ENTROPIC UNCERTAINTY RELATIONS

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New entropic uncertainty relations for angle-angular momentum and position-momentum, derived recently by Partovi, are related to older relations of a similar type, which were proved by Bialynicki-Birula and Mycielski. Significantly improved lower bounds are obtained in both cases.

In two recent papers, Deutsch [1] and Partovi [2] have introduced new uncertainty relations in quantum mechanics, expressed in terms of the information entropy. The purpose of this note is to significantly sharpen their results in the case of canonically conjugate observables and to remark, also, on the history of this subject.

The entropic uncertainty relation for two observables, A and B, has the form [2]

$$S^A + S^B \geq b. \quad (1)$$

The entropies $S^A$ and $S^B$ are defined by the standard formula

$$S^A = -\sum_i p_i^A \ln p_i^A, \quad (2)$$

where $p_i^A$ is the probability to find the value of the observable $A$ in the $i$th interval of the spectrum of $A$, or in the $i$th bin, in the terminology of ref. [2].

Deutsch studied the simpler case of observables with purely discrete spectra. Partovi extended these methods to the general case of discrete and continuous spectra and applied them to the most important pairs of observables: angle-angular momentum and position-momentum. In these two cases, he obtained certain expressions for the lower bounds $b$, but his results are not optimal.

In what follows, I shall derive the optimal bound for the angle-angular momentum pair and improve the lower bound for the position-momentum pair.

In the first case, the left-hand side of (1) is

$$S^\phi + S^{L_z} = -\sum_i \int d\phi |\psi(\phi)|^2 \times \ln \left( \int_{\Delta \phi_i} d\phi |\psi(\phi)|^2 \right)-\sum_m |c_m|^2 \ln |c_m|^2, \quad (3)$$

where $\psi(\phi)$ is the angular wave function and $c_m$ are its expansion coefficients into the eigenfunctions of $L_z$,

$$\psi(\phi) = (2\pi)^{-1/2} \sum_m c_m e^{im\phi}. \quad (4)$$

Since $x \ln x$ is a convex function, the following inequality holds $^+$ [3]

$$\frac{1}{\Delta \phi} \int d\phi |\psi|^2 \ln |\psi|^2 \geq \frac{1}{\Delta \phi} \int d\phi |\psi|^2 \ln \left( \frac{1}{\Delta \phi} \int d\phi |\psi|^2 \right). \quad (5)$$

I shall assume that all the integration intervals in (3) are of equal length and next I will use (5) to obtain the inequality.

$^+$ The value of a convex function at the mean value of the argument does not exceed the mean value of the function.
\[ S^\phi + S^{L_z} \geq -\int_0^{2\pi} d\phi |\psi|^2 \ln |\psi|^2 - \ln \Delta \phi \]
\[ - \sum_m |c_m|^2 \ln |c_m|^2. \]  
(6)

At this point I can use, for the sum of the two entropies appearing on the right-hand side of (6), the following inequality proved by Mycielski and myself some time ago [4]:
\[ -\int_0^{2\pi} d\phi |\psi|^2 \ln |\psi|^2 - \sum_m |c_m|^2 \ln |c_m|^2 \geq \ln 2\pi. \]  
(7)

This is the crucial step in my derivation. It leads to the optimal entropic uncertainty relation,
\[ S^\phi + S^{L_z} \geq -\ln (\Delta \phi/2\pi), \]  
(8)
for the angle-angular momentum pair. The sum of the two entropies attains its lower bound on the eigenstates of \( L_z \).

In a similar fashion, I can improve the lower bound for the position-momentum pair. To this end, I shall use another entropic uncertainty relation,
\[ -\int dx |\bar{\psi}|^2 \ln |\bar{\psi}|^2 - \int dp |\bar{\psi}|^2 \ln |\bar{\psi}|^2 \hbar \geq 1 + \ln \pi, \]  
(9)
where
\[ \bar{\psi}(p) = (2\pi\hbar)^{-1/2} \int dx \exp(-ipx/\hbar) \psi(x). \]  
(10)

This inequality has an interesting history. It has been conjectured (but not proven) by Everett in 1957 in an extended version of his famous Ph.D. thesis [5], in which he introduced the many-worlds interpretation of quantum mechanics and also independently, at the same time, by Hirschman [6]. The first proof of this inequality was given by Bialynicki-Birula and Mycielski [4] and independently by Beckner [7]. It was also used to obtain the lower energy energy bound for the logarithmic Schrödinger equation [8].

Following the same line of reasoning, as in the derivation of (8), one can obtain with the help of (9):
\[ S^{x} + S^{p} \geq 1 - \ln 2 - \ln \gamma, \]  
(11)
where \( \gamma = \Delta x \Delta p/\hbar \). This is not the optimal bound. However, it becomes one in the limit, when \( \gamma \) tends to zero. For example, for \( \gamma = 0.01 \), the relative difference between the left- and the right-hand side of (11) is only 3.5\%, for gaussian wave functions, which saturate (9). Thus for small bin sizes, my lower bound significantly improves the result \( 2 \ln [2/(1 + \sqrt{\gamma})] \), obtained in ref. [2].

References