Quantum electrodynamics of qubits

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Qubits are physical objects

- Quantum computing differs from classical computing
- Bits and bytes can be treated as mathematical objects because in classical physics we can disregard the influence of the observer and of the environment
- Classical computers are robust even at room temperature — Bit error rates (BER) are 10^{-12} or better
- They certainly do not suffer from decoherence
- Therefore, we can disregard the physical nature of bits
- Quantum computers are fragile
- We must worry about the interactions of qubits
- Electromagnetically controlled qubits always interact with the omnipresent quantized electromagnetic field

Standard Perturbation Theory

- Perturbation theory is perhaps the most useful and universal tool in quantum mechanics
- There are various versions of perturbation theory in standard quantum mechanics:
    - Time independent PT with nondegerate levels and with degenerate levels
    - Raleigh-Schrödinger PT and Brillouin-Wigner PT
    - Time-dependent Dirac PT: interaction representation
- All these versions of perturbation theory lack the simplicity, elegance, and versatility of the perturbation theory that has been developed and successfully applied in relativistic quantum field theory
- This theory can also be used in a nonrelatistic theory

Feynman perturbation theory

- Feynman perturbation theory was invented for relativistic field theory
  - I will show that it can also be successfully applied to quantum mechanics of qubits
- The main ingredient of Feynman perturbation theory are Feynman propagators
  - Feynman diagrams are a visualization of expressions built from Feynman propagators
- Question:
  Can one introduce Feynman propagators for qubits?
- Answer:
  Yes, in framework of the second quantization formalism

Advantages of Feynman PT

- One never needs to calculate the ground state
- Wick theorem allows for the expression of all quantities in terms of free one-particle propagators
- The number of terms is greatly reduced as compared to the standard perturbation theory
- Simple algebraic properties of the integrands allow for the calculation of integrals by the method of residues
- Analytic properties of the full Feynman propagators allow for the extension of the scattering theory
  - To linear response theory by the analytic continuation in the energy variable
  - To the theory at finite temperature by the analytic continuation in the time variable

Second quantization of qubits

- In the standard description the state of a qubit is described by a vector in a two-dimensional space

$$\psi = \begin{pmatrix} \psi_e \\ \psi_g \end{pmatrix}$$

- In the formalism of second quantization the components of the wave function become annihilation operators $\boldsymbol{\psi}_\alpha$ their complex conjugates become creation operators $\boldsymbol{\psi}_\alpha^\dagger$

$$\{\boldsymbol{\psi}_\alpha, \boldsymbol{\psi}_\beta^\dagger\} = \delta_{\alpha\beta} \quad \{\boldsymbol{\psi}_\alpha, \boldsymbol{\psi}_\beta\} = 0$$

- We assumed the fermionic anticommutation relations in order to keep the Fock space as small as possible

Fock space

- Fock space generated from the vacuum state by two fermionic creation operators has four dimensions
- Four basis vectors are

$$|0\rangle \qquad |g\rangle = \boldsymbol{\psi}_g^\dagger|0\rangle \quad |e\rangle = \boldsymbol{\psi}_e^\dagger|0\rangle \quad |ge\rangle = \boldsymbol{\psi}_e^\dagger\boldsymbol{\psi}_g^\dagger|0\rangle$$

  - Qubit space is spanned by the one-particle vectors $|g\rangle$ and $|e\rangle$
- The vacuum vector $|0\rangle$ and two-particle vector $|ge\rangle$ are still needed to define the action of the creation and annihilation operators on the vectors that belong to the qubit subspace

$$\boldsymbol{\psi}_g^\dagger|e\rangle = |ge\rangle \quad \boldsymbol{\psi}_e^\dagger|g\rangle = -|ge\rangle \quad \boldsymbol{\psi}_e|e\rangle = |0\rangle \quad \boldsymbol{\psi}_g|g\rangle = |0\rangle$$

Feynman propagator for the qubit field

- Feynman propagators are defined in terms of time-ordered products of field operators
- In order to define free propagators for the qubit field we need the time-dependent operators

$$H_0 = m\boldsymbol{\psi}_e^\dagger \boldsymbol{\psi}_e - m\boldsymbol{\psi}_g^\dagger \boldsymbol{\psi}_g$$

$$\boldsymbol{\psi}_e(t) = e^{iH_0t}\boldsymbol{\psi}_e e^{-iH_0t} = e^{-imt}\boldsymbol{\psi}_e$$

$$\boldsymbol{\psi}_g(t) = e^{iH_0t}\boldsymbol{\psi}_g e^{-iH_0t} = e^{imt}\boldsymbol{\psi}_g$$

$$S_{F\alpha\beta}(t-t') = -i\langle g|T\left(\boldsymbol{\psi}_\alpha(t)\boldsymbol{\psi}_\beta^\dagger(t')\right)|g\rangle$$

$$= \begin{cases} \alpha = \beta = e & -i\theta(t-t')e^{-im|t-t'|} \\ \alpha = \beta = g & i\theta(t'-t)e^{-im|t-t'|} \end{cases}$$

Feynman propagator in momentum space

- Feynman propagators in momentum space are more useful

$$S_{F\alpha\beta}(t-t') = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} S_{F\alpha\beta}(p_0) e^{-ip_0(t-t')}$$

$$S_F(p_0) = \frac{\sigma_z}{p_0\sigma_z - (m - i\epsilon)} = \frac{1}{p_0 - (m - i\epsilon)\sigma_z}$$

- This is to be compared with the electron propagator in full QED

$$S_F(p_0) = \frac{\gamma^0}{p_0\gamma^0 - p_i\gamma^i - (m - i\epsilon)}$$

The spatial part of momentum disappears and $\gamma_0 \rightarrow \sigma_z$

Hamiltonian for the spin qubit

- The standard qubit Hamiltonian is built from Pauli matrices that act in the two-dimensional qubit space

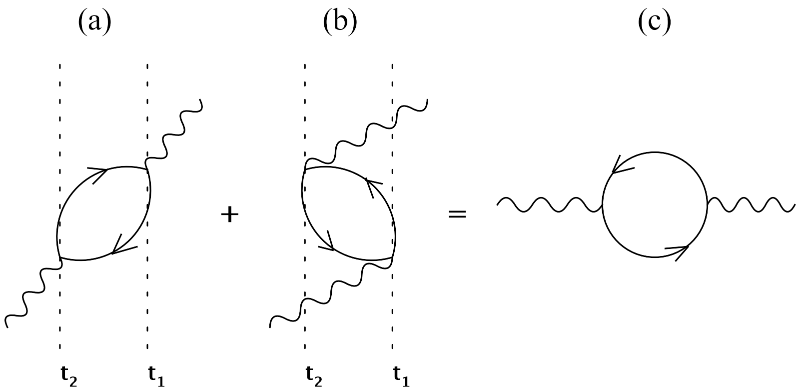
$$H = m\sigma_z + \sum_{i=x,y,z} \int_0^\infty dk\, \hbar\omega\, a_i^\dagger(k) a_i(k)$$
$$+ \mu_B \sum_{i=x,y,z} \sigma_i \int_0^\infty dk\, g(k)\phi_i(k) \quad m = \mu B_0$$

- After second quantization this Hamiltonian has the form

$$H = m\boldsymbol{\psi}^\dagger \sigma_z \boldsymbol{\psi} + \sum_{i=x,y,z} \int_0^\infty dk\, \hbar\omega\, a_i^\dagger(k) a_i(k)$$
$$+ \mu_B \sum_{i=x,y,z} \boldsymbol{\psi}^\dagger \sigma_i \boldsymbol{\psi} \int_0^\infty dk\, g(k)\phi_i(k)$$

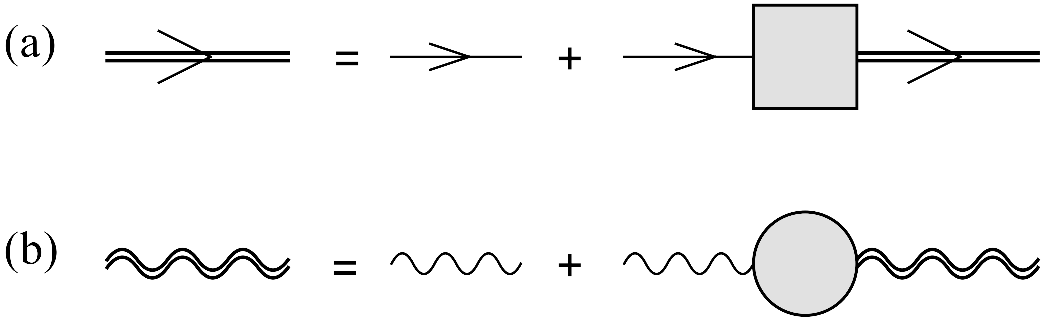
Vacuum polarization

- Ground state of the spin system corresponds to the vacuum state in QED
- Dirac sea: All negative energy states are occupied
- One Feynman diagram stands for several processes that differ in the time sequence of interactions
- Example: Lowest order correction to photon propagator



Tools of QED

- Relations between propagators and self-energy parts
- Double-lines represent full propagators and the gray box and circle represent the self-energy parts

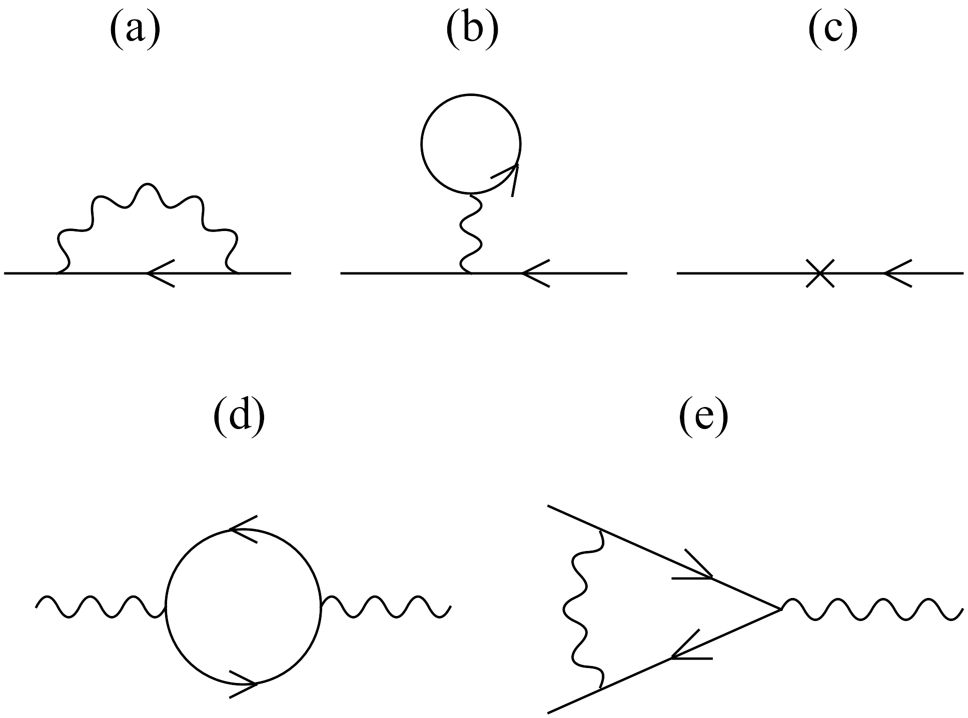


$$G_F(p_0) = S_F(p_0) + S_F(p_0)\Sigma(p_0)S_F(p_0)$$

$$+ S_F(p_0)\Sigma(p_0)S_F(p_0)\Sigma(p_0)S_F(p_0) + \ldots = \frac{1}{S_F^{-1}(p_0) - \Sigma(p_0)}$$

Lowest order corrections

- Radiative corrections to propagators in the second order of perturbation theory



- They determine the shifts and widths of the resonances

Feynman integrations

- There is no need to combine denominators
- All integrations over the loop energies can be performed analytically in every order of perturbation
  In contrast to QED no special functions are necessary
- Example: Fourth-order photon self-energy parts

$$\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\int_{-\infty}^{\infty}\frac{dl_0}{2\pi}\frac{1}{p_0+k_0\pm(m-i\epsilon)}\frac{1}{p_0+k_0+l_0\pm(m-i\epsilon)}$$
$$\times\frac{1}{p_0+l_0\pm(m-i\epsilon)}\frac{1}{p_0\pm(m-i\epsilon)}\frac{1}{l_0^2-k^2+i\epsilon}$$

- Depending on the combination of signs the residues will lie in the upper or in the lower half of the complex plane

Two-level atoms

- Two-level atoms have more than two relevant states
- Nevertheless they can be described in a similar way

Summary

- Field-theoretic methods have a great power
    - They simplify the calculations
    - They allow for a summation of perturbation series
    - They enable one to construct general formulas
    - They lead to a new interpretation
    - They unify the description of qubits and the EM field