# *On the linearity of the Schrödinger equation*

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# Nonlinear Schrödinger equation

- Almost 30 years ago we (my collaborator Jerzy Mycielski and myself) embarked on a program to test the linearity of the Schrödinger equation
  *Nonlinear wave mechanics, Annals of Physics, 100, 62 (1976)*
- In order to carry out this task we needed some nonlinear equation that could be used to quantify (possible) departures from the strictly linear regime
- We required this equation to be as close as possible to the linear equation to make sure that we would only violate the linearity but that we would keep two other important properties of quantum mechanics intact:
  - the locality and
  - the separability of noninteracting subsystems

# Logarithmic Schrödinger equation

- We discovered that there is a rather simple nonlinear equation that has the required properties: The Schrödinger equation with a logarithmic nonlinearity (in any number of dimensions)

$$i\hbar\partial_t\psi(\vec{r},t) = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{r},t) - b\ln\frac{|\psi(\vec{r},t)|^2}{\rho_0}\right)\psi(\vec{r},t)$$

  $b$ measures the violation of linearity on the energy scale

- The locality clearly holds here
- The separability follows from the basic property of the logarithm: If there are no forces acting between the two parts of the system $V(\vec{r},t) = V(\vec{r}_1,t) + V(\vec{r}_2,t)$ then the nonlinearity should not lead to any correlations

# Additional properties

- In other words: the product wave function made of the solutions $\psi(\vec{r}_1, t)$ and $\psi(\vec{r}_2, t)$ of the Schrödinger equations for the subsystems is a solution of the equation for the whole system
- The nonlinear term does not introduce any unwanted coupling between the noninteracting parts
- As additional bonuses we get the validity of the probabilistic interpretation (normalization of $\psi(\vec{r}, t)$) and of the Planck's relation $E = \hbar\omega$ for all stationary states $\psi(\vec{r}, t) = e^{-i\omega t}\psi(\vec{r})$, where $E$ is the mean energy (including the nonlinear contribution)
- The logarithmic Schrödinger equation in the absence of external forces possesses analytic solutions in the form of Gaussons — stable Gaussian wave packets

# Proposal by Abner Shimony

- In 1979 Abner Shimony came up with an ingenious proposal how to find evidence against nonlinear theories in neutron diffraction experiments. Since the nonlinear terms involve $|\psi|^2$, the attenuation of the beam will lead to a phase shift that can be seen in the interference experiments
- In his paper Shimony estimated that the existing double-crystal neutron interferometer developed by Zeilinger, Shull, Horne, and Squires would enable one to set an upper limit of $1.5 \cdot 10^{-12}$eV on the constant $b$ if the interference fringes were not shifted. In an experiment carried our a year later to test this prediction, Shull, Atwood, Arthur, and Horne have shown that this limit is $3.4 \cdot 10^{-13}$eV

# Gähler-Klein-Zeilinger experiment

- Further reduction of the allowed value of $b$ and by full two orders of magnitude became possible with the use of a different method. In their experiment carried out at the Laue-Langevin Institute in Grenoble, Gähler, Klein, and Zeilinger observed the Fresnel diffraction neutrons on the sharp edge. Neutrons were moving at the speed of only 2 meters/sec and they traveled 10 meters before being detected.
- By comparing their results with the hypothetical nonlinear term in the Schrödinger equation, they were able to put the upper limit on $b$ at the value $3.3 \cdot 10^{-15}$eV
- This was a very small value
- But that was not yet the end!

# Weinberg's nonlinear theory

- In 1989 Steven Weinberg published a Physical Review Letter in which he has shown that the most sensitive test of the linearity of quantum mechanics is in the domain of hyperfine transitions. Such a test uses the truncated Schrödinger equation including only a few hyperfine levels
- In his paper Weinberg estimated on the basis of then available experimental information that the upper limit of the nonlinear term is of the order of $10^{-15}$eV
- This information came from experiments on hyperfine transitions in Beryllium ions carried out by the Wineland's group at the Time and Frequency Division of the NBS in Boulder

# Most stringent limits

- Within a year after the appearance of Weinberg's paper, three groups performed dedicated experiments to establish stringent upper limits on the nonlinear terms
  - Bollinger, Heinzen, Itano, Gilbert, and Wineland using $^{3}$Be$^{+}$ obtained $2.4 \cdot 10^{-20}$eV
  - Chupp and Hoare using $^{21}$Ne obtained $1.3 \cdot 10^{-19}$eV
  - Majumder, Venema, Lamoreaux, Heckel, and Fortson using $^{201}$Hg obtained $10^{-20}$eV
- These are incredibly stringent limits
  - $10^{-20}$eV = $2.4\mu$Hz
- In view of this overwhelming evidence we must conclude that the Schrödinger equation is linear

# Frobenius and Perron

- Every nonlinear set of equations can be made linear by raising the level of the mathematical description
- This idea goes back to Frobenius and Perron and more recently it has been used in the theory of chaos
- Let us take classical mechanics
  - For definitness, I shall consider an anharmonic oscillator in one dimension

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q - \lambda q^3 \end{aligned}$$

  - It is easy to replace these nonlinear equations by an equivalent linear equation

# Nonlinear vs. linear

- To this end, let us define a complex “wave function”

$$\psi(\xi,\eta,t) = e^{i(\xi q(t)+\eta p(t))}$$

- Note that $q \to -i\partial_\xi$ and $p \to -i\partial_\eta$
- With the use of the classical equations of motion

$$i\partial_t\psi(\xi,\eta,t) = i(\xi\partial_\eta - \eta\partial_\xi + \lambda\eta\partial_\xi^3)\psi(\xi,\eta,t)$$

- Of course, this is just an illustration of the concept but not a solution of the problem
- Perhaps a similar mechanism can account also for the linearity of the Schrödinger equation but this is only my wild guess