-space and in the physical space

Can one restore the full space-time symmetry?

This can be done by replacing the 3+3 dimensional phase-space
by the 4+4 dimensional phase-space

All four space-time variables will be treated on equal footing
but they will be completely separated from the phase-space coordinates

$$q \to q^i,\ i = 1, \dots, 4 \quad p \to p_i,\ i = 1, \dots, 4$$

$$t \to x^\mu = (t, x, y, z)$$

Since there is no mass here, I assume nonstandard dimensions

$$[q] = \text{Length}\sqrt{\text{Momentum}}, \quad [p] = \sqrt{\text{Momentum}}$$

The analogy

The analogs of nonrelativistic Wigner functions $W(\vec{r}, \vec{p}, t)$ will be functions on the 4+4 dimensional phase-space: $W(q^i, p_i, x^\mu)$

There are several questions concerning this analogy:

- What is the counterpart of the free motion $\vec{r}(t) = \vec{r} + \vec{p}\,t/m$?
- What are the geometric properties of the new phase-space?
- What is the connection between (q^i, p_i) and x^μ?

The space-time translation (analog of free motion) should look like

$$q^i(x^\mu) = q^i + \mathcal{G}^{ik}_\mu x^\mu p_k, \;\; p_k(x^\mu) = p_k$$

This formula suggests the interpretation of the new phase-space as the space of twistors because a twistor $(\phi^A, \kappa_{\dot{A}})$ transforms as

$$'\phi^A = \phi^A + g_\mu{}^{A\dot{B}} \kappa_{\dot{B}} x^\mu, \;\; '\kappa_{\dot{A}} = \kappa_{\dot{A}}$$

Twistors

I choose the following relation of twistors to the phase-space points

$$\boxed{\begin{aligned} \phi^1 = \frac{q^1 + iq^2}{\sqrt{2}}, \quad \phi^2 = \frac{q^3 + iq^4}{\sqrt{2}}, \\ \kappa_{\dot{1}} = \frac{p_1 + ip_2}{\sqrt{2}}, \quad \kappa_{\dot{2}} = \frac{p_3 + ip_4}{\sqrt{2}} \end{aligned}}$$

This identification enables one to treat the pair $(\phi^A, \kappa_{\dot{A}})$
as the canonically conjugate variables with the Poisson brackets

$$\{q^i, p_k\} = \delta^i_k, \quad \rightarrow \quad \{\phi^A, \kappa_B\} = \delta^A_B, \quad \{\phi^{\dot{A}}, \kappa_{\dot{B}}\} = \delta^{\dot{A}}_{\dot{B}}$$

The symplectic structure based on these Poisson brackets allows for the introduction of the 36-parameter group of symplectic transformations

The group $U(2,2)$ is a subgroup of the symplectic group
It covers the inhomogeneous conformal group in Minkowski space

Generators of the conformal group

The generators are the following 16 real quadratic forms
of the canonical variables

$$\begin{aligned}
P_\mu &= \kappa_A g_\mu{}^{A\dot{B}} \kappa_{\dot{B}} && \text{Translations} \\
K_\mu &= \phi^{\dot{A}} g_{\mu\,\dot{A}B} \phi^B && \text{Proper conformal transformations} \\
M_{\mu\nu} &= \frac{1}{2}\left(\kappa_A S_{\mu\nu}{}^A{}_B \phi^B + \phi^{\dot{A}} S_{\mu\nu\dot{A}}{}^{\dot{B}} \kappa_{\dot{B}}\right) && \text{Rotations \& Lorentz tr.} \\
D &= \frac{1}{2}(\kappa_A \phi^A + \phi^{\dot{A}} \kappa_{\dot{A}}) && \text{Dilation} \\
S &= \frac{i}{2}(\kappa_A \phi^A - \phi^{\dot{A}} \kappa_{\dot{A}}) && \text{Phase transformations}
\end{aligned}$$

They generate linear canonical transformations
through the standard canonical Poisson brackets $\{f, G\}$

In particular, $\{\phi^A, P_\mu\} = g_\mu{}^{A\dot{B}} \kappa_{\dot{B}}$, $\{\phi^{A\dot{B}}, P_\mu\} = g_\mu{}^{\dot{A}B} \kappa_B$

Canonical quantization

The standard rules for the canonical operators

$$\left[\hat{\phi}^A, \hat{\kappa}_B\right] = i\hbar\{\phi^A, \kappa_B\} = i\hbar\delta^A_B, \quad \left[\hat{\phi}^{\dot{A}}, \hat{\kappa}_{\dot{B}}\right] = i\hbar\{\phi^{\dot{A}}, \kappa_{\dot{B}}\} = i\hbar\delta^{\dot{A}}_{\dot{B}}$$

The Schrödinger (coordinate) representation

$$\begin{aligned} \hat{\phi}^A\Upsilon(\phi,\phi^\star) &= \phi^A\Upsilon(\phi,\phi^\star), \quad \hat{\phi}^{\dot{A}}\Upsilon(\phi,\phi^\star) = \phi^{\dot{A}}\Upsilon(\phi,\phi^\star), \\ \hat{\kappa}_A\Upsilon(\phi,\phi^\star) &= \frac{\hbar}{i}\frac{\partial}{\partial\phi^A}\Upsilon(\phi,\phi^\star), \quad \hat{\kappa}_{\dot{A}}\Upsilon(\phi,\phi^\star) = \frac{\hbar}{i}\frac{\partial}{\partial\phi^{\dot{A}}}\Upsilon(\phi,\phi^\star) \end{aligned}$$

The Schrödinger equation in three dimensions

$$i\hbar\partial_t\psi(\vec{r},t) = -\hbar^2\frac{\Delta_3}{2m}\psi(\vec{r},t)$$

according to the formula $P_0 = \kappa_A {g_0}^{A\dot{B}}\kappa_{\dot{B}}$ is replaced by
the Schrödinger equation in four dimensions

$$i\hbar\partial_t\Upsilon(\phi,\phi^\star,t) = -c\hbar^2(\partial_1\partial_{\dot{1}} + \partial_2\partial_{\dot{2}})\Upsilon(\phi,\phi^\star,t) = -c\hbar^2\frac{\Delta_4}{2}\Upsilon(\phi,\phi^\star,t)$$

Full symmetry between space and time

In addition to the time translation we have space translations
In contrast to nonrelativistic wave mechanics
all four translations lead to the Schrödinger-like equations

$$\boxed{i\hbar\partial_\mu\Upsilon(\phi,\phi^\star,x) = -\hbar^2 g_\mu{}^{A\dot{B}}\frac{\partial}{\partial\phi^A}\frac{\partial}{\partial\phi^{\dot{B}}}\Upsilon(\phi,\phi^\star,x)}$$

We may solve these four equations simultaneously
because all operators P_μ commute among themselves

Full symmetry between space and time is restored

The simplest solutions of these equations are plane waves in ϕ^A and $\phi^{\dot{A}}$

$$\Upsilon(\phi,\phi^\star,x) = e^{i(\phi^A\kappa_A+\kappa_{\dot{A}}\phi^{\dot{A}})}e^{\pm i\kappa_A x^{A\dot{B}}\kappa_{\dot{B}}}, \quad x^{A\dot{B}} = g_\mu{}^{A\dot{B}}x^\mu$$

All other solutions can be viewed as superpositions of such plane waves

The wave equation

Every solution of our four Schrödinger-like equations
is a solution of the d'Alembert wave equation

$$\Box\Upsilon(\phi,\phi^\star,x)=0$$

With the help of the standard relations for the ${g_\mu}^{A\dot B}$'s we obtain

$$P_\mu P^\mu=\kappa_A {g_\mu}^{A\dot B}\kappa_{\dot B}\kappa_C g^{\mu C\dot D}\kappa_{\dot D}=2\kappa_A\kappa^A\kappa_{\dot A}\kappa^{\dot A}=0$$

Therefore for every solution of the four equations we have

$$-\hbar^2\partial_\mu\partial^\mu\Upsilon(\phi,\phi^\star,x)=P_\mu P^\mu\Upsilon(\phi,\phi^\star,x)=0$$

Wave packets of plane waves are solutions of the four Schrödinger-like equations

$$\boxed{\Upsilon(\phi,\phi^\star,x)=\int d\kappa d\kappa^* f(\kappa,\kappa^*)e^{i(\phi^A\kappa_A+\kappa_{\dot A}\phi^{\dot A})}e^{\pm i\kappa_A x^{A\dot B}\kappa_{\dot B}}}$$

Solutions of the equations for massless particles

All wave equations for massless particles with spin can be written as

$$g^{\mu\dot{A}A}\partial_\mu\psi_{AB,\ldots,X}(x) = 0, \quad \text{or} \quad g^{\mu A\dot{A}}\partial_\mu\psi_{\dot{A}\dot{B},\ldots,\dot{X}}(x) = 0,$$

where $\psi_{AB,\ldots,X}(x)$ and $\psi_{\dot{A}\dot{B},\ldots,\dot{X}}(x)$ are symmetric spinor fields

The simplest of these equations are the Weyl and Maxwell equations

$$g^{\mu\dot{A}A}\partial_\mu\psi_A(x) = 0, \quad \text{and} \quad g^{\mu\dot{A}A}\partial_\mu\psi_{AB}(x) = 0$$

The construction of the symmetric spinors obeying these equations is surprisingly very simple

$$\psi_{AB,\ldots,X}(x) = \frac{\partial}{\partial\phi^A}\cdots\frac{\partial}{\partial\phi^X}\Upsilon(\phi,\phi^\star,x^\mu)$$

The only condition on $\Upsilon(\phi,\phi^\star,x^\mu)$ is that it must be a solution of the four Schrödinger-like equations

Solutions of the Maxwell equations

We can transform Maxwell equations to spinorial form

$$\psi_{00} = -E_x - B_y - i(E_y - B_x),\ \psi_{01} = E_z + iB_z,\ \psi_{11} = E_x - B_y - i(E_y + B_x)$$

The inverse transformation reads

$$E_x = \mathrm{Re}\frac{\psi_{11} - \psi_{00}}{2},\ E_y = -\mathrm{Im}\frac{\psi_{11} + \psi_{00}}{2},\ E_z = \mathrm{Re}(\psi_{01}),$$

$$B_x = -\mathrm{Im}\frac{\psi_{11} - \psi_{00}}{2},\ B_y = -\mathrm{Re}\frac{\psi_{11} + \psi_{00}}{2},\ B_z = \mathrm{Im}(\psi_{01})$$

The prescription for generating solutions of the Maxwell equations is

- Choose a function $\Upsilon(\phi, \phi^\star)$ of ϕ^A and $\phi^{\dot{A}}$
- Propagate Υ in space-time with the four Schrödinger-like equations
- Differentiate $\Upsilon(\phi, \phi^\star, x^\mu)$ with respect to ϕ^A and ϕ^B to obtain ψ_{AB}
- Express the electromagnetic field vectors in terms of ψ_{AB}

Example 1: Circularly polarized electromagnetic wave

- Choose the plane-wave solution of the four Schrödinger-like equations
$$\Upsilon_P(\phi,\phi^\star,x) = e^{i(\phi^A\kappa_A+\kappa_{\dot A}\phi^{\dot A})}e^{-ik_\mu x^\mu}$$
where κ_A is an arbitrary spinor and $k_\mu = \kappa_A {g_\mu}^{A\dot B}\kappa_{\dot B}$
- Differentiations lead to $\psi_{AB} = -\kappa_A\kappa_B e^{i(\phi^A\kappa_A+\kappa_{\dot A}\phi^{\dot A})}e^{-ik_\mu x^\mu}$
- In order to make the final formulas for the electromagnetic field simple I choose the coordinate system in such a way that $\kappa_A = \{0,\kappa\}$
- With this choice of κ_A the components of $\psi_{AB}(x)$ become
$$\psi_{00}(x) = 0,\ \ \psi_{01}(x) = 0,\ \ \psi_{11}(x) = \kappa^2 e^{ik(z-ct)}$$
- The resulting expressions for the components of the electric and magnetic field coincide with the textbook results (I choose $\kappa = \sqrt{2A}$)
$$E_x = A\cos[k(z-ct)],\ E_y = A\sin[k(z-ct)],\ E_z = 0$$
$$B_x = A\sin[k(z-ct)],\ B_y = -A\cos[k(z-ct)],\ B_z = 0$$

Example 2: Gaussian wave packet

- Let us choose the Gaussian wave packet like in NR quantum mechanics

$$\Upsilon_G(\phi,\phi^\star,x) = \frac{1}{\pi^2}\int d\kappa d\kappa^* e^{-\kappa_A a^{A\dot{B}}\kappa_{\dot{B}}} e^{i(\phi^A\kappa_A+\kappa_{\dot{A}}\phi^{\dot{A}})} e^{-i\kappa_A x^{A\dot{B}}\kappa_{\dot{B}}}$$

  where a is a vector lying in the future light cone

- Four simple Gaussian integrations lead to

$$\Upsilon_G(\phi,\phi^\star,x) = -\frac{1}{s^2} e^{-\phi^{\dot{A}} s_{\dot{A}B}\phi^A/s^2}, \quad s_{\dot{A}B} = g_{\mu\,\dot{A}B}s^\mu$$

  where $s^\mu = x^\mu - ia^\mu$. In the simplest case when $\phi^A = 0 = \phi^{\dot{A}}$ we recover the well known fundamental solution $1/s^2$ of the wave equation

- Differentiations lead to

$$\psi_{AB} = -\frac{\phi^{\dot{A}} s_{\dot{A}A}\,\phi^{\dot{B}} s_{\dot{B}B}}{s^6}\, e^{-\phi^{\dot{C}} s_{\dot{C}D}\phi^D/s^2}$$

  This is a fairly complicated electromagnetic field so it would be very hard to find directly this solution working with the Maxwell equations

Characteristic properties

The field vectors are quite complicated, for example,

$$\begin{gathered}
\text{den } E_x = a^8 + 2\left(r^2 - 2t^2 + x^2 - 5tz\right) a^6 + 12txya^5 - \\
2\left(5t^4 - 5zt^3 + \left(15x^2 - 9r^2\right) t^2 + 9r^2 zt - 3r^2 x^2\right) a^4 - \\
8t\left(5t^2 - 3r^2\right) xya^3 - \\
2\left(t^2 - r^2\right)\left(2t^4 - 9zt^3 + \left(11r^2 - 15x^2\right) t^2 - 3r^2 zt + r^2\left(3x^2 - r^2\right)\right) a^2 + \\
12t\left(r^2 - t^2\right)^2 xya - \left(r^2 - t^2\right)^3 \left(y^2 - x^2 + (t - z)^2\right)
\end{gathered}$$

$$\text{den} = 2\left(a^4 + 2\left(t^2 + r^2\right) a^2 + \left(-t^2 + r^2\right)^2\right)^3$$

This is a null field: $\vec{E}^2 = \vec{B}^2$ and $\vec{E} \cdot \vec{B} = 0$

Energy density and the energy flux are simpler
than the field components because to calculate them
we do not have to separate the real and imaginary parts

Energy density and the Poynting vector

$$\text{EnergyDensity}(\rho, z, t) = \frac{1}{4} \frac{a^2 + \rho^2 + (t-z)^2}{\left((a^2 + t^2 - \rho^2 - z^2)^2 + 4a^2(\rho^2 + z^2)\right)^3}$$

$$\text{PoyntingVector}(x, y, z, t) = \frac{\text{EnergyDensity}}{a^2 + x^2 + y^2 + (t-z)^2}$$

$$\times \left\{2(x(t-z) - ay), 2(y(t-z) + ax), \left(x^2 + y^2 - a^2 - (t-z)^2\right)\right\}$$

Total energy is finite and, of course, time-independent

Gaussian wave packets with an embedded vortex ring

Null fields may carry true vortices because we may construct new solutions of Maxwell equations in the form

$$\vec{F}' = \phi\vec{F}$$

where ϕ must obey some differential equations

Choosing appropriate multiplier functions we may construct solutions with embedded vortices

Such a solution will be shown in pictures because the formulas are too complex to fit on the screen

Summary

Twistors are usually introduced in the highly
sophisticated abstract field theories
In the twistor program of Roger Penrose they are supposed
to play a fundamental role in physics

I have shown how twistors can be employed in
standard physics to generate new solutions of the Maxwell
and other wave equations.

The solutions are obtained from a solution of
the four Schrödinger-like equations
All that one has to do next is to differentiate and to
separate the real and imaginary parts

It remains to be seen whether the 4+4 phase space parametrized by
twistors is more than just a convenient mathematical construction