75 years of Born-Infeld electrodynamics

Nonlinear theory of the electromagnetic field

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Maxwell theory

Maxwell equations are universal

$$\begin{aligned}
\partial_t \boldsymbol{B}(\boldsymbol{r},t) &= -\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)\\
\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t) &= 0\\
\partial_t \boldsymbol{D}(\boldsymbol{r},t) &= \boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)\\
\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) &= 0
\end{aligned}$$

Typical constitutive equations are linear

$$\boldsymbol{B} = \mu\boldsymbol{H} \qquad \boldsymbol{D} = \varepsilon\boldsymbol{E}$$

Standard Maxwell equations follow from the Lagrangian

$$\mathcal{L}_{\mathrm{Maxwell}} = \frac{1}{2}\left(\varepsilon\boldsymbol{E}^2 - \boldsymbol{B}^2/\mu\right)$$

Nonlinear electrodynamics of Born

Born used the analogy with relativistic mechanics

$$\mathcal{L}_{\text{particle}} = mc^2 \left[1 - \sqrt{1 - (\boldsymbol{v}^2)/c^2} \right]$$

Maximal velocity *c* $\Longleftrightarrow$ Maximal field strength *b*

$$\mathcal{L}_{\text{Born}} = b^2 \left[1 - \sqrt{1 - (\boldsymbol{E}^2 - \boldsymbol{B}^2)/b^2} \right]$$

From now on I use the units $\varepsilon = 1 = \mu$

$c \to \infty$ Correspondence with Newton theory

$b \to \infty$ Correspondence with Maxwell theory

Nonlinear electrodynamics of Born and Infeld

Born and Infeld used false arguments but obtained a unique theory

$$\mathcal{L}_{\rm Born-Infeld} = b^2 \left[\sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu} + f_{\mu\nu}/b)} \right]$$

Born-Infeld electrodynamics has earned its longevity through its elegant, compact, determinantal form

S. Deser and G. W. Gibbons (1998)
in *Born-Infeld-Einstein Actions*

Lagrangian density in Minkowski space

$$\mathcal{L}_{\rm Born-Infeld} = b^2 \left[1 - \sqrt{1 - (\boldsymbol{E}^2 - \boldsymbol{B}^2)/b^2 - (\boldsymbol{E}\cdot\boldsymbol{B})^2/b^4} \right]$$

Constitutive relations for Born-Infeld electrodynamics

$$\boldsymbol{D} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{E}} \qquad \boldsymbol{H} = -\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}}$$

$$\begin{aligned}
\boldsymbol{D} &= \frac{\partial \mathcal{L}}{\partial \boldsymbol{E}} = \frac{\boldsymbol{E} + (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{B}/b^2}{\sqrt{1-(\boldsymbol{E}^2-\boldsymbol{B}^2)/b^2-(\boldsymbol{E}\cdot\boldsymbol{B})^2/b^4}} \\
\boldsymbol{H} &= -\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}} = \frac{\boldsymbol{B} - (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{E}/b^2}{\sqrt{1-(\boldsymbol{E}^2-\boldsymbol{B}^2)/b^2-(\boldsymbol{E}\cdot\boldsymbol{B})^2/b^4}}
\end{aligned}$$

One can invert these relations and obtain

$$\begin{aligned}
\boldsymbol{E} &= \frac{\boldsymbol{D} + \boldsymbol{B}\times(\boldsymbol{D}\times\boldsymbol{B})/b^2}{\sqrt{1+(\boldsymbol{D}^2+\boldsymbol{B}^2)/b^2+(\boldsymbol{D}\times\boldsymbol{B})^2/b^4}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{D}} \\
\boldsymbol{H} &= \frac{\boldsymbol{B} - \boldsymbol{D}\times(\boldsymbol{D}\times\boldsymbol{B})/b^2}{\sqrt{1+(\boldsymbol{D}^2+\boldsymbol{B}^2)/b^2+(\boldsymbol{D}\times\boldsymbol{B})^2/b^4}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{B}}
\end{aligned}$$

The Hamiltonian

Mechanical analogy:

$$\boldsymbol{D} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{E}} \sim \boldsymbol{p} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}} \qquad \boldsymbol{E} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{D}} \sim \boldsymbol{v} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}$$

Lagrange variables $\boldsymbol{E}$ and $\boldsymbol{B}$ ($\boldsymbol{E} \sim$ velocity)
Hamilton variables $\boldsymbol{D}$ and $\boldsymbol{B}$ ($\boldsymbol{D} \sim$ momentum)
Hamiltonian density is obtained by Legendre transformation $\boldsymbol{E} = f(\boldsymbol{D}, \boldsymbol{B})$
and it agrees with the evaluation of $\boldsymbol{E}$ and $\boldsymbol{H}$ as derivatives of $\mathcal{H}(\boldsymbol{D}, \boldsymbol{B})$

$$\begin{aligned} \mathcal{H}(\boldsymbol{D}, \boldsymbol{B}) &= \boldsymbol{E}(\boldsymbol{D}, \boldsymbol{B}) \cdot \boldsymbol{D} - \mathcal{L}(\boldsymbol{D}, \boldsymbol{B}) \\ &= b^2 \left[\sqrt{1 + (\boldsymbol{D}^2 + \boldsymbol{B}^2)/b^2 + (\boldsymbol{D} \times \boldsymbol{B})^2/b^4} - 1 \right] \end{aligned}$$

$\mathcal{H}$ is like $c\sqrt{m^2c^2 + \boldsymbol{p}^2}$ and it exists for all $\boldsymbol{D}$ and $\boldsymbol{B}$

Energy of a point charge

Energy density is integrable for a point particle

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot D &= e\delta^{(3)}(\boldsymbol{r}) \qquad \boldsymbol{B}=0 \\
\boldsymbol{D} &= \frac{e\boldsymbol{r}}{4\pi|\boldsymbol{r}|^3} \qquad \boldsymbol{D}^2/b^2 = \frac{q^2}{r^4} \\
E_{\rm tot} &= 4\pi b^2 \int_0^\infty dr\, r^2 \left(\sqrt{1+q^2/r^4}-1\right) \\
&= \frac{4\Gamma^2(5/4)\sqrt{e^3 b}}{3\pi} = 1.2361\sqrt{e^3 b}
\end{aligned}
$$

Finite energy solutions of Born-Infeld theory are now often called BIons

Electromagnetic field of a moving charge is found by Lorentz transformation

No birefringence

A lot of other fully relativistic nonlinear theories of the electromagnetic field exist
Every function of the two electromagnetic invariants can serve as a Lagrangian

$$\mathcal{L}(\boldsymbol{E}, \boldsymbol{B}) = \sqrt{-g}\, l(S, P)$$

$$S = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) \qquad P = -\frac{1}{4} f_{\mu\nu} \check{f}^{\mu\nu} = (\boldsymbol{E} \cdot \boldsymbol{B})$$

Wave propagation in a nonlinear theory of the electromagnetic field
is usually characterized by the birefringence:
The propagation velocity is different for different wave polarizations

There are only two theories that have no birefringence:
Maxwell electrodynamics and Born-Infeld electrodynamics

No birefringence $\equiv$ Single lightcone

Dispersion law

Assuming that a weak wave propagates on a background field $F_{\mu\nu}$
one can obtain the conditions for the absence of birefringence
In the general case the dispersion equation has the following form:

$$k^2 = a^2\,\lambda_\pm \qquad a^\mu = F^{\mu\nu} k_\nu$$

$$\lambda_\pm = \frac{-\gamma_S(\gamma_{SS}+\gamma_{PP})+2\left(\gamma_{SS}\gamma_{PP}-\gamma_{SP}^2\right)\pm\Delta}{2\left[-\gamma_S^2+2\gamma_S(S\gamma_{SS}-P\gamma_{SP})+P^2\left(\gamma_{SS}\gamma_{PP}-\gamma_{SP}^2\right)\right]}$$

$$\Delta = \left[\gamma_S\left(\gamma_{SS}-\gamma_{PP}\right) - 2S\left(\gamma_{SS}\gamma_{PP}-\gamma_{SP}^2\right)\right]^2$$

$$+\,4\left[\gamma_S\gamma_{SP} - P\left(\gamma_{SS}\gamma_{PP}-\gamma_{SP}^2\right)\right]^2 = [\mathrm{Eq1}]^2 + [\mathrm{Eq2}]^2$$

$$\gamma_S = \frac{\partial\mathcal{L}}{\partial S} \qquad \gamma_{SS} = \frac{\partial^2\mathcal{L}}{\partial S^2} \qquad \gamma_{PP} = \frac{\partial^2\mathcal{L}}{\partial P^2} \qquad \gamma_{SP} = \frac{\partial^2\mathcal{L}}{\partial S\partial P}$$

No birefringence condition $\Delta = 0$ leads to two equations: Eq1=0 and Eq2=0
Maxwell theory and Born-Infeld electrodynamics are the only regular solutions

Open problems

Except for a point charge and plane waves there are no exact solutions

The most interesting problems are the radiation reaction and the equations of motion
In Born-Infeld electrodynamics the pathological behavior should not occur

Runaway solutions are not possible due to the total energy conservation

Interesting problem: Two identical charged particle are initially at rest

Assuming that there is no field except for the Coulomb field

Find the the equations of motion and the final velocity of the two particles

The difference between the initial and final energy of the particles

Determines the total energy radiated by the system

New Age

Strings, p-branes and the Born-Infeld theory

$$\begin{aligned}
\textbf{Point particle}: \ \mathcal{L} &= -mc^2 \int d\tau \sqrt{d\xi^\mu/d\tau \cdot d\xi_\mu/d\tau} \\
\textbf{String}: \ \mathcal{L} &= T \int d\tau d\sigma \sqrt{\det\left(\partial\xi^\mu/\partial\tau^A \cdot \partial\xi_\mu/\partial\tau^B\right)} \\
\textbf{Dirichlet p-brane}: \ \mathcal{L} &= \int d^{p+1}x \sqrt{-\det(\partial_\mu x^M \partial_\nu x_M + f_{\mu\nu}/b)}
\end{aligned}$$

All these, like Born-Infeld electrodynamics, are clearly “determinantal” theories as described picturesquely by Deser and Gibbons

There is no evidence so far that the theory of Born and Infeld has any direct connection with the physical reality, but it stands out as a masterpiece of mathematical physics: an exceptional structure of many intricate theoretical features, as charming now as it appeared to its discoverers/creators 50 years ago