The Riemann-Silberstein vector as a *photon wave function*

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September 2005

Ludwik Silberstein

- Ludwik Silberstein (1872-1948), Polish theoretical physicist, emigrated from Poland to Italy, England, Canada, and finally to the United States
- On November 6, 1919 the results of the Brazilian expedition to measure the deflection of light were reported at the joint meeting of the Royal Society and the Royal Astronomical Society. Eddington recalled that, as the meeting was dispersing, Ludwig Silberstein came up to him and said: “Professor Eddington, you must be one of three persons in the world who understands general relativity.” On Eddington demurring to this statement, Silberstein responded, “Don’t be modest Eddington.” And the Eddington’s reply was, “On the contrary, I am trying to think who the third person is!”

Complex form of Maxwell equations

- Almost 100 years ago Ludwik Silberstein published a paper under the title *Elektromagnetische Grundgleichungen in bivectorieller Behandlung* (Annalen der Physik 22, 579 (1907))
- In this paper he introduced a complex vector

  $$\boldsymbol{F} = \sqrt{\epsilon_0/2}(\boldsymbol{E} + ic\boldsymbol{B})$$

  This vector (I named it the Riemann-Silberstein vector) satisfies the equation

  $$i\partial_t \boldsymbol{F}(\boldsymbol{r}, t) = c\nabla \times \boldsymbol{F}(\boldsymbol{r}, t)$$

- This equation looks like the Schrödinger equation

  $$i\hbar\partial_t \boldsymbol{F}(\boldsymbol{r}, t) = c(\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{p}})\boldsymbol{F}(\boldsymbol{r}, t) = \hat{H}\boldsymbol{F}(\boldsymbol{r}, t)$$

Why do we need wave functions?

- Superposition of quantum states is the central property of the quantum world
- Dirac called the Principle of Superposition of States “the most fundamental and the most drastic” law of nature
- Superposition means the ability to add quantum states
- Why wave functions? Can’t we work with state vectors?
- Abstract state vectors are not enough for doing physics
- State vectors must be labeled to tell them apart
- Wave functions are labels on the state vectors
- Photon is undeniably a quantum particle
- Therefore, there must exist a photon wave function

Candidate for the photon wave function

- Silberstein noticed that important physical quantities are given as bilinear combinations of $\boldsymbol{F}$ and $\boldsymbol{F}^*$
- The total energy of the field is the norm of $\boldsymbol{F}$

$$E = ||\boldsymbol{F}||^2 = \int d^3r\, \boldsymbol{F}^* \cdot \boldsymbol{F}$$

- The total momentum and the angular momentum of the field also look nice

$$\boldsymbol{P} = -i \int d^3r\; \boldsymbol{F}^* \times \boldsymbol{F} \qquad \boldsymbol{M} = -i \int d^3r\; \boldsymbol{r} \times (\boldsymbol{F}^* \times \boldsymbol{F})$$

- This observation was made 20 years before the discovery of quantum mechanics!

Complexification saves space and time

- Complex solutions of the Maxwell equations are very often employed as a technical trick
    - Normally, however, only the real part is used and the imaginary part is discarded
    - In contrast, in the case of the RS vector both parts have a physical significance of electric and magnetic fields
- Fourier analysis of complex functions is simpler since there are no constrains that guarantee reality
- In particular we have the relation

$$\int d^3r\ \boldsymbol{F}_1^*(\boldsymbol{r})\boldsymbol{F}_2(\boldsymbol{r}) = \int \frac{d^3k}{(2\pi)^3}\ \tilde{F}_1^*(\boldsymbol{k})\tilde{F}_2(\boldsymbol{k})$$

Expansion into plane waves

- Every solution of the Maxwell equations for $\boldsymbol{F}$ can be expanded into plane waves

$$\boldsymbol{F}(\boldsymbol{r},t) = \int d^3k\,\boldsymbol{e}(\boldsymbol{k})\left(\alpha_+(\boldsymbol{k})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} + \alpha_-(\boldsymbol{k})e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

- The complex unit polarization vector $\boldsymbol{e}(\boldsymbol{k})$ is

$$\boldsymbol{e}(\boldsymbol{k}) = N(\boldsymbol{k})\,(-ik_y\kappa + k_zk_+, ik_x\kappa + ik_zk_+, -\kappa k_z)$$

$$\kappa = k_x + ik_y,\;\; k_+ = \omega/c + k_z,\;\; N^{-1}(\boldsymbol{k}) = \sqrt{2}k_+\omega/c$$

- What is the classical physical interpretation of $\alpha_\pm(\boldsymbol{k})$?
- The functions $\alpha_\pm(\boldsymbol{k})$ are the amplitudes of the right and left circularly polarized waves with the wave vector $\boldsymbol{k}$

How to handle negative energy solutions?

- Solutions of the Maxwell equations contain, in general, positive and negative frequencies
- In the spirit of the relativistic quantum mechanics we interpret the negative energy solutions as complex conjugate wave functions of antiparticles
- From that point of view left-handed photons are antiparticles of right-handed photons
- Since photons do not carry conserved quantum numbers, one can take superpositions (there are no superselection rules) of particle and antiparticle states and obtain photons with linear polarization
- Obvious analogy with the Dirac wave function ψ

Quantum mechanics of photons

- Can $\boldsymbol{F}(\boldsymbol{r},t)$ serve as a photon wave function?
- Yes, it can but with some reservations
- Eigenvalue problems for observables are well posed

$$\text{Energy}: \hat{H}\boldsymbol{F} = -ic\hbar(\hat{\boldsymbol{s}}\cdot\boldsymbol{\nabla})\boldsymbol{F} = E\boldsymbol{F}$$

$$\text{Momentum}: \; -i\hbar\partial_i\boldsymbol{F} = p_i\,\boldsymbol{F}$$

$$\text{Angular momentum}: \; (-i\hbar(\boldsymbol{r}\times\nabla)_z + \hbar\hat{s}_z)\,\boldsymbol{F} = M_z\,\boldsymbol{F}$$

$$\hat{s}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \; \hat{s}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \; \hat{s}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Probabilistic interpretation

- The modulus squared of the RS vector $\boldsymbol{F}^*\boldsymbol{F}$ has the dimension energy density: Energy/Volume
- It cannot serve as a probability density
- So, how does one define expectation values?
- A clue comes from the formulas

$$E = \int d^3r\,\boldsymbol{F}^*\cdot\boldsymbol{F} = \int d^3r\,\boldsymbol{F}^*\cdot\frac{|\hat{H}|}{|\hat{H}|}\boldsymbol{F} = \langle|\hat{H}|\rangle$$

- We define (R. H. Good 1957) the expectation values as

$$\langle\hat{O}\rangle = N^{-1}\int d^3r\,\boldsymbol{F}^*(\boldsymbol{r},t)\frac{1}{\sqrt{|\hat{H}|}}\cdot\hat{O}\frac{1}{\sqrt{|\hat{H}|}}\boldsymbol{F}(\boldsymbol{r},t)$$

Pauli against Landau-Peierls

- It seems that when it comes to expectation values the wave function is not $\boldsymbol{F}$ but rather $\sqrt{1/|\hat{H}|}\boldsymbol{F}$
- This is a nonlocal operation $\sqrt{\hbar c/|\hat{H}|}\boldsymbol{F} = (-\Delta)^{-1/4}\boldsymbol{F}$

$$((-\Delta)^{-1/4}f)(\mathbf{r}) = \pi\int\frac{d^3r'}{(2\pi|\mathbf{r}-\mathbf{r}'|)^{5/2}}f(\mathbf{r}')$$

- The nonlocal photon wave function was first proposed by Landau and Peierls in 1930 and criticized by Pauli:
- It is totally unacceptable because “their value at a point depends on the whole spatial behavior and not merely on the behavior of the function in the neighborhood of the point considered”

Wave function in momentum space

- The criticism by Pauli is not convincing because the wave function is by its nature a nonlocal object; to know the state, we must know the wave function everywhere
- The problem of nonlocality is less visible, when we deal with the photon wave function in momentum space
- Momentum space wave functions have two components $\{f^+(\boldsymbol{k}), f^-(\boldsymbol{k})\}$ that describe right-handed and left-handed photons
- They are related to the RS vector through the formula

$$\boldsymbol{F}(\boldsymbol{r},t) = \int d^3k\,\boldsymbol{e}(\boldsymbol{k})\sqrt{\frac{\hbar\omega}{(2\pi)^3}}\left(f^+(\boldsymbol{k})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} + f^{-*}(\boldsymbol{k})e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

The RS vector and QED

- In QED the RS vector becomes a field operator

$$\hat{\boldsymbol{F}}(\boldsymbol{r},t)=\int d^3k\;\boldsymbol{e}(\boldsymbol{k})\sqrt{\frac{\hbar\omega}{(2\pi)^3}}\left(\hat{a}(\boldsymbol{k})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}+\hat{b}^\dagger(\boldsymbol{k})e^{i\omega t-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

- Operators $\hat{a}(\boldsymbol{k})$ and $\hat{b}(\boldsymbol{k})$ annihilate right-handed and left-handed photons with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$
- Energy, momentum and angular momentum

$$\hat{\mathcal{E}} = \int d^3k\,\hbar\omega\left(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}\right)\quad \hat{\mathcal{P}}=\int d^3k\,\hbar\boldsymbol{k}\left(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}\right)$$

$$\hat{\mathcal{M}}_z = \int d^3k\,\hbar\left(\hat{a}^\dagger\frac{1}{i}\frac{\partial}{\partial_\varphi}\hat{a}+\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\frac{1}{i}\frac{\partial}{\partial_\varphi}\hat{b}-\hat{b}^\dagger\hat{b}\right)$$

Wave functions and the field operator

- A general expansion formula for the field operator $\hat{\boldsymbol{F}}$

$$\hat{\boldsymbol{F}}(\boldsymbol{r},t) = \sum_{n=1}^{\infty}\left(\hat{a}_n f_n^{+}(\boldsymbol{r},t) + \hat{b}_n^{\dagger} f^{-*}(\boldsymbol{r},t)\right)$$

- The coefficient functions $f_n^{\pm}(\boldsymbol{r},t)$ are the wave functions of photons
- They must form complete sets but are otherwise arbitrary
- There are several choices of such sets that play a role in optics
- Multipole field and Bessel beams are the most interesting cases

Summary

- All quantum particles should be treated the same way
- Photon wave functions are needed for completeness of the quantum mechanical description
- The Riemann-Silberstein vector is the most natural candidate:
  - It obeys a Schrödinger-like equation
  - It gives average values as bilinear forms
  - It may be used to solve important eigenvalue problems
  - It gives a simple relation between the position-space and momentum-space wave functions
  - It gives a succinct description of right-handed and lift-handed photons