Variations on a theme of electrodynamics

EM field as a huge oscillator

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OUTLINE

- Photon number operator (traditional approach)
- Photon number in classical theory
- Nonlocal invariant and its significance
- Wigner function in quantum mechanics
- Wigner function of the EM field
- Wigner function at finite temperature
- Excursion into squeezing
- Some numbers

Photon number according to textbooks

Creation and annihilation operators appear in the Fourier expansion of $\hat{\vec{A}}(\vec{r},t)$

$$\hat{\vec{A}}(\vec{r},t) = \sum_{\vec{k},\lambda} \sqrt{\frac{\hbar}{2\epsilon V \omega_k}} \left(\vec{\varepsilon}_\lambda e^{i\vec{k}\cdot\vec{r} - i\omega_k t} \hat{a}_{\vec{k},\lambda} + h.c. \right)$$

$\vec{k}$ - wave vector $\quad$ λ - polarization index

The number of photons is introduced through the number operator $\hat{N}_{\vec{k},\lambda}$

$$\hat{N}_{\vec{k},\lambda} = \hat{a}^\dagger_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} = \hat{H}_{\vec{k},\lambda}/\hbar\omega_{\vec{k}}$$

The total number of photons is the sum

$$\hat{N} = \sum_{\vec{k},\lambda} \hat{a}^{\dagger}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$

Number of photons can be defined also
for a classical electromagnetic field
We replace the annihilation and creation operators
by the corresponding Fourier coefficients

$$\left(\hat{a}_{\vec{k},\lambda}, \hat{a}^{\dagger}_{\vec{k},\lambda}\right) \rightarrow \left(\alpha_{\vec{k},\lambda}, \alpha^{*}_{\vec{k},\lambda}\right)$$

$$N = \sum_{\vec{k},\lambda} \alpha^{*}_{\vec{k},\lambda} \alpha_{\vec{k},\lambda}$$

We can construct a pure number N

(Ya. B. Zeldovich 1965)

$$N = \frac{1}{4\pi^2\hbar} \int d^3r \int d^3r' \left[\sqrt{\frac{\epsilon}{\mu}} \frac{\vec{B}(\vec{r}) \cdot \vec{B}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} + \sqrt{\frac{\mu}{\epsilon}} \frac{\vec{D}(\vec{r}) \cdot \vec{D}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \right]$$

This number turns out to be

invariant under all symmetry transformations

of free Maxwell theory:

Translations, rotations, Lorentz transformations,

and conformal transformations

A simple calculation to explain it all

$$\frac{1}{2\pi^2\hbar c|\vec{r}-\vec{r}\,'|^2} = \int\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{\hbar c|\vec{k}|}$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{\hbar\omega} = \frac{\delta^{(3)}(\vec{r}-\vec{r}\,')}{\text{“}\hbar\omega\text{”}}$$

$$N = \frac{1}{2}\int d^3r\left(\frac{1}{\epsilon}\,\vec{D}(\vec{r})\frac{1}{\text{“}\hbar\omega\text{”}}\vec{D}(\vec{r}) + \frac{1}{\mu}\,\vec{B}(\vec{r})\frac{1}{\text{“}\hbar\omega\text{”}}\vec{B}(\vec{r})\right)$$

$$N = \frac{\textbf{Field Energy}}{\text{“}\hbar\omega\text{”}} = \textbf{Number of photons}$$

A reminder

Wigner function in quantum mechanics

$$W(\vec{r},\vec{p}) = \int \frac{d^n\eta}{(2\pi\hbar)^n} e^{i\vec{\eta}\cdot\vec{p}/\hbar}\psi(\vec{r}-\vec{\eta}/2)\psi^*(\vec{r}+\vec{\eta}/2)$$

Ground state of harmonic oscillator in 1D

$$\psi(x) = Ce^{-\frac{m\omega x^2}{2\hbar}} \longrightarrow W_G(x,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{2H(p,x)}{\hbar\omega}\right)$$

$$H(p,x) = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Wave functional of the vacuum state

(J. A. Wheeler 1968)

The Hamiltonian of the EM field is

$$\hat{H} = \int d^3r \left[-\frac{\hbar^2}{2\epsilon} \frac{\delta^2}{\delta \vec{A}(\vec{r})^2} + \frac{1}{2\mu} (\nabla \times \vec{A}(\vec{r}))^2 \right].$$

The lowest energy eigenstate (vacuum state)

$$\Psi_0 \left[\vec{A} \right] = C \exp \left[-\frac{1}{4\pi^2 \hbar} \int d^3r \int d^3r' \sqrt{\frac{\epsilon}{\mu}} \frac{\vec{B}(\vec{r}) \cdot \vec{B}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^2} \right]$$

Wigner functional of the full EM field

$$\Psi_0\left[\vec{A}\right] \rightarrow W_0\left[\vec{A},\vec{D}\right] = \exp(-2N\left[\vec{A},\vec{D}\right])$$

$$N\left[\vec{A},\vec{D}\right] = \frac{1}{4\pi^2\hbar}\int d^3r \int d^3r'$$

$$\times\left[\sqrt{\frac{\epsilon}{\mu}}\frac{(\nabla\times\vec{A}(\vec{r}))\cdot(\nabla\times\vec{A}(\vec{r}\,'))}{|\vec{r}-\vec{r}\,'|^2} + \sqrt{\frac{\mu}{\epsilon}}\frac{\vec{D}(\vec{r})\cdot\vec{D}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right]$$

The same integral that represents
the number of photons
determines the statistical properties of
the vacuum state

- Quantum electrodynamics is intrinsically nonlocal
- There are long range correlations between the field values in different regions
- It is more likely that the nearby fields are antiparallel than that they are parallel
- In a dielectric (or magnetic) medium larger values of D (or B) are more likely than in the vacuum

Another reminder

Thermal state of harmonic oscillator in 1D

The density operator is

$$N \exp\left(-\frac{\hat{H}(p,x)}{kT}\right)$$

The corresponding Wigner function is

$$W_{\mathrm{T}}(x,p) = \frac{\tanh(\frac{\hbar\omega}{2kT})}{\pi\hbar} \exp\left(-2\frac{\tanh(\frac{\hbar\omega}{2kT})\,H(p,x)}{\hbar\omega}\right)$$

Wigner functional of the electromagnetic field at finite temperature

Only the kernel must be modified

$$\frac{1}{2\pi^2\hbar c|\vec{r}-\vec{r}\,'|^2} = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \frac{1}{\hbar\omega}$$

$$\to \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \frac{\tanh(\hbar\omega_k/2kT)}{\hbar\omega_k}$$

$$= \frac{kT}{\pi\hbar^2 c^2|\vec{r}-\vec{r}\,'|\sinh(2\pi kT|\vec{r}-\vec{r}\,'|/\hbar c)}$$

Correlations become short range

Range is determined by
the thermal length $\lambda_T = \hbar c/(2\pi kT)$

At room temperature $\lambda_T = 1.24 \cdot 10^{-6}\mathrm{m}$
At CBR temperature $\lambda_T = 1.33 \cdot 10^{-4}\mathrm{m}$

At macroscopic distances the fields
become statistically independent

Thermal fluctuations wipe out correlations

Beyond the vacuum state

We may easily extend the functional description to cover general squeezed states

Instead of the Wheeler formula

$$\Psi_0\left[\vec{A}\right] = C\exp\left[-\frac{1}{4\pi^2\hbar}\int d^3r\int d^3r'\sqrt{\frac{\epsilon}{\mu}}\frac{\vec{B}(\vec{r})\cdot\vec{B}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\right]$$

we can take a displaced Gaussian

$$\Psi_d\left[\vec{A},t\right] = C\exp\left[-\frac{1}{4\pi^2\hbar}\int d^3r\int d^3r'\sqrt{\frac{\epsilon}{\mu}}\frac{\vec{B}_d(\vec{r})\cdot\vec{B}_d(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} + i\Phi\right]$$

$$\Phi = \frac{1}{\hbar}\int d^3r\,\vec{A}(\vec{r})\cdot\vec{\mathcal{D}}(\vec{r},t) \quad \vec{B}_d(\vec{r}) = \vec{B}(\vec{r}) - \vec{\mathcal{B}}(\vec{r},t)$$

Full analogy with harmonic oscillator

This state is an analog of a Gaussian wave packet for a one dimensional oscillator

$$\psi(x,t) = C\exp\left[-\frac{m\omega}{2\hbar}(x-\xi(t))^2 + \frac{i}{\hbar}x\pi(t)\right]$$

where functions $\xi(t)$ and $\pi(t)$ must obey the classical equations of motion

In electrodynamics $\vec{\mathcal{D}}(\vec{r},t)$ and $\vec{\mathcal{B}}(\vec{r},t)$ must obey the Maxwell equations to satisfy the Schrödinger equation for $\Psi_s\left[\vec{A},t\right]$

We may extend this analogy to squeezed states

Squeezed states

In quantum mechanics of an oscillator
every Gaussian function describes a squeezed state

In electrodynamics we have Gaussian functionals

$$\Psi_0\left[\vec{A},t\right]=C\exp\left[-\frac{1}{2}\int d^3r\int d^3r'\,\vec{B}(\vec{r})\cdot K\left(\vec{r},\vec{r}\,',t\right)\cdot\vec{B}(\vec{r}\,')\right]$$

To satisfy the Schrödinger equation $K\left(\vec{r},\vec{r}\,',t\right)$
must obey a certain nonlinear equation

These Gaussians may also be displaced
by an arbitrary solution of Maxwell equations

Number of photons in the Coulomb field

Consider elementary charge

$$N = \frac{e^2}{16\pi^3\, 4\pi\epsilon_0\hbar c}\int_\lambda^{R_s} d^3r \int_\lambda^{R_s} d^3r' \left(\frac{\vec{r}\cdot\vec{r}\,'}{r^3 r'^3 |\vec{r}-\vec{r}\,'|^2}\right)$$

Ultraviolet cutoff at the electron

Compton wave length λ (vacuum polarization)

Infrared cutoff at a screening length R_s

$$N = \frac{\alpha}{2\pi}\ln\left(\frac{R_s}{\lambda}\right)$$

$$\lambda = 10^{-13}\text{m} \quad R_s = 1\text{m} \quad N = 3\cdot 10^{-3}\ \text{photons}$$

Number of photons in constant magnetic field

Consider a sphere of radius R

$$N = \frac{\epsilon \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}/\mu}{4\pi^2 \, \hbar c} \int_0^R d^3 r \int_0^R d^3 r' \frac{1}{|\vec{r} - \vec{r}\,'|^2}$$

$$N = \left(\frac{4\pi R^3}{3} \, \frac{\epsilon E^2 + B^2/\mu}{2} \right) \Big/ \left(\frac{8\hbar c}{3R} \right)$$

This number scales as R^4 and B^2

Therefore it is huge for large volumes!

At 1 Tesla there are $5 \cdot 10^{31}$ photons in 1m^3

The galactic magnetic field is 10^{-10} T

The radius of Milky Way is $3 \cdot 10^{20}$m

Therefore in Milky Way there are
$4 \cdot 10^{93}$ magnetic photons!

Density of CBR photons: $N_{\text{CBR}} = 4.12 \cdot 10^8/\text{m}^3$

The total number is “only” 10^{70} CBR photons

Conclusions

Electrodynamics takes place in space-time

Mode decomposition obscures
space-time relations

The analogy with one-dimensional harmonic
oscillator enables one to write down many
formulas without doing any hard work