

ENTROPIC UNCERTAINTY RELATIONS

Iwo Białynicki-Birula

Center for Theoretical Physics Polish Academy of Sciences, Lotnikow 32/46, 02-668 Warsaw, Poland

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I. INTRODUCTION

Every notable physical theory has its trademark - an eye-catching symbol, picture or equation — that can be placed on a book cover or on a T-shirt. In Newtonian mechanics it is

$$\vec{F} = m\vec{a}, \quad (1)$$

in relativity theory it is

$$E = mc^2, \quad (2)$$

in the theory of elementary particles it is a Feynman diagram



$$\rangle \text{---} \text{---} \text{---} \langle \quad (3)$$

and in quantum mechanics it is the celebrated Heisenberg relation

$$\Delta x \cdot \Delta p \sim \hbar, \quad (4)$$

expressing the uncertainty principle. It is worth noting that this relation was discovered by Heisenberg in 1927 in the aftermath of the outburst of quantum mechanics. The first formulation of the uncertainty relation appeared in the paper "On the intuitive content of the quantum-theoretic kinematics and mechanics" (Heisenberg 1927) where Heisenberg explained this relation in the following way:

"The more precisely the position of the electron is determined, the less precisely the momentum is known, and vice versa."

"Determined" here means that a measurement has been performed. In his uncertainty relation Heisenberg saw a "direct and intuitive interpretation" of the commutation relations between the position and momentum operators that were in those early days of quantum mechanics still full of mystery.

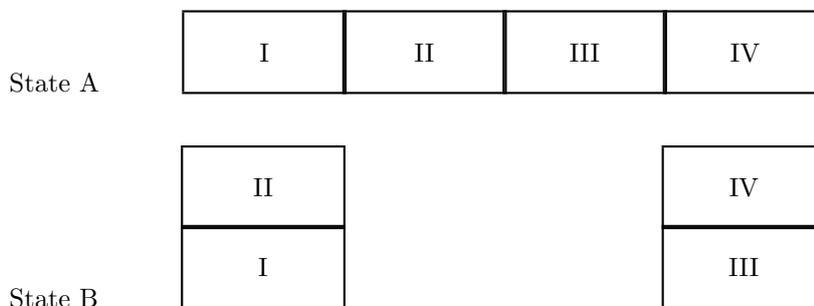
The derivation of the uncertainty relation from the mathematical formalism of quantum mechanics was given by Kennard in the same year 1927 (Kennard 1927). In the derivation given by Kennard the quantities Δx and Δp are already precisely defined as standard deviations and the uncertainty relation takes on the form of a mathematical inequality

$$\Delta x \cdot \Delta p \geq \hbar/2. \quad (5)$$

II. CRITIQUE OF THE STANDARD FORM OF UNCERTAINTY RELATIONS

The standard deviation is used a lot in the statistical analysis of experiments. It is a reasonable measure of the spread when the distribution in question is of a simple "hump" type. In particular it is a very good characteristic for a Gaussian distribution since it measures the half-width of this distribution. However, when the distribution of the values has more than one hump or when it is not of any simple type, the standard deviation loses some of its usefulness especially in connection with the notion of uncertainty. In order to explain, why the standard deviation is not the best measure of uncertainty, let me consider two very simple examples taken from quantum mechanics of a particle moving in one dimension. In the first example let us compare two states of a particle. One state describing the particle localized with a uniformly distributed probability in a box of length L and the other describing the particle localized with equal probabilities in two smaller boxes each of length $L/4$.

The probability distributions corresponding to these two states are graphically represented as follows:



The wave functions describing these two situations can be taken in the form:

$$\text{State A} \quad \psi = \begin{cases} \frac{1}{\sqrt{L}} & \text{inside the box,} \\ 0 & \text{outside the box,} \end{cases} \quad (6)$$

$$\text{State B} \quad \psi = \begin{cases} \sqrt{\frac{2}{L}} & \text{inside the box,} \\ 0 & \text{outside the box.} \end{cases} \quad (7)$$

Before continuing let us ponder in which case, A or B, the uncertainty concerning the particle's position is greater. According to our intuitive notion, the uncertainty is greater in the case A. In the case B we know more about the position; we know that the particle is not in the regions II and III. However, when we calculate the standard deviation Δx we obtain the opposite result:

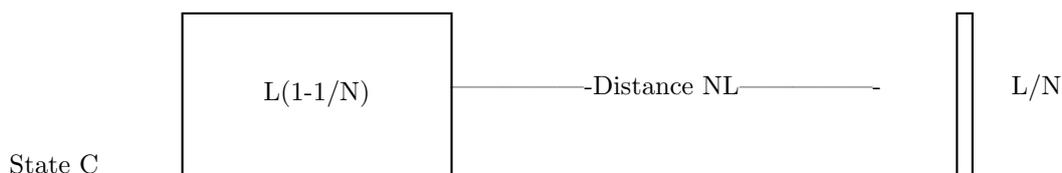
Case A

$$\Delta x_A = \frac{L}{\sqrt{12}} \quad (8)$$

Case B

$$\Delta x_B = \sqrt{\frac{7}{4}} \frac{L}{\sqrt{12}} \quad (9)$$

The second, somewhat more dramatic example of the situation where the standard deviation does not give a sensible measure of uncertainty is provided by the following distribution of probability.



The wave function does not vanish and is constant in two regions I and II separated by a large distance NL (N is a large number). The region I is of the size $L(1 - 1/N)$ and the region II is of the size L/N .

$$\text{State C} \quad \psi = \begin{cases} \frac{1}{\sqrt{L(1-1/N)}} & \text{in region I,} \\ \frac{1}{\sqrt{L/N}} & \text{in region II,} \\ 0 & \text{elsewhere.} \end{cases} \quad (10)$$

For large N the standard deviation Δx is approximately equal to:

Case C

$$\Delta x_C \sim (1 + 12N) \frac{L}{\sqrt{12}}, \quad (11)$$

so that it tends to infinity with N in spite of the fact that the probability of finding the particle in the region I tends to 1. The second example shows most vividly what is wrong with the standard deviation. It gets very high contributions from distant regions, because these contribution enter with a large weight: the distance from the mean value squared.

I have spoken about the position of the particle but the same criticism applies to the standard deviation used as a measure of uncertainty for momentum or other physical quantities. In order to illustrate once again the fact that the weighing of the probability density with the distance squared overemphasizes the contributions from distant regions, I would like to consider the standard deviation for the momentum variable corresponding to the wave function describing the state A in my first example. The momentum representation of this wave function is

$$\tilde{\psi}(p) = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \exp(ipx/\hbar) = \frac{2\hbar}{\sqrt{L}} \frac{\sin(pL/2\hbar)}{p}. \quad (12)$$

The probability density for the momentum defined by this wave function diminishes to zero when p tends to infinity only as $1/p^2$, so that the second moment of this distribution is infinite rendering the Heisenberg uncertainty relations expressed in terms of the standard deviations meaningless.

Can we overcome the deficiencies of the traditional approach to uncertainty relations by changing the measure of uncertainty while still keeping the spirit of the Heisenberg ideas intact? Not only is the answer to this question in the affirmative but we can do it using a very profound definition of uncertainty, much more fundamental than the one based on the second moment. Such definition comes from information theory. Let us take a look at uncertainty from the point of view of information. What is uncertainty? It is just the lack of information; a hole in our knowledge or missing information. Therefore, the uncertainty can be measured in exactly the same manner as the information is measured.

III. INFORMATION ENTROPY AS A MEASURE OF UNCERTAINTY

Since 1948 we have a very good measure of information introduced by Shannon (Shannon 1948) in his classic treatise entitled "The mathematical theory of communication". It is called the information entropy and it is defined by the following formula closely resembling the definition of the physical entropy,

$$H = - \sum_i p_i \ln p_i, \quad (13)$$

where p_i is the probability of the occurrence of the i -th event or the a priori probability of the i -th message. In information theory one uses the logarithms to the base of 2 (H is then measured in bits), but a change of the base results only in a change of the scale; H is multiplied by a constant factor. The information entropy may serve as a precise measure of uncertainty and it has even been described already by Shannon as a measure of "information, choice or uncertainty". The quantity H may serve at the same time as a measure of information and as a measure of uncertainty, because the two notions are directly related. The information gained in the measurement removes the uncertainty about its outcome. The origins of the Shannon formula go back to Boltzmann, who had thought of a similar definition for the thermodynamic entropy.

The information uncertainty is in many ways a much better measure of uncertainty than the standard deviation. First of all one may mention that the form of H has been derived by Shannon from a plausible set of simple assumptions. The uniqueness of H is further emphasized by the so called noiseless coding theorem (cf., for example, Ash 1965):

Average number of questions needed to discover "the truth" (that is to remove uncertainty) hidden in one of the boxes with the probability distribution p_i is bounded from below by H and by a proper choice of the strategy one may approach H arbitrarily close.

Another confirmation that H is the natural measure of uncertainty may be found in experimental psychology. For example, it has been established (Hyman 1955) that the reaction time of human subjects as a function of the uncertainty of the stimulus is proportional to H . In such experiments sets of lamps were being turned on randomly according to certain patterns that were governed by a probability distribution p_i . Persons who were the subjects of these experiments were asked to react in a certain specified way to each pattern. Their reaction time turned out to be a linear function of H . Finally, the fundamental role played in physics by the thermodynamic entropy which is in turn closely related to the information entropy seems to justify the claim that H is a better universal measure of uncertainty than the standard deviation.

In all three cases A,B, and C considered before, the definition (13) gives sensible results. The entropic uncertainty is in the case A by a factor of 2 bigger than in the case B. In the case C the entropic uncertainty does not practically depend on N .

IV. ENTROPIC UNCERTAINTY RELATION FOR POSITION AND MOMENTUM

Before introducing the uncertainty relation for the position and momentum in quantum mechanics with the information entropy as the measure of uncertainty. Let me begin with a few words on the history of the subject. It all started almost 40 years ago when a physicist Everett (Everett 1957), a mathematician Hirschman (Hirschman 1957), and two information scientists (Burrell 1958, Leipnik 1959, 1960) discovered independently the existence of the following inequality satisfied by any function ψ and its Fourier transform $\tilde{\psi}$:

$$- \int dx |\psi|^2 \ln |\psi|^2 - \int dp |\tilde{\psi}|^2 \ln (|\tilde{\psi}|^2) \geq 1 + \ln \pi. \quad (14)$$

Their conjecture was based on the observation that this inequality is saturated by all Gaussian functions and that infinitesimal variations around the Gaussian function increase the left hand side of the inequality. It is worth mentioning that Everett discovered this inequality while working on the many-worlds interpretation of quantum mechanics and this inequality has been published in the expanded version of his PhD Thesis published much later in a volume devoted to the many-worlds interpretation. The first proof of the inequality (14) was given by Mycielski and myself (Bialynicki-Birula and Mycielski 1975) and independently by Beckner (Beckner 1975) almost 20 years after its discovery. The inequality (14) represents an important mathematical relation but it can not be treated by itself as an entropic uncertainty relation, because the integrals appearing in it do not have a direct physical interpretation as measures of uncertainty. This inequality is, however, instrumental in the derivation of the proper entropic uncertainty relation for x and p . The general framework for such uncertainty relations has been set by Deutsch (Deutsch 1983) and Partovi (Partovi 1983) (see also Blankenbeckler and Partovi 1985). Let us consider a physical quantity (observable) \mathcal{A} described in the formalism of quantum mechanics by the operator A . The measurement of this observable involves an apparatus which always introduces a partitioning of the spectrum of \mathcal{A} into non overlapping subsets usually called bins. In the mathematical formalism of quantum theory to this partitioning of the spectrum there corresponds the partitioning of the Hilbert space into orthogonal subspaces, each subspace representing one bin. With each subspace, and therefore with each bin we can associate the corresponding projection operator \hat{P}_i^A , that is cutting out the appropriate subspace of the Hilbert space. For a given state of the system, pure or mixed, to each projection operator and thus to each bin we can assign the probability p_i that the measurement will give a value belonging to the i -th bin. The formula for p_i in the general case reads

$$p_i^A = \text{Tr}\{\rho\hat{P}_i^A\}, \quad (15)$$

while for pure states it reduces to

$$p_i^A = \langle\Psi|\hat{P}_i^A|\Psi\rangle. \quad (16)$$

From the set of all the probabilities evaluated for a given observable and for a given state of the system we can construct the entropic measure of the uncertainty according to Shannon's formula. An important difference between this measure of uncertainty and the one given by the standard deviation $\Delta\mathcal{A}$ is that the entropic uncertainty will depend not only on the observable and the on the state of the system but also on the partition of the spectrum into bins that is on the properties of the measuring device, namely, on its resolving power. The application of this general framework to position and momentum requires the partitioning of the spectrum of x and p into bins which for simplicity will be taken all equal to δx and δp . The probabilities p_i^x and p_i^p to find the value in the i -th bin are (for the pure state) given by

$$p_i^x = \int_{\delta x} dx |\psi|^2, \quad (17)$$

and

$$p_i^p = \int_{\delta p} dp |\tilde{\psi}|^2, \quad (18)$$

where the integrations are over the i -th bin. In this way we arrive at two entropic measures of uncertainty H^x and H^p . Each of them separately can vanish if the state of the particle is described by a wave function having support in only one bin (in position space or in momentum space). However, one can not achieve this simultaneously for both uncertainties. The existence of a lower bound of the sum of the two entropic uncertainties is the expression of the uncertainty principle for position and momentum in terms of entropic uncertainties. On dimensional grounds, this lower bound may depend only on one dimensionless parameter in the problem: $\gamma = \delta x \delta p / h$. The lower bound has not so far been calculated exactly, but the inequality (14) enabled me to give a fairly good approximation to it, valid for small values of γ ,

$$H^x + H^p \geq 1 - \ln 2 - \ln(\delta x \delta p / h). \quad (19)$$

The right hand side of this inequality gives the leading part of the expansion of the exact bound for small γ ; the corrections are of the order of γ . This bound is shown in the Fig. 1. Of course, the negative values of the bound in this figure are meaningless, since the left hand side is nonnegative. They reflect the fact that the bound is only relevant for small values of γ . The physical content of the entropic uncertainty relation is still the same as for the traditional uncertainty relations. It can still be expressed by the Heisenberg's words: "the more precisely the position...".

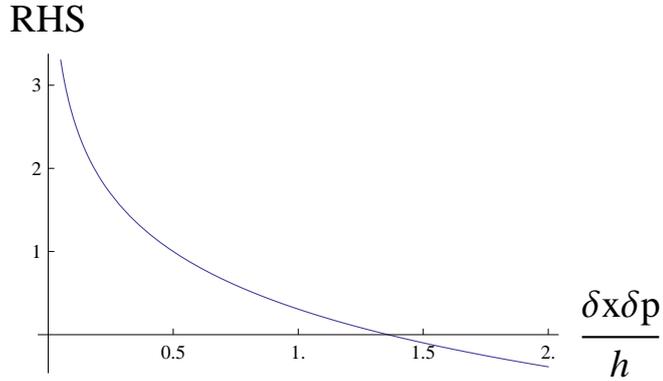


FIG. 1. The plot of the r.h.s. of the entropic uncertainty relation (19). The area above the curve represents the allowed values of the sum of the two entropies.

V. ENTROPIC UNCERTAINTY RELATIONS FOR THE ANGLE AND ANGULAR MOMENTUM OR FOR THE PHASE AND THE PHOTON NUMBER

The most successful application of the entropic uncertainty relations is to the conjugate pairs of the variables: angle—angular momentum or phase—number of photons. This case is of particular interest since the description in terms of standard deviations fails (Carruthers and Nieto 1968). The reason for that failure is the absence of a self-adjoint operator that could represent the azimuthal angle or the phase (multiplication by the angle variable produces a function which is no longer periodic). In contrast, the entropic uncertainty relations work in this case even better than for the position-momentum pair.

The starting point of that derivation is again an analog of the inequality (14) for the wave functions (Bialynicki-Birula and Mycielski 1975) which reads:

$$-\int_0^{2\pi} d\varphi |\Phi|^2 \ln |\Phi|^2 - \sum_{m=-\infty}^{\infty} |c_m|^2 \ln |c_m|^2 \geq \ln(2\pi), \quad (20)$$

where ψ is the wave function depending on the angular (or phase) variable φ , and c_m 's are its expansion coefficients into the set of harmonic functions $\exp(im\varphi)$,

$$\Phi = \sum_{m=-\infty}^{\infty} c_m \exp(im\varphi). \quad (21)$$

With the help of this mathematical inequality, we obtain the following entropic uncertainty relation:

$$H^\varphi + H^{L_z} \geq -\ln(\delta\varphi/2\pi), \quad (22)$$

where $\delta\varphi$ is the size of the bin in the measurement of the angle. Both entropic uncertainties in this formula were constructed according to the same general principle as in the case of position and momentum but this time the discreteness of the spectrum of L_z leads not to integrals of the probability density but directly to a discrete set of numbers $|c_m|^2$ which determine the probability to find a given value of angular momentum. In contrast to the position-momentum case, one can easily find states for which the entropic uncertainty relation for the angle and the angular momentum is saturated. These are simply the eigenstates of angular momentum. For these states the probability density in the angular variable is distributed uniformly over the whole unit circle, that is:

$$H^\varphi = -\ln(\delta\varphi/2\pi), \quad (23)$$

and the uncertainty in the angular momentum vanishes.

Entropic uncertainty relation can also be formulated for the double pair of conjugate variables: azimuthal and polar angles — one vector component and the square of the total angular momentum. In this case the approach based on the standard deviations fails completely, whereas the entropic uncertainty relations are conceptually and even technically not more complicated than in previously discussed cases.

Let us consider the angular wave function $\Psi(\varphi, \theta)$ describing the angular distribution of the probability. After partitioning the sphere into bins (this time it would be not very natural to try to make them all equal) representing a collection of detectors, we can calculate the probability of detection p_i by each detector. Then with the use of Shannon's formula we obtain the entropic uncertainty of the angular distribution $H^{(\varphi, \theta)}$. The calculation of the complementary entropic uncertainty in the measurements of L_z and L^2 is equally simple. All we have to do is to decompose the angular wave function into spherical harmonics,

$$\Psi(\varphi, \theta) = \sum_{lm} c_{lm} Y_{lm}(\varphi, \theta), \quad (24)$$

and use the expansion coefficients to calculate the entropic uncertainty in L_z and L^2 .

The entropic uncertainty relation for $H^{(\varphi, \theta)}$ and $H^{(L_z, L^2)}$ reads:

$$H^{(\varphi, \theta)} + H^{(L_z, L^2)} \geq -\ln(\delta\bar{\varphi}/2\pi), \quad (25)$$

where this time the angle $\delta\bar{\varphi}$ measures the largest opening angle of the detectors as seen from the center of the sphere.

I hope that the simplicity of the form and of the physical interpretation of the entropic uncertainty relations for angular variables and their conjugate angular momenta will induce lecturers to include them as standard material in the courses of quantum mechanics even at the undergraduate level.

VI. CONCLUSIONS

Entropic uncertainty relations combine interesting mathematical properties showing up in their proofs with novel physical conceptions enabling one to extend the range of applicability of these relations. We find two important differences when the entropic relations are compared with the traditional uncertainty relations based on the standard deviations.

1. First, in order to calculate the entropic uncertainty it is necessary to know the resolving power of the measuring device. I consider this dependence of the uncertainty on the precision of the experiment especially appealing from the physical point of view.
2. Second, in contrast to standard deviations the entropic measures of uncertainty do not depend on the notion of the distance between different points of the spectrum.

We have defined the entropic uncertainties for angular variables where the distance could have been probably introduced with some effort, but we can also calculate the entropic uncertainty in the cases where the notion of the distance would not make much sense, like, for example, in sociological studies of human attitudes.

Entropic uncertainty relations are becoming recently more and popular. They feature prominently in two recent reviews (Dodonov and Manko 1987, Uffink 1990) devoted to uncertainty relations.

REFERENCES

- Ash, R.B., 1965, *Information Theory*, Interscience, New York.
- Beckner, W., 1975, *Inequalities in Fourier analysis*, *Ann.Math.*102, 159.
- Bialynicki-Birula, I. and J. Mycielski, 1975, *Uncertainty relations for information entropy in wave mechanics*, *Comm.Math.Phys.* 44, 129.
- Bialynicki-Birula, I., 1984a, *Entropic uncertainty relations*, *Phys.Lett.*103A, 253.
- Bialynicki-Birula, I., 1984b, *Entropic uncertainty relations in quantum mechanics*, in *Quantum Probability and Applications*, Eds. L. Accardi and W. von Waldenfels, *Lecture Notes in Mathematics* 1136, Springer, Berlin.
- Bialynicki-Birula, I. and J. Madajczyk, 1985, *Entropic uncertainty relations for angular distributions*, *Phys.Lett.*108A, 384.
- Blankenbecler R., and M.H. Partovi, 1985, *Uncertainty, entropy and the statistical mechanics of microscopic systems*, *Phys.Rev.Lett.*54, 373.
- Borrei, R., 1958, *A note on the information theoretic form of the uncertainty principle*, *Information and Control* 1, 398.
- Carruthers, P., and M.M. Nieto, 1968, *Phase and angle variables in quantum mechanics*, *Rev.Mod.Phys.*40, 411.

- Deutsch, D., 1983, Uncertainty in quantum measurements, *Phys.Rev.Lett.*50, 631.
- Dodonov V.V., and V.I. Manko, 1987, Generalizations of uncertainty relations in quantum mechanics, *Lebedev Institute Papers* 183, 5.
- Everett, H., 1973, The theory of the universal wave function, in *The Many- Worlds Interpretation of Quantum Mechanics*, Eds. B.S. DeWitt and N. Graham, Princeton University Press, Princeton.
- Heisenberg, W., 1927, Ueber den anschaulichen Inhalt der quantenmechanischen Kinematik und Mechanik, *Z.Phys.*43, 172.
- Hirschman, I.I., 1957, A note on entropy, *Am.J.Math.*79, 152.
- Hyman, R., 1955, Stimulus information as a determinant of reaction time, *J.Exper.Psychology.*45, 188.
- Kennard, E.H., 1927, Zur Quantenmechanik einfacher Bewegungstypen, *Z.Phys.*44, 326.
- Leipnik, R., 1959, Entropy and the uncertainty principle, *Information and Control* 2, 64.
- Leipnik, R., 1960, The extended entropy uncertainty principle, *Information and Control* 3, 18.
- Partovi, H.M., 1983, Entropic formulation of uncertainty for quantum measurements, *Phys.Rev.Lett.*50, 1883.
- Shannon C.E., 1948, The mathematical theory of communication, *Bell Syst.Tech.J.*27, 379, 623. Reprinted in C.E. Shannon and W. Weaver, *The Mathematical Theory of Communication*, Univ. of Illinois Press, Urbana 1949.
- Uffink J.B.M., 1990, Measures of Uncertainty and the Uncertainty Principle, PhD.Thesis, University of Utrecht.