Theorem Concerning Gauge Invariance in Quantum Electrodynamics

I. BIALYNICKI-BIRULA
Institute of Theoretical Physics, Warsaw University, Warsaw, Poland
(Received 3 October 1966)

It is pointed out that the theorem on gauge invariance of transition amplitudes in quantum electrodynamics given by Feynman and used by many others is incorrect. The invariance of physical scattering amplitudes under the gauge transformations of the photon propagator is briefly discussed.

In his classical paper[1] Feynman proves the following theorem on the gauge invariance of the transition amplitudes. Transition amplitudes obtained directly from Feynman diagrams are not influenced on the mass shell by any terms proportional to either k_μ or k_ν in the photon propagator $D_{\mu\nu}{}^F(k)$. Since 1949, this theorem has been widely used by various authors.[2]

The purpose of this article is to show that the theorem is false and to explain why all physical scattering amplitudes are invariant under gauge transformations of the photon propagator.

$$D_{\mu\nu}f(k) \rightarrow D_{\mu\nu}f(k)+k_u k_\nu f(k^2).$$

Feynman's proof is based on some identities,[3] which are derived under a tacit assumption that none of the factors $(m-\mathbf{p}_k)^{-1}$ becomes singular on the mass shell. This assumption is not valid whenever there are radiative corrections to external electron lines. Such corrections are present in all higher orders of perturbation theory, and therefore, the theorem is valid only in the lowest order of perturbation theory (for example, for Møller or Bhabha scattering).

As a simple example, consider the scattering of an electron in an external electromagnetic field A_μ in the second order of perturbation theory.[4] In this case, the longitudinal part $k_\mu k_\nu f(k^2)$ of the photon propagator contributes on the mass shell, the following terms to the transition amplitude:

$$-\gamma^\mu \tilde{A}_\mu(p_1-p_2)\int d_4k\, f(k^2).$$

This difference from a zero result is to be expected on the basis of renormalization theory. Unrenormalized S-matrix elements obtained from Feynman diagrams have no direct physical meaning. They appear in physical-scattering amplitudes always multiplied by renormalization constants Z_2 and Z_3. While Z_3 is gauge invariant, Z_2 is gauge dependent.[5] The unrenormalized transition amplitudes, proved erroneously in Ref. 1 to be invariant, must, therefore be, gauge-dependent in order to secure the gauge invariance of the product: renormalization constants times unrenormalized matrix elements. Gauge-transformation properties of the propagators in quantum electrodynamics[6] enable one to prove directly that this invariance is, indeed, obtained.

Finally, we would like to mention that various proofs of the invariance of the unrenormalized S operator under the gauge transformations of the photon propagator are also incorrect. All such proofs[7] employ integration by parts and the boundary terms are not taken properly into account. When these terms are computed correctly, they contribute precisely the factors needed to make the renormalized S-matrix elements invariant under the gauge transformations of the photon propagator.

---

[1] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[2] F. J. Dyson, Phys. Rev. **77**, 420 (1950); Z. Koba, N. Mugibayashi, and S. Nakai, Progr. Theoret. Phys. (Kyoto) **6**, 322 (1951); J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), p. 195; S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), p. 556; T. T. Wu, Phys. Rev. **129**, 1420 (1963); S. Weinberg, *ibid.* **138**, B988 (1965); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965), p. 198.

[3] Reference 1, p. 781.

[4] Reference 1. Fig. 4.

[5] It can even be made finite in every order of perturbation theory by a proper choice of the longitudinal part of the photon propagator [I. Bialynicki-Birula, Nuovo Cimento **17**, 122 (1960)].

[6] L. D. Landau and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **29**, 89 (1955) [English transl.: Soviet Phys.—JETP **2**, 69 (1956)]; B. Zumino, J. Math. Phys. **1**, 1 (1960); I. Bialynicki-Birula, Nuovo Cimento **17**, 951 (1960).

[7] F. Coester and J. M. Jauch, Phys. Rev. **78**, 149 (1950); S. T. Ma, *ibid.* **80**, 729 (1950); R. Utiyama, T. Imamura, S. Sunakawa, and T. Dodo, Progr. Theoret. Phys. (Kyoto) **6**, 587 (1951); P. Rastall, Nucl. Phys. **30**, 664 (1962); A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience Publishers, Inc., New York, 1965), Sec. 24; G. Gallavotti, Nuovo Cimento **45A**, 847 (1966).