The Field Theory with Yukawa Coupling in One Dimension.

I. Bialynicki-Birula

Institute of Theoretical Physics, Warsaw University - Warsaw

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In a very recent paper (1) V. GLASER has found the solution of the equation of motion and the $S$ matrix in the Thirring model (2a). The aim of this note is to show that one can solve exactly the equations of motion and find the $S$ matrix also in the case of a zero rest mass spinor field interacting with a vector field. Depending on whether one considers a boson field with vanishing (electromagnetic field) or non-vanishing rest mass one has either

\begin{equation}
H_{\gamma} = e\bar{\psi} \gamma_{\mu} \psi A_{\mu}.
\end{equation}

or

\begin{equation}
H_{\gamma} = g\bar{\psi} \gamma_{\mu} \psi A_{\mu} + g^2 \bar{\psi} \psi \psi \psi.
\end{equation}

The contact term arises in the usual way (4).

In the following we shall deal with the electromagnetic field only. The case of the vector meson field can be examined in a very similar manner. The operator $U(\sigma, \sigma_0)$ which transforms the states in the interaction picture is defined usually by the equations

\begin{equation}
\begin{aligned}
i \frac{\delta}{\delta\sigma(x)} U(\sigma, \sigma_0) &= H_{\gamma}(x) U(\sigma, \sigma_0), \\
-i \frac{\delta}{\delta\sigma_0(x)} U(\sigma, \sigma_0) &= U(\sigma, \sigma_0) H_{\gamma}(x).
\end{aligned}
\end{equation}

If we take into account the commutation rules for the current operator $j_{\mu}$ (1) and the commutation rules for the potentials

\begin{equation}
[A_{\mu}(x), A_{\nu}(x')] = \frac{1}{i} g_{\mu\nu} D(x - x'),
\end{equation}

(1) V. GLASER, Nuovo Cimento 9, 990 (1958).
(3) W. THIRRING, Ann. of Phys. 9, 91 (1958).
(4) H. UMEZAWA, Quantum Field Theory (Amsterdam, 1956).
we can obtain the solution of the equations (3) in the form

\[ U(\sigma, \sigma_0) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} j_{\mu} A^\mu \, d^2x \right] \exp \left[ -ie^2 \int_{\sigma_0}^{\sigma} j_{\mu}(x) D(x - x') j^\mu(x') \, d^2x \, d^2x' \right]. \]

Here

\[ g_{00} = 1, \quad g_{11} = -1, \]

\[ D(x - x') = (2\pi)^{-2} \int \frac{d^2k}{k} k^2 \exp \left[ -ik_{\mu} x^\mu \right] = D_A - D_R, \]

\[ D(x - x') = (2\pi)^{-2} \int \frac{d^2k}{k} k^2 \exp \left[ -ik_{\mu} x^\mu \right] = \frac{1}{2} (D_A + D_R). \]

With the notations

\[ j_+ = \psi_1^* \psi_1, \quad j_- = \psi_2^* \psi_2, \quad A_\pm = A^0 \pm A^1, \quad x_\pm = x^0 \pm x^1, \]

the formula (5) gives

\[ D(\sigma, \sigma_0) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} (j_+ A_+ + j_- A_-) \, d^2x \right] \exp \left[ -2ie^2 \int_{\sigma_0}^{\sigma} j_{\mu}(x) D(x - x') j^\mu(x') \, d^2x \, d^2x' \right]. \]

\[ U(\sigma, \sigma_0) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} A_+ \, d^2x \right] \exp \left[ -ie \int_{\sigma_0}^{\sigma} A_- \, d^2x \right]. \]

\[ \cdot \exp \left[ -2ie^2 \int_{\sigma_0}^{\sigma} j_+(x) D_R(x - x') j_+(x') \, d^2x \, d^2x' \right], \]

\[ U(\sigma, \sigma_0) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} A_- \, d^2x \right] \exp \left[ -ie \int_{\sigma_0}^{\sigma} A_+ \, d^2x \right]. \]

\[ \cdot \exp \left[ -2ie^2 \int_{\sigma_0}^{\sigma} j_-(x) D_R(x - x') j_-(x') \, d^2x \, d^2x' \right]. \]

The S matrix can be obtained from (5) by the usual limiting process

\[ S = \lim_{\sigma_0 \to -\infty, \sigma \to \infty} U(\sigma, \sigma_0). \]

To obtain the solutions of the field equations in terms of the free field operators
it is sufficient to compute the expressions

\begin{equation}
U^{-1}(\sigma, \sigma_0) A_\mu(x) U(\sigma, \sigma_0) = A_\mu(x),
\end{equation}

\begin{equation}
U^{-1}(\sigma, \sigma_0) \psi(x) U(\sigma, \sigma_0) = \psi(x).
\end{equation}

This can be done for arbitrary \( x, \sigma, \sigma_0 \) and yields

\begin{equation}
A_\mu(x) = A_\mu(x) + e \int_{\sigma_0}^{\sigma} D(x - x') d^2x' \hat{j}_\mu(x'),
\end{equation}

\begin{align}
\begin{cases}
\psi_1(x) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} \delta(x_+ - x'_+) d^2x' \left( A_+(x') + 2e \int_{\sigma_0}^{\sigma} D_R(x' - x^r) j_+(x^r) d^2x^r \right) \right] \psi_1(x), \\
\psi_2(x) = \exp \left[ -ie \int_{\sigma_0}^{\sigma} \delta(x_+ - x'_-) d^2x' \left( A_-(x') + 2e \int_{\sigma_0}^{\sigma} D_R(x' - x^r) j_-(x^r) d^2x^r \right) \right] \psi_2(x).
\end{cases}
\end{align}

It is worthwhile to notice that the assumption concerning the vanishing of the fermion rest mass is in this model, like in the Thirring model, the necessary condition to solve the problem in a simple manner. Further considerations on this model will be published later.