Hydrodynamics of relativistic probability flows *

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Abstract

A hydrodynamic form of relativistic wave equations is derived. The formulation is quite general and can be applied to any set of first order wave equations. The most interesting examples are the Weyl equation and the Maxwell equations. The set of hydrodynamic variables includes the density field, the velocity field, and the momentum field. The reduction in the number of independent variables requires a quantization condition that relates the curl of the momentum field to a vector built from the remaining fields.

I. INTRODUCTION

The hydrodynamic formulation of wave mechanics is almost as old as the Schrödinger equation [1]. This approach has been extended by Takabayasi [2–5] and others [6–8] to the nonrelativistic wave equation for spinning particles (Pauli equation). A review of the hydrodynamic formulation of nonrelativistic wave mechanics including a large number of references can be found in [9] (for a textbook treatment, see [12]). A hydrodynamic formulation of relativistic wave mechanics of the Dirac particle was given by Takabayasi [10]. His formulation, however, can not be applied to the massless case treated in the present work.

In the present paper I shall present a general derivation that can be applied to any set of first order equations and yields a hydrodynamic form of these equations. The application of this method to Maxwell equations offers a new look at the dynamics of photons. The hydrodynamic formulation of wave mechanics enables one to visualize the flow of probability by providing us with an intuitive picture of this flow described in terms of the familiar variables of classical hydrodynamics. Such a visualization is of particular interest for the Maxwell theory owing to the unquestionable dominance of electromagnetic phenomena in many areas of physics. There were many attempts to provide a “mechanical picture” of electromagnetism and the hydrodynamic formulation may also serve a role in this category. The main purpose of my study, however, is to exhibit an intricate relationship between linear (Weyl equation and Maxwell equations) and nonlinear (hydrodynamic-like equations)

theories. The Maxwell equations and their hydrodynamic version offer a very good platform to discuss this relationship. On the one side there is a very well understood theory describing the propagation of the electromagnetic field governed by linear field equations while on the other side there is a nonlinear, hydrodynamic-like theory describing the transport of various physical quantities (energy, momentum, etc.).

In the hydrodynamic formulation of nonrelativistic wave mechanics the flow of probability is described by the hydrodynamic variables: the probability density \( \rho \) and the velocity of the probability flow \( \vec{v} \). For spinning particles we need additional variables describing the internal degrees of freedom. In general, the number of hydrodynamic variables exceeds the number of components of the wave function. In order to make the hydrodynamic formulation completely equivalent to the original one, one must impose a condition on the initial values of the hydrodynamic variables. The condition restricting the number of the degrees of freedom is related to the Bohr-Sommerfeld quantization condition of quantum theory.

II. GENERAL APPROACH TO THE HYDRODYNAMIC FORMULATION

In the simplest case of a nonrelativistic spinless particle, described by the Schrödinger equation, the hydrodynamic variables and the wave function are related through the formulas

\[
\rho = |\psi|^2, \quad m\vec{v} = \vec{p} = \nabla S, \tag{1}
\]

where \( S \) is the phase of the wave function (\( R \) is assumed real)

\[
\psi = R \exp(\frac{i}{\hbar} S). \tag{2}
\]

The equations of motion for the hydrodynamic variables \( \rho \) and \( \vec{v} \) can be obtained by the repeated use of the Schrödinger equation. For the force-free case they have the form

\[
\partial_t \rho + (\vec{v} \cdot \nabla) \rho = -\partial_k v_k \rho, \tag{3}
\]

\[
\partial_t v_i + (\vec{v} \cdot \nabla) v_i = \frac{\hbar^2}{4m^2 \rho} \partial_k (\rho \partial_k \partial_i \ln \rho). \tag{4}
\]

The requirement that the wave function be single-valued imposes the following restriction — the quantization condition — on the velocity field

\[
\oint_C d\vec{l} \cdot \vec{v} = \frac{2\pi \hbar}{m} n, \tag{5}
\]

where \( C \) is an arbitrary closed contour. By the Stokes theorem, the quantization condition can also be expressed in terms of a surface integral representing the vorticity flux,

\[
\int_S d\vec{S} \cdot (\nabla \times \vec{v}) = \frac{2\pi \hbar}{m} n, \tag{6}
\]

where \( S \) is any surface spanned by the closed contour \( C \). Thus, the quantization condition states that the motion of the probability fluid is irrotational almost everywhere except possibly at a discrete set of vortex lines whose strength is quantized in units of \( 2\pi \hbar/m \). This
condition reduces the number of independent components to two: the density \( \rho \) and the longitudinal part of the velocity vector \( \vec{v} \). The transverse part of the momentum vector is fixed.

The starting point of my analysis will be a general relativistic wave equation, obeyed by an \( N \)-component wave function \( \phi_a \), of the form

\[
\partial_t \phi = -\alpha^k \partial_k \phi - i\beta \phi.,
\]  

(7)

where \( \alpha^k \) and \( \beta \) are \( N \times N \) matrices. Every set of linear equations with higher derivatives can be reduced to this form by introducing additional components of the wave function. For example, for the Klein-Gordon equation,

\[
(\Box + (mc/\hbar)^2)\psi = 0,
\]  

(8)

one can define

\[
\chi = (\hbar/mc^2)\partial_t \psi,
\]  

(9)

\[
\chi_k = (\hbar/mc)\partial_k \psi,
\]  

(10)

to rewrite (8) as

\[
\partial_t \psi = (mc^2/\hbar)\chi,
\]  

(11)

\[
\partial_t \chi = c\nabla \cdot \vec{\chi} - (mc^2/\hbar)\psi,
\]  

(12)

\[
\partial_t \vec{\chi} = c\nabla \chi.
\]  

(13)

This set of equations is clearly of the form (7). In the present case one also needs an auxiliary condition

\[
\nabla \psi = (mc/\hbar)\vec{\chi}
\]  

(14)

imposed on the initial data and preserved in time. Similar auxiliary conditions will be present in other cases.

In what follows I shall assume that all four matrices \( \alpha^k \) and \( \beta \) are Hermitian. This is required by the conservation of probability (the Hamiltonian must be Hermitian) and is, of course, satisfied in all cases of interest. I shall derive a hydrodynamic form of the general wave equation (7) in four steps.

In the first step, I define the following hydrodynamic variables (summation convention is used throughout):

- a scalar density of the “probabilistic fluid” \( \rho \),

\[
\rho = \phi^*_a \phi_a,
\]  

(15)

- a Hermitian matrix \( w_{ab} \) of trace 1,

\[
\rho w_{ab} = \phi^*_a \phi_b,
\]  

(16)
• and the momentum vector $\vec{u}$

$$\rho \vec{u} = \frac{1}{2i} \phi_a^* \nabla \phi_a. \quad (17)$$

The matrix $w$ carries all the information about the direction of the complex vector $\phi$ in the $N$-dimensional space. The length of this vector is determined by $\rho$. In order to complete the reconstruction one only needs the phase of $\phi$. In complete analogy with the treatment of the Schrödinger equation, one may obtain the information about the phase from the longitudinal part of the momentum vector $\vec{u}$. I shall keep referring to this vector as momentum even though its dimension — inverse of length — is that of a wave vector. In order to obtain the correct dimension I would have to multiply $\vec{u}$ by the Planck constant but such an operation would only introduce $\hbar$ into various formulas. Thus, my basic hydrodynamic variables in the general case will comprise the density $\rho$, $2N - 2$ independent components of $w$ and the longitudinal part of $\vec{u}$.

In the second step, with the help of (7), I derive the following two evolution equations

$$\partial_t (\rho w) = - (\vec{\alpha} \cdot \nabla)(\rho w) + \rho (\vec{\alpha} \cdot \vec{z}^\dagger - \vec{z}^\dag \cdot \vec{\alpha}) - i\rho [\beta, w], \quad (18)$$

$$\partial_t (\rho z_k) = - (\vec{\alpha} \cdot \nabla)(\rho z_k) + \rho (\vec{\alpha} \cdot z_k \vec{z}^\dagger - z_k \vec{z}^\dag \cdot \vec{\alpha}) - i\rho [\beta, z_k], \quad (19)$$

where the two auxiliary vector matrices $\vec{z}$ and $\vec{z}^\dagger$ are defined through the formulas

$$\rho \vec{z}_{ab} = (\nabla \phi_a) \phi_b^*, \quad \rho \vec{z}^\dagger_{ab} = \phi_a \nabla \phi_b^*, \quad (20)$$

and the matrix multiplication is understood whenever applicable. The use of the vectors $\vec{z}$ and $\vec{z}^\dagger$ greatly simplifies the formulas.

In the third step I derive the following formulas that relate the matrices $\vec{z}$ and $\vec{z}^\dagger$ to the basic hydrodynamic variables

$$\vec{z} = (\frac{1}{2\rho} \nabla \rho + i\vec{u} + \nabla w)w, \quad \vec{z}^\dagger = w(\frac{1}{2\rho} \nabla \rho - i\vec{u} + \nabla w). \quad (21)$$

These relations can be verified by substituting the definitions of $\rho$, $\vec{u}$ and $w$ on the right hand side and performing all the differentiations. I shall not carry out the substitution of (21) into (18) and (19) in the general case because it does not lead to anything simple and transparent. I shall do it in the special cases in the forthcoming sections when the Eqs. (7) take on a specific form.

Finally, in the fourth step, I derive the quantization condition that fixes the transverse part of $\vec{u}$. This condition can be expressed in terms of $w$ by using again $\vec{z}$ and $\vec{z}^\dagger$ in the intermediate steps. From the definition of $\vec{u}$ one obtains

$$\nabla \times \rho \vec{u} = i\rho \text{Tr}\{\vec{z} \times \vec{z}^\dagger\}. \quad (22)$$

With the help of (21) one arrives at a very simple formula

$$\nabla \times \vec{u} = \text{Tr}\{w \nabla w \times \nabla w\}. \quad (23)$$

This relation holds everywhere except on quantized vortex lines, where $\nabla \times \vec{u}$ has a surface delta-function singularity. Therefore, the quantization condition in the general case has the form
\[ \int_S d\vec{S} \cdot (\nabla \times \vec{u} - \text{Tr}\{w \nabla w \times \nabla w\}) = 2\pi n. \] (24)

In the following sections, this general formalism will be applied to the Weyl equation and to the Maxwell equations.

III. HYDRO_DYNAMIC FORM OF THE WEYL EQUATION

The wave equation describing the time evolution of the wave function for a massless, spin one-half particle (the Weyl equation) has the form

\[ \partial_t \phi = -c \vec{\sigma} \cdot \nabla \phi, \] (25)

where \( \phi \), has two complex components. The hydrodynamic variables for neutrino are defined as follows

\[ \rho = \phi^*_a \phi_a, \quad \rho \vec{v} = c \phi^*_a \vec{\sigma}_{ab} \phi_b, \quad \rho \vec{u} = \frac{1}{2} \phi^*_a \nabla \phi_a. \] (26)

The variable \( \vec{v} \) appearing in the second equation has a fixed length \( c \) and can be interpreted as the particle velocity. For spinning particles velocity \( \vec{v} \) and momentum \( \vec{u} \) are distinct — in general, they have different directions.

In the present case the matrix \( w_{ab} \) has three independent components that can be expressed by the components of the velocity vector \( \vec{v} \)

\[ w_{ab} = \frac{1}{2} (\delta_{ab} + \vec{\sigma}_{ab} \cdot \vec{v}/c). \] (27)

This leads to the following formula for \( \vec{z}_{ab} \)

\[ \vec{z}_{ab} = \frac{1}{4} ([\nabla \ln \rho + 2i\vec{u}](1 + \vec{\sigma} \cdot \vec{v}/c) + (\nabla v_k - i\varepsilon_{ijk} v_i \nabla v_j)\sigma_{k}{}^{ab}, \tag{28} \]

and its Hermitian conjugate for \( \vec{z}^\dagger \). With their help I obtain the following set of evolution equations for the hydrodynamic variables, see [11],

\[ \partial_t \rho + (\vec{v} \cdot \nabla) \rho = -(\nabla \cdot \vec{v}) \rho, \] (29)

\[ \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \vec{v} \times (\vec{v} \times \nabla \ln \rho - 2c \vec{u} - 2\nabla \times \vec{v}), \] (30)

\[ \partial_t \vec{u} + (\vec{v} \cdot \nabla) \vec{u} = \frac{1}{2c \rho} \partial_i (\rho \varepsilon_{ikl} \nabla v_l). \] (31)

These equations have a characteristic hydrodynamic form with substantial derivatives of the variables appearing on the left hand side, as in the nonrelativistic case.

In order to restrict the solutions of the hydrodynamic equations (29–31) to only those that are obtained from a two-component wave function, I impose constraints on the initial conditions. The first constraint is purely algebraic; it determines the length of the velocity vector, \( |\vec{v}| = c \). The second constraint — the quantization condition — is obtained from the general formula (24). In terms of the hydrodynamic variables, this condition reads

\[ \int_S d\vec{S} \cdot [\nabla \times \vec{u} - \frac{1}{4c^3} \varepsilon_{ijk} v_i (\nabla v_j) \times (\nabla v_k)] = 2\pi n. \] (32)

Thus, the time evolution of the neutrino wave function is described in the hydrodynamic formulation as a flow of probability with velocity \( \vec{v} \). The velocity vector is precessing around the vector \( \vec{v} \times \nabla \ln \rho - 2(c \vec{u} + \nabla \times \vec{v}). \)
IV. HYDRODYNAMIC FORM OF THE MAXWELL EQUATIONS

In order to find the hydrodynamic form of the Maxwell equations I shall rewrite them in a form (7) in order to apply my general formalism. This is done with the use of the photon wave function that was described in detail in [13,14]. The photon wave function \( \vec{F} \) is a complex three-vector and is constructed from the fields \( \vec{D} \) and \( \vec{B} \) as follows

\[
\vec{F}(\vec{r}, t) = \frac{\vec{D}(\vec{r}, t)}{\sqrt{2\varepsilon}} + i \frac{\vec{B}(\vec{r}, t)}{\sqrt{2\mu}}. \tag{33}
\]

Maxwell equations written in terms of \( \vec{F} \) have the form

\[
\partial_t \vec{F}(\vec{r}, t) = -ic\nabla \times \vec{F}(\vec{r}, t), \tag{34}
\]

\[
\nabla \cdot \vec{F}(\vec{r}, t) = 0. \tag{35}
\]

The important physical quantities can be expressed in terms of \( \vec{F} \) as bilinear combinations:

- **Energy density**

\[
\mathcal{E} = \vec{F}^* \cdot \vec{F} = \frac{\vec{D}^2}{2\varepsilon} + \frac{\vec{B}^2}{2\mu}. \tag{36}
\]

- **Energy flux (Poynting vector)**

\[
\vec{P} = \frac{c}{2i} \vec{F}^* \times \vec{F} = c^2 \vec{D} \times \vec{B}, \tag{37}
\]

- **Maxwell stress tensor**

\[
T_{ij} = F_i^* F_j + F_j^* F_i - \delta_{ij} \vec{F}^* \cdot \vec{F} = \frac{D_i D_j}{\varepsilon} + \frac{B_i B_j}{\mu} - \delta_{ij} (\frac{\vec{D}^2}{2\varepsilon} + \frac{\vec{B}^2}{2\mu}). \tag{38}
\]

The well known continuity equations for the energy density and the momentum density

\[
\partial_t \mathcal{E} = -\nabla \cdot \vec{P}, \quad \partial_t P_i = c^2 \partial_j T_{ij}, \tag{39}
\]

may serve as hints how to introduce convenient hydrodynamic variables. Five hydrodynamic variables will be derived from the energy density, the energy flux and the stress tensor. The sixth variable determines the phase of \( \vec{F} \) and it is defined as in my general treatment.

\[
\rho = \mathcal{E} = \frac{\vec{F}^2}{2\varepsilon} + \frac{\vec{B}^2}{2\mu}, \quad \rho \vec{v} = \vec{P} = c^2 \vec{D} \times \vec{B}, \tag{40}
\]

\[
\rho t_{ij} = c T_{ij} + \delta_{ij} c \rho = \frac{c D_i D_j}{\varepsilon} + \frac{c B_i B_j}{\mu}, \tag{41}
\]

\[
\rho \vec{u} = \frac{1}{2t} F_i^* \nabla F_i = \frac{c}{2} (D_i \nabla B_i - B_i \nabla D_i). \tag{42}
\]
Only one component of $t_{ij}$ is free to choose. The remaining five are fixed by the conditions (summation convention)

$$t_{ii} = 2c, \quad v_i t_{ik} = 0, \quad t_{ij} t_{ij} = 4c^2 - 2\vec{v}^2.$$  \hspace{1cm} (43)

The momentum vector $\vec{u}$ is defined in analogy with wave mechanics but there is one difference; it transforms as a pseudovector under space-reflections. This is due to the fact that $\vec{F}$ is complex conjugated under reflections since $\vec{B}$ is a pseudovector.

The curl of $\vec{u}$ is subject to the quantization condition that can be obtained from the general formula (24),

$$\int dS \cdot [\nabla \times \vec{u} - \frac{1}{8c^3} \varepsilon_{ijk}(v_i \nabla v_j \times \nabla v_k + v_i \nabla t_{jl} \times \nabla t_{kl} - 2t_{il} \nabla t_{jl} \times \nabla v_k)] = 2\pi n.$$  \hspace{1cm} (44)

In addition to the quantization condition, the initial values of the hydrodynamic variables must satisfy two equations that guarantee that the fields $\vec{D}$ and $\vec{B}$ are divergenceless

$$\frac{1}{2} \partial_k (\rho t_{ik}) + \rho \varepsilon_{ijk} v_j u_k + \frac{\rho}{4c} \varepsilon_{ikl} t_{lj} - t_{ij} \partial_k t_{ik} + v_k \partial_k v_i - v_i \partial_k v_k = 0,$$  \hspace{1cm} (45)

$$\frac{1}{2} \partial_k (\rho \varepsilon_{ikl} v_l) + \rho t_{ik} u_k + \frac{\rho}{4c} \varepsilon_{ikl} (t_{il} \partial_k v_j - v_j \partial_k t_{il}) + \varepsilon_{ijkl} (t_{ik} \partial_l v_j - v_j \partial_k t_{kl}) = 0.$$  \hspace{1cm} (46)

Having completed all preparatory steps, I may write now the evolution equations in terms of the hydrodynamic variables

$$\partial_t \rho = -\nabla \cdot (\rho \vec{v})$$  \hspace{1cm} (47)

$$\partial_t (\rho v_i) = \partial_j (-c^2 \rho \delta_{ij} + \rho t_{ij})$$  \hspace{1cm} (48)

$$\partial_t (\rho t_{ij}) = \delta_{ij} \rho v_k \partial_k t_{kl} - \delta_{ij} \epsilon_{klm} \partial_l t_{ml} + \frac{c}{2} (v_i \partial_j + v_j \partial_i) \rho$$

$$+ \rho \varepsilon_{ikl} u_k v_l - \rho \varepsilon_{ikl} u_k t_{li} + \rho (v_k \partial_k t_{ij} - t_{ij} \partial_k v_k)$$

$$+ \frac{\rho}{2} (t_{ik} \partial_k v_j - t_{ik} \partial_k v_j - v_j \partial_k t_{ki} - v_j \partial_k t_{ki} - v_k \partial_k t_{kj} - v_k \partial_k t_{kj})$$  \hspace{1cm} (49)

$$\partial_t (\rho u_i) = \partial_j [-\rho u_j + \frac{\rho}{4c} \varepsilon_{ikl} (t_{km} \partial_l t_{ml} + v_k \partial_m v_l)].$$  \hspace{1cm} (50)

These equations can also be written in a form clearly exhibiting their hydrodynamic structure, with all substantial derivatives on the left

$$\partial_t \rho + (\vec{v} \cdot \nabla) \rho = -\rho (\nabla \cdot \vec{v})$$  \hspace{1cm} (51)

$$\partial_t v_i + (\vec{v} \cdot \nabla) v_i = \frac{1}{\rho} \partial_j (-c^2 \rho \delta_{ij} + \rho v_i v_j + \rho t_{ij})$$  \hspace{1cm} (52)

$$\partial_t t_{ij} + (\vec{v} \cdot \nabla) t_{ij} = \frac{1}{\rho} (t_{ij} v_k \partial_k \rho - c \delta_{ij} v_k \partial_k \rho + c \varepsilon_{ikl} u_k t_{li})$$

$$+ \partial_j v_k \partial_k t_{ij} + 2v_k \partial_k t_{ij} + c \varepsilon_{ikl} u_k t_{li}$$

$$+ (v_k \partial_k t_{ij} - t_{ij} \partial_k v_k) + \frac{1}{2} (t_{ik} \partial_k v_j + t_{jk} \partial_k v_i)$$

$$- \frac{1}{2} (v_i \partial_k t_{kj} + v_j \partial_k t_{ki} + v_k \partial_i t_{kj} + v_k \partial_j t_{ki})$$  \hspace{1cm} (53)

$$\partial_t u_i + (\vec{v} \cdot \nabla) u_i = \frac{1}{4c\rho} \partial_j [\rho \varepsilon_{jkl} (t_{km} \partial_l t_{ml} + v_k \partial_m v_l)].$$  \hspace{1cm} (54)
The hydrodynamic form of Maxwell equations has a dual interpretation. On the one hand, one may use purely classical notions of the energy flow and momentum flow of the classical electromagnetic field. On the other hand, one may use the quantum notion of the flow of the probability density and the additional variables characterizing the probabilistic fluid. I prefer the second interpretation because of the intimate connection with the hydrodynamic formulation of the Schrödinger equation. In particular the variable $\vec{u}$ with its quantization condition is deeply rooted in quantum theory; its role in the classical interpretation being purely formal.

V. CONCLUSIONS

The main conclusion of this study is that the flow of probability associated with the Weyl equation and the Maxwell equations can be described in purely hydrodynamic terms. The probabilistic fluid moves with the speed $\vec{v}$ and is endowed with additional degrees of freedom — the longitudinal part of a vector field $\vec{u}$ and an additional variable for the Maxwell theory. As compared with relativistic dynamics of a perfect fluid, the flow of the probability fluid corresponding to relativistic wave equations is fairly complex. In the face of these complications, are there any advantages of using the hydrodynamic description? It certainly offers a totally different look at the wave function and its time evolution. The new form involves only observable quantities bilinear in the wave function. Moreover, the hydrodynamic description clearly separates the local dynamical laws and a nonlocal, global quantization condition. This important property has been utilized in the past to derive the quantization condition for the magnetic charge [15] and to clarify the interpretation of the Aharonov-Bohm effect [16]. Whether it will lead to some new insights into the quantum properties of massless particles, remains an open question.

Finally, I would like to mention a possible connection of the hydrodynamic formulation of wave mechanics of massless particles to the string theory. In the hydrodynamic formulation an essential role is assigned to quantized vortex lines. Such vortex lines are very similar to relativistic strings. They move with the speed of light, but in contradistinction to free strings they interact mutually, just like vortex lines do in ordinary fluid dynamics. It might be possible to extract some interesting relativistic dynamics of the vortex lines — the strings — from the hydrodynamic equations of the probabilistic fluid.

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